MINIMAL SURFACES
WS 05/06

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Contents

References

Books surveying minimal surfaces:

[J] Jost : Differentialgeometrie und Minimalflächen, Springer 1994 (The default text for this class are Sections 1 to 4. Differential geometry and analysis is the focus of the book.)

[O] Osserman: Minimal surfaces, Dover (Unlike Jost, here the focus lies on the complex methods, in particular the discussion of the Gauss map. A nice short text.)

[DHKW] Dierkes, Hildebrandt, Küster, Wohlrab: Minimal surfaces I + II, Springer 1992 (This is a broad layout of the theory, accompanied with many examples and pretty pictures. Next edition to appear.)

[O1] Oprea: The mathematics of soap films: Explorations with Maple, AMS 2000 (A more elementary text.)

[L] Lawson: Minimal submanifolds, Publish or Perish 1980 (The first chapter deals with minimal submanifolds of Riemannian manifolds. The second chapter is much more specific and contains the solution of the Plateau problem in $\mathbb{R}^3$.)

[N] Nitsche: Vorlesungen über Minimalflächen, Springer 1975 (An English translation of the first half appeared at Cambridge. A text not aimed at students. The exposition seems not accessible on the first place; but with longtime usage the values of this opus are obvious.)

[BC] Barbosa, Colares: Minimal surfaces in $\mathbb{R}^3$, Springer Lecture Notes 1195 (Finite total curvature minimal surfaces are studied; written before some major advances of the theory.)

[Ka] Karcher: Construction of minimal surfaces, Tokyo 1989

[Ku] Kuwert: Vorlesung Minimalflächen 1998
`http://home.mathematik.uni-freiburg.de/analysis/lehre/skripten/`

[St] Struwe: Plateau's prolem and the calculus of variations, Princeton 1988

Articles surveying modern research:

[HK] Hoffman, Karcher: In Geometry V

[R] H. Rosenberg: Some recent developments in the theory of minimal surfaces in 3-manifolds, Publicações Matemáticas, Impa 2003

Books introducing general theory being used for minimal surface:

[GT] Gilbarg, Trudinger: Elliptic partial differential equations of second order, Springer 1977, 1983 (contains all the results about partial differential equations we need.)

[A] Ahlfors: Complex analysis, McGraw-Hill 1953, 1979

[FL] Fischer-Lieb: Funktionentheorie, Vieweg

[FK] Farkas-Kra: Riemann surfaces, Springer 1980, 1992

[Sy] Sauvigny: Partielle Differentialgleichung in der Physik und Geometrie 1+2, Springer 2004

[M] Morgan: Geometric measure theory, Academic Press 1988, 2000

Popular books:

[HT] Hildebrandt, Tromba: Kugel, Kreis und Seifenblasen, Birkhäuser 1996

Introduction

The present lectures cover a two term class given at Darmstadt. Students from the third and fourth year attended. While the class was given in German, wishes from the audience lead to the notes being written in English.

The class provides an introduction to minimal surfaces. The prerequisites are only the material taught in a traditional German analysis course. All necessary facts from differential geometry and partial differential equations (mostly harmonic functions) are derived in these notes.

Part 1 is a survey of surface geometry. The main goal is to introduce mean curvature.

Part 2 serves as an introduction to minimal surfaces. First, various equations for minimality are derived. Then I present the classical examples of minimal surfaces; I present them alongside with properties which make them interesting. Only after that, first and second variation are discussed. Foliation techniques are then introduced to study stability questions; as a consequence, minimal surfaces are shown to be locally area minizing. The maximum principle is then presented to give the basic uniqueness statements for minimal surfaces.

In Part 3 the Plateau problem is discussed, mostly following Jost's book. Our proof gives that the Plateau solution is critical for area, but we do not verify it is in fact the minimizer of area. The properties of harmonic functions needed for the proof are supplied in a separate section. The second term started with a discussion of branch points. Osserman's proof on the absence of branch points in the interior of a Plateau solution is sketched. Then I proved how the Plateau problem can be used to construct complete periodic minimal surfaces: A polygonal boundary bounds a Plateau solution which may be refleted across its boundary arcs. The Schwarz P- and D surfaces provide examples.

Part 4 is devoted to the Weierstrass representation, which is used in its original version due to Weierstrass. I follow Karcher's lecture notes [K]. Three examples are discussed: Higher order Enneper surfaces, the simply periodic Scherk surface. Ultimately, the Costa surface is introduced along the necessary theory of elliptic functions. Only the local version is used, but there is a section included on the global version. The lectures conclude with the conjugate surfaces technique.

I regret that in the last part a number of topics is used but not introduced thoroughly. This concerns in particular integrability issues: The existence of Weierstrass data for an arbitrary minimal surface is not established. Also I used fundamental group and covering theory for the Weierstrass representation. Geodesics play an important rôle in the later symmetry discussions, but also have not been covered. Another ommission is the mapping degree. Would it be introduced, it would streamlight various proofs.

I thank Steffen Fröhlich and Matthias Bergner for discussions, suggestions, and corrections which improved the class and these notes. Besides the accessible sources listed in the references, I made use of notes of a lecture given by F. Sauvigny in 1997.

Darmstadt, July 06

Part 1. **Differential geometry of surfaces**

`2. Lecture, Tuesday 1.11.05`

Originally, minimal surfaces received their name according to their property to minimize area for prescribed boundary values. Indeed a soap film, being the physical model of a minimal surface, has precisely this property.

Much success with the analysis of real-valued functions comes from the approach to consider all critical points. The subsequent investigation of critical points yields the well-known and somewhat subtle sufficient and necessary conditions for minima.

Similarly so for "minimal" surfaces: The modern definition of a minimal surface is that the surface is critical for the area functional. This turns out to be equivalent to the mean curvature to vanish.

We will investigate the variational problem only in the next part. In the present part, we introduce the notions of classical differential geometry, which will allow us to formulate the variational problem in terms of geometric quantities.

1. Parameterized curves and surfaces

The notion of curvature for curves will be used lateron to define the curvature of surfaces. For this reason we start with a short subsection on curves.

1.1. **Curvature of curves.**

Definition. (i) A *parameterized curve* is a smooth (arbitrarily often differentiable) map $c\colon I \to \mathbb{R}^n$, $n \geq 2$, where $I \subset \mathbb{R}$ is an interval.
(ii) The curve c is *regular* if its *tangent vectors* $c'(t)$ are nonzero for all $t \in I$.

In physics a parametrized curve c describes a motion: t is time, $c(t)$ is the object in motion (point with mass), and $c'(t)$ is the velocity vector.

Examples. 1. The parametrized curves $t \mapsto (t^2, t^3)$ and $t \mapsto (t^3, t^3)$ are not regular.
2. A *helix* with height increment $2\pi h \in \mathbb{R}$ and radius $r \in \mathbb{R}$ can be parameterized

$$c\colon [a,b] \to \mathbb{R}^3, \qquad c(t) = \big(r\cos t, r\sin t, ht\big).$$

For $h = 0$ the helix becomes a circle, for $r = 0$ a vertical line. As $c'(t) = \big(-r\sin t, r\cos t, h\big)$ all tangents have the same length $\ell := \sqrt{r^2 + h^2}$; they run on a circle of latitude on the sphere of radius r. In particular, only for $h = r = 0$ is c not regular.

3. Consider the unit circle $c(t) = (\cos t, \sin t, 0)$. We let $v(t)$ be the unit vector obtained by a rotation of $c(t)$ by the angle $\frac{t}{2}$ about the axis $c'(t)$, that is,

$$v(t) := \Big(\cos\frac{t}{2}\cos t,\ \cos\frac{t}{2}\sin t,\ \sin\frac{t}{2}\Big).$$

For $|\varepsilon| < 1$ define

$$c_\varepsilon\colon [0, 4\pi] \to \mathbb{R}^3, \qquad c_\varepsilon(t) = c(t) + \varepsilon v(t),$$

which is regular (check!). Note that $(\varepsilon, t) \mapsto c_\varepsilon(t)$ is a *Moebius strip*. This curve is *closed*, i.e., c and its derivatives agree at the endpoints; moreover, it is *embedded*, that is, $c|_{[0,4\pi)}$ is injective.

By a change of variable, each regular curve can be parameterized with unit speed:

Proposition 1. *Let $c\colon [a,b] \to \mathbb{R}^n$ be a regular curve of* length $L := \int_a^b |c'(t)|\,dt$. *Then there is a diffeomorphism $\varphi\colon [0,L] \to [a,b]$ such that $\tilde c := c \circ \varphi\colon [0,L] \to \mathbb{R}^n$ is* parametrised by arc length, *i.e., $|\tilde c'| \equiv 1$.*

Proof. By regularity, the auxiliary function

$$\ell\colon [a,b] \to [0,L], \qquad \ell(t) := \int_a^t |c'(s)|\,ds$$

satisfies $\ell'(t) = |c'(t)| > 0$. Thus it has an inverse $\varphi := \ell^{-1}$ with derivative

$$\varphi'(t) = \frac{1}{\ell'(\varphi(t))} = \frac{1}{|c'(\varphi(t))|} > 0.$$

This gives

$$\big|(c \circ \varphi)'(t)\big| = \big|c'\big(\varphi(t)\big)\varphi'(t)\big| = 1.$$

□

We define the curvature of a curve c as the change of the unit tangent direction. Hence we assume c has an arc length parametrization and define its *curvature [Krümmung]* by

$$\kappa\colon I \to [0,\infty), \qquad \kappa(t) := |c''(t)|.$$

Our notion of curvature is unoriented, meaning that a change of the sense of parametrisation leaves κ unchanged.

The arc length parametrization gives

$$0 = \frac{d}{dt}|c'|^2 = 2\langle c', c''\rangle \quad\Rightarrow\quad c'' \perp c'.$$

Thus in physics, curvature becomes the acceleration of a point with mass, as long as it moves with unit velocity.

Occasionally, we call the vector $c''(t)$ the *curvature vector*. At points with nonzero curvature, we call its unit direction

$$n(t) := \frac{c''(t)}{|c''(t)|}$$

the *normal* of the curve c.

We state three facts involving curvature without proof:

The curvature is the inverse radius of the best fitting circle at $c(t)$,

$$\kappa(t) = \frac{1}{r(t)}, \tag{1}$$

where $r(t) = \infty$ in the case that the circle becomes a line. This is the most intuitive way to define the curvature. To make this statement rigorous, let us consider triples $t_- < t < t_+$. the points $c(t_-), c(t), c(t_+)$ define a circle (or line). The circle (or line) which fits the curve best is the limiting circle (or line) for $t_-, t_+ \to t$. It can be seen to be contained in the plane spanned by $c'(t)$ and $n(t)$, and its radius $r(t) \geq 0$ satisfies (1).

Problems. 1. A graph $c(t) = \big(t, f(t)\big)$ is a particular curve. Show that at a point with horizontal tangent, $f'(t_0) = 0$, we have $\kappa(t_0) = |f''(t_0)|$.

2. For fixed endpoints, a curve with minimal length in $\mathbb{R}^n$ has curvature 0.

1.2. **Surfaces.** We use U to denote a *domain [Gebiet]* of $\mathbb{R}^n$, that is, the subset is open and (arc-)connected. A mapping $f \in C^1(U, \mathbb{R}^m)$ is an *immersion*, if the differential $df_p\colon \mathbb{R}^n \to \mathbb{R}^m$ has rank n for each $p \in U$ (why does $n \leq m$ hold?). Conforming to the standards of analysis, a mapping $f \in C^1(V, \mathbb{R}^m)$ for $V \subset \mathbb{R}^n$ arbitrary is called a surface if there is an open set $U \supset V$ such that f has a C^1-extension to V which is an immersion. Similar to the case of curves, immersions are also called *regular surfaces [regulären Flächen]*. Note, however, that there are different uses of the term *regular* for mappings.

As for curves, we will allow for self-intersections of our surfaces and define:

Definition. A *(parametrised) surface [(parametrisierte) Fläche]* is a smooth immersion $f\colon U \to \mathbb{R}^m$.

Each (smooth) diffeomorphism $\varphi\colon V \to U$ gives rise to a *change of variables* of a parameterised surface $f\colon U \to \mathbb{R}^m$. In *differential geometry* we are interested in those properties of surfaces which remain invariant under change of variables. To do this in a formally satisfying manner, note that change of variables gives rise to an equivalence relation; its equivalence classes are the *surfaces [Flächenstücke]*.

Usually we do not depend on the smoothness of our surfaces but we need they are twice differentiable. If the immersion happens to be an embedding, that is, a homeomorphism onto its image, then $f(U)$ is a submanifold of $\mathbb{R}^m$. Nevertheless, we do wish to admit the case of self-intersections as well.

To each $p \in U$ and $X \in \mathbb{R}^n$ we can associate the *tangent vector* $df_p(X)$. Their set forms the *tangent space* in p,

$$T_p f := \{df_p(X) \mid X \in \mathbb{R}^n\},$$

which is an n-dimensional subspace of $\mathbb{R}^m$. Since f need not be an embedding the indexing of the tangent spaces by points p of the parametrising domain U means that we can distinguish possibly different tangent spaces at the same image point in $\mathbb{R}^m$, the so-called multiple points.

The immersion property implies:

Lemma 2. *For each $p \in U$ the tangent map $df_p\colon \mathbb{R}^n \to T_p f$ is an isomorphism of vector spaces.*

`3. Lecture, Wednesday 1.11.05` ______

We can always write a tangent vector $df_p(X)$ to a surface as a tangent vector to a curve contained in the surface: For instance, we can pick $\gamma\colon (-\varepsilon, \varepsilon) \to U$, $\gamma(t) := p + tX$, and $c := f \circ \gamma$, to obtain

$$c'(0) = df_{\gamma(0)}\big(\gamma'(0)\big) = df_p(X).$$

1.3. **First fundamental form.** Parametrisation by arc length is useful for curves. But not every surface admits a length preserving parametrisation, as is wellknown from cartography. For this reason, we must use length distorting parametrisations. We can regard the distortion as a property of the domain:

Definition. Let $f\colon U \to \mathbb{R}^m$ be a parametrised surface, and $p \in U$. The *first fundamental form [erste Fundamentalform]* is the mapping

$$g\colon U \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \qquad g_p(X,Y) := \big\langle df_p(X), df_p(Y)\big\rangle.$$

We call $\|X\|_p := |df_p(X)| = \sqrt{g_p(X,X)}$ the *(Riemannian) length* of X.

The first fundamental form is bilinear and symmetric in (X,Y); it is positive definite, $g_p(X,X) = \big|df_p(X)\big|^2 > 0$ for $X \neq 0$, using the immersion property of f. Thus the first fundamental form is a scalar product which depends smoothly on p.

To introduce the matrix notation for the first fundamental form, let us denote the standard basis on $U \subset \mathbb{R}^n$ by $e_1, \ldots, e_n$. Then we set

$$g_{ij} := g_p(e_i, e_j) = \Big\langle \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j} \Big\rangle, \qquad 1 \le i,j, \le n$$

and see that (g) is symmetric. For $f = \mathrm{id}$ or $f(x) = (x, 0)$ we have the standard metric $g_{ij} = \delta_{ij}$.

Consider two *vector fields* $X, Y \in C^\infty(U, \mathbb{R}^n)$. We write

$$X(p) = \big(X^1(p), \ldots, X^n(p)\big) = \sum_{i=1}^n X^i(p) e_i$$

and similarly for $Y(p)$, to obtain

$$g(X,Y) = g\big(\textstyle\sum_i X^i e_i, \sum_j Y^j e_j\big) = \displaystyle\sum_{i,j=1}^n X^i Y^j g(e_i, e_j) = \sum_{i,j=1}^n g_{ij} X^i Y^j.$$

Examples. 1. The *helicoid [Wendelfläche]* is the parametrized surface

$$f\colon \mathbb{R}^2 \to \mathbb{R}^3, \qquad f(x,y) := (x \sin y, -x \cos y, y) \tag{2}$$

Note that for each fixed x the curves $y \mapsto f(x,y)$ are helices of radius x and with height increment 2π. For each fixed y the curves $x \mapsto f(x,y)$ are straight lines. We compute

$$\frac{\partial f}{\partial x}(x,y) = \begin{pmatrix} \sin y \\ -\cos y \\ 0 \end{pmatrix}, \qquad \frac{\partial f}{\partial y}(x,y) = \begin{pmatrix} x\cos y \\ x \sin y \\ 1 \end{pmatrix}$$

Thus

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 + x^2 \end{pmatrix}. \tag{3}$$

Note that g_{22} gives the squared length of the tangent vector to the helices, which increases with x.

2. For a given (height) function $u\colon U \subset \mathbb{R}^n \to \mathbb{R}$ we consider the graph $f(x) := \big(x, u(x)\big)$ with

$$g_{ij} = \langle \partial_i f, \partial_j f \rangle = \left\langle \begin{pmatrix} e_i \\ \partial_i u \end{pmatrix}, \begin{pmatrix} e_j \\ \partial_j u \end{pmatrix} \right\rangle = \delta_{ij} + \partial_i u \, \partial_j u. \tag{4}$$

Thus a tangent vector X has squared length

$$\|X\|^2 = \sum_{i,j} X^i \big(\delta_{ij} + \partial_i u \, \partial_j u\big) X^j = |X|^2 + \langle \nabla u, X\rangle^2.$$

Indeed, perpendicularly to the gradient ∇u, the tangent plane is a length preserving graph, while vectors in the tangent plane parallel to ∇u are increased in length.

3. By means of the first fundamental form we can represent the arc length of a curve $c := f \circ \gamma$ as an integral in the parametrising domain:

$$L(c) = \int_a^b |c'(t)| \, dt = \int_a^b |df_{\gamma(t)} \gamma'(t)| \, dt = \int_a^b \big\|\gamma'(t)\big\|_{\gamma(t)} \, dt$$

1.4. **Conformality, isometry, area.** We discuss three properties determined by the first fundamental form.

Angle preserving parametrisations will become important for us:

Definition. A surface $f\colon U \subset \mathbb{R}^n \to \mathbb{R}^m$ has a *conformal [konform]* parametrisation if there is $\varphi\colon U \to (0,\infty)$ such that for all $1 \le i,j \le n$

$$g_{ij}(p) = \varphi(p)\delta_{ij}(p) \qquad \text{for all } p \in U.$$

A two-dimensional surface f is conformal, if and only if $g = \left(\begin{smallmatrix}\varphi & 0\\ 0 & \varphi\end{smallmatrix}\right)$ for $\varphi(x,y) > 0$, or

$$|f_x| \equiv |f_y| > 0, \qquad \langle f_x, f_y\rangle \equiv 0.$$

According to a deep result, any two-dimensional has a conformal parameterization, see [Sy2, XII,§8, Satz 2].

Example. By (3) the helicoid $f(y,x) = (x\sin y, -x\cos y, y)$ is not conformal. However, the reparameterization

$$\tilde f(x,y) = \begin{pmatrix}\sinh x \sin y\\ -\sinh x\cos y\\ y\end{pmatrix} \tag{5}$$

with

$$\tilde f_x(x,y) = \begin{pmatrix}\cosh x\sin y\\ -\cosh x\cos y\\ 0\end{pmatrix}, \qquad \tilde f_y(x,y) = \begin{pmatrix}\sinh x\cos y\\ \sinh x\sin y\\ 1\end{pmatrix}$$

satisfies

$$\tilde g_{11} = |\tilde f_x|^2 = \cosh^2 x = 1 + \sinh^2 x = |\tilde f_y|^2 = \tilde g_{22} \quad \text{and} \quad \tilde g_{12} = \langle \tilde f_x, \tilde f_y\rangle = 0. \tag{6}$$

Thus $\tilde f$ is conformal.

More generally, we will call a mapping $f\colon U^n \to \mathbb{R}^m$ *conformal*, if

$$|f_x| \equiv |f_y|, \qquad \langle f_x, f_y\rangle \equiv 0. \tag{7}$$

That is, a conformal map for which $df \neq 0$ is a conformally parameterized surface.

Example. A holomorphic map $f\colon U \subset \mathbb{C} \to \mathbb{C}$ is conformal (exercise).

Another important notion is isometry.

Definition. Two surfaces $f\colon U \to \mathbb{R}^m$ and $\tilde f\colon \tilde U \to \mathbb{R}^m$ are *isometric [isometrisch]* if there is a diffeomorphism $\varphi\colon U \to \tilde U$, such that the two surfaces $f, \tilde f\circ\varphi\colon U \to \mathbb{R}^m$ have first fundamental forms g, $\tilde g$ which agree in the following sense:

$$g(X,Y) = \tilde g\big(d\varphi X, d\varphi Y\big) \qquad \text{for all } X, Y \in \mathbb{R}^n.$$

An isometry means that the Riemannian lengths and angles agree. This does not mean that the surfaces agree up to ambient motion A particular case is for $\varphi = \mathrm{id}$; then the condition is simply $g = \tilde g$. this may hold although $f \neq \tilde f$.

Imagine we have a physical model of an (embedded) surface made from an unelastic material such as metal, or paper; clearly, we might need some very special curved paper! If thin enough, the material will allow for bending (but not for streching). Such a model will fit any isometric surface as well.

Example. 1. Cylinder and plane are isometric.
2. The *catenoid [Katenoid]*,

$$f\colon \mathbb{R}^2 \to \mathbb{R}^3, \qquad f(x,y) := \begin{pmatrix}\cosh x\cos y\\ \cosh x \sin y\\ x\end{pmatrix}, \tag{8}$$

with

$$f_x(x,y) = \begin{pmatrix}\sinh x\cos y\\ \sinh x\sin y\\ 1\end{pmatrix}, \qquad f_y(x,y) = \begin{pmatrix}-\cosh x\sin y\\ \cosh x\cos y\\ 0\end{pmatrix}$$

satisfies

$$g_{11} = |f_x|^2 = \sinh^2 x + 1 = \cosh^2 x = |f_y|^2 = g_{22} \quad \text{and} \quad g_{12} = \langle f_x, f_y\rangle = 0.$$

Thus g agrees with the first fundamental form of the conformal helicoid; the two surfaces are isometric! Note that the surfaces have a very different appearence, however.

As known from Analysis III, the first fundamental form determines the n-dimensional area content $A(f) = A_U(f)$ of a surface $f\colon \mathbb{R}^n \to \mathbb{R}^{n+k}$ by

$$A_U(f) = \int_U \sqrt{\det g}\, dx.$$

Note that in general $A_U(f)$ is not the area of the set $f(U)$, but counts the area of $f(U)$ with multiplicity. For an embedding, however, the two quantities agree. By means of a partition of unity, we can define the area of a hypersurface which needs more than one parametrising neighbourhood

We call $\det g = \det(df^T df)$ the *Gram determinant*. We recall the computation proving that f remains invariant under change of parameters: For $\varphi\colon V \to U$ a diffeomorphism and $\tilde f := f \circ \varphi$ we have according to chain rule and transformation formula

$$\begin{aligned} A_V(\tilde f) &= \int_V \sqrt{\det(d\tilde f_\varphi^T d\tilde f_\varphi)}\,dx = \int_V \sqrt{\det(d\varphi^T df_\varphi^T df_\varphi d\varphi)}\,dx \\ &= \int_V \sqrt{\det d\varphi^T \det(df_\varphi^T df_\varphi)\det d\varphi}\,dx = \int_V \sqrt{\det(df_\varphi^T df_\varphi)}\,|\det d\varphi|\,dx \\ &= \int_U \sqrt{\det(df^T df)}\,dx = A_U(f(U)); \end{aligned}$$

here, the notation $df_\varphi(x)$ means the differential df is evaluated at the point $\varphi(x)$.

The important special case $n = 2$ with $f = f(x, y)$ has the alternative representation

$$A_U(f) = \int_U \sqrt{g_{11}g_{22} - g_{12}^2}\,dxdy = \int_U \sqrt{|f_x|^2|f_y|^2 - \langle f_x, f_y\rangle^2}\,dxdy = \int_U \big|f_x \times f_y\big|\,dxdy,$$

that is, the area is given by integrating the (outer) vector product. Here, we used $|v|^2|w|^2 = \langle v, w\rangle^2 + |v \times w|^2$ for any two vectors $v, w \in \mathbb{R}^n$.

2. The normal mapping of hypersurfaces and its derivatives

We specialize from now on to the case of codimension 1, that is, $m = n + 1$. Surfaces $f : U \to \mathbb{R}^{n+1}$ are called *hypersurfaces*.

2.1. **Normal curvatures and the Gauss map.** For hypersurfaces the *normal space*

$$N_pf := T_pf^\perp, \qquad p \in U,$$

is one-dimensional. It is useful to span this space by a continuous choice of unit vectors:

Definition. Let $f : U \to \mathbb{R}^{n+1}$ be a parametrised hypersurface. A continuous (and hence differentiable) map $\nu : U \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ with $\langle \nu(p), df_p(X)\rangle = 0$ for all $X \in \mathbb{R}^n$ is called the *Gauss map* or *normal mapping*.

The outer product can be used to define a normal. For instance, in case $n = 2$,

$$\nu := \frac{f_x \times f_y}{\big|f_x \times f_y\big|},$$

where $v \times w = \big(v_2w_3 - v_3w_2,\, v_3w_1 - v_1w_3,\, v_1w_2 - v_2w_1\big)$. This formula shows that ν is smooth.

Example. For a graph $f(x) = \big(x, u(x)\big)$ the upper normal is

$$\nu(x) = \frac{1}{\sqrt{1 + |\nabla u|^2}}\big(-\nabla u, 1\big). \tag{9}$$

Remark. The two normals are distinguished by the determinant of $\big(df(e_1), \dots, df(e_n), \nu\big)$. If it is positive, we say (f, ν) is *positively oriented*, else negative.

Let us study curves in surfaces with respect to their curvature. So let $\gamma : (a, b) \to U$ be a curve and $c := f \circ \gamma$. We suppose that c is parametrized by arc length, meaning that γ has the Riemannian length $\|\gamma'\| \equiv 1$. To analyse how the space curve c bends with respect to the surface (f, ν), we decompose the curvature vector c'' orthogonally in a component tangent and another normal to the tangent space:

$$c'' = c''^\top + c''^\perp = c''^\top + \langle c'', \nu \circ \gamma\rangle(\nu \circ \gamma) \quad \in T_\gamma f \oplus N_\gamma f. \tag{10}$$

The moduli of the vectors on the right hand side are the essential expressions of surface geometry. We are interested in the *normal curvature* $\langle c'', \nu \circ \gamma\rangle$. It measures the curvature of the surface (in direction c') relative to the ambient space. On the other hand, the *geodesic curvature* $|c''^\top|$ measures how the curve bends within the surface; it is of lesser importance in the context of minimal surfaces.

`4. Lecture, Tuesday 8.11.05`

Examples. 1. If the surface is a plane then any curve only has geodesic curvature.
2. A great circle in the sphere is an example where the curvature is entirely normal curvature.

For hypersurfaces we have the following expression for the normal curvature

$$\kappa_{\text{norm}}(c) := \langle c'', \nu \circ \gamma\rangle \overset{\langle c', \nu\rangle = 0}{=} -\big\langle c', d\nu_\gamma(\gamma')\big\rangle = -\big\langle df(\gamma'), d\nu_\gamma(\gamma')\big\rangle.$$

This formula leads us to study the differential $d\nu$ of the Gauss map more closely.

2.2. **The shape operator.** When we run in U along a curve $\gamma(t)$, what is the effect on the surface normal $\nu \circ \gamma$? First of all, $d\nu_\gamma(\gamma')$ is tangential:

Lemma 3. *The map $d\nu_p$ maps the space $\mathbb{R}^n$ onto T_pf for each $p \in U$.*

Proof. We calculate $d\nu_p(X)$ by restricting ν to a curve γ in U with $\gamma(0) = p$ and $\gamma'(0) = X$. Since

$$0 = \langle \nu \circ \gamma, \nu \circ \gamma\rangle' = 2\langle d\nu_\gamma(\gamma'), \nu \circ \gamma\rangle$$

we conclude for $t = 0$ that $0 = \langle d\nu_p(X), \nu(p)\rangle$, that is, $d\nu_p(X) \perp \nu(p)$. $\square$

We would like to measure the effect of $d\nu$ on the parameterizing domain U. By Lemma 2 we can solve the equation $d\nu_p(X) = df_p(Y)$ with a unique $Y \in \mathbb{R}^n$. Geometrically the equation means that upon moving in direction X in the parameterizing domain the Gauss map of f will change in direction $df(Y)$. It is useful to consider the map $X \mapsto Y$:

Definition. The *shape operator [Weingartenabbildung]* of (f,ν) is the mapping

$$S\colon U\times\mathbb{R}^n\to\mathbb{R}^n,\qquad X\mapsto S_p(X):=-(df_p)^{-1}\big(d\nu_p(X)\big).$$

The shape operator is well-defined by Lemma 2 and 3. It has the following properties:

1. $X\mapsto S_pX$ is linear at each $p\in U$.

2. We have $df_p(S_pX)=-d\nu_p(X)$. In the expression for S the essential part is $d\nu$ while df plays only the rôle of the necessary transient from the parameterizing domain to the image. Note that S changes its sign with ν.

3. Under a change of parameters $\tilde f:=f\circ\varphi$ the shape operator remains similar:

$$\tilde S_{\tilde p}:=-(d\tilde f_{\tilde p})^{-1}\cdot d\tilde\nu_{\tilde p}=-(d\varphi_{\tilde p})^{-1}\cdot(df_p)^{-1}\cdot d\nu_p\cdot d\varphi_{\tilde p}=(d\varphi_{\tilde p})^{-1}\cdot S_p\cdot d\varphi_{\tilde p}\tag{11}$$

Here, $p=\varphi(\tilde p)$, $\tilde\nu=\nu\circ\varphi$, etc.

With respect to the standard basis $e_1,\dots,e_n$, the shape operator S has a matrix representation $S(e_j)=\sum_{i=1}^n S^i_j e_i$. Here, in order to have the index of summation appear in one upper and one lower position, we lifted one of the indices of S.

Examples. 1. A plane $f\colon\mathbb{R}^2\to\mathbb{R}^3$, $f(x,y)=(x,y,0)$ has a constant normal $\nu\equiv(0,0,1)$. Hence $d\nu=0$ and the shape operator vanishes, $S\equiv0$.

2. For the upper half-cylinder

$$f(x,y):=\big(x,y,\sqrt{1-x^2}\big)\tag{12}$$

we have

$$\frac{\partial f}{\partial x}(x,y)=\begin{pmatrix}1\\0\\\dfrac{-x}{\sqrt{1-x^2}}\end{pmatrix},\quad\frac{\partial f}{\partial y}(x,y)=\begin{pmatrix}0\\1\\0\end{pmatrix}\quad\Rightarrow\quad\nu(x,y)=\begin{pmatrix}x\\0\\\sqrt{1-x^2}\end{pmatrix},$$

meaning we choose the outer normal. Then

$$d\nu_{(x,y)}(e_1)=\Big(1,0,\frac{-x}{\sqrt{1-x^2}}\Big)=df_{(x,y)}(e_1),\qquad\text{and}\qquad d\nu_{(x,y)}(e_2)=0$$

so that $S_pe_1=-e_1$ and $S_pe_2=0$ for all p. In the following we will usually skip the indices denoting the footpoint. In matrix notation, $S=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$.

3. For the helicoid (5) in conformal parameters $f(x,y):=(\sinh x\sin y,-\sinh x\cos y,y)$ we compute

$$\frac{\partial f}{\partial x}(x,y)=\begin{pmatrix}\cosh x\sin y\\-\cosh x\cos y\\0\end{pmatrix},\qquad\frac{\partial f}{\partial y}(x,y)=\begin{pmatrix}\sinh x\cos y\\\sinh x\sin y\\1\end{pmatrix},$$

so that

$$\nu(x,y)=\frac{1}{\cosh x}\begin{pmatrix}\cos y\\\sin y\\-\sinh x\end{pmatrix}.$$

Along the straight x-lines $x\mapsto f(x,y)$ the normal can only rotate so that we expect that Se_1 is some multiple of $df(e_2)$. Indeed, making use of $(\frac{1}{\cosh})'=-\frac{\sinh}{\cosh^2}$ and $(\frac{\sinh}{\cosh})'=\frac{1}{\cosh^2}(\cosh^2-\sinh^2)=\frac{1}{\cosh^2}$, we find at any $(x,y)\in\mathbb{R}^2$

$$df(Se_1)=-d\nu(e_1)=-\frac{\partial\nu}{\partial x}=\frac{1}{\cosh^2x}\begin{pmatrix}\sinh x\cos y\\\sinh x\sin y\\1\end{pmatrix}=\frac{1}{\cosh^2x}df(e_2).\tag{13}$$

Similarly,

$$df(Se_2)=-d\nu(e_2)=-\frac{\partial\nu}{\partial y}=\frac{1}{\cosh x}\begin{pmatrix}\sin y\\-\cos y\\0\end{pmatrix}=\frac{1}{\cosh^2x}df(e_1),\tag{14}$$

which means that along the helices $x\mapsto f(x,y)$ the normals do not tilt but they purely rotate. Certainly along the straight axis $x\mapsto f(x,0)$ we expect this behaviour. This gives a matrix representation of S

$$S_{(x,y)}=\frac{1}{\cosh^2x}\begin{pmatrix}0&1\\1&0\end{pmatrix}\tag{15}$$

2.3. **The second fundamental form.** To study the shape operator in detail, we will relate it to a form which is more useful for computation:

Definition. The bilinear form on $\mathbb{R}^n$

$$b(X,Y):=\big\langle\nu,d^2f(X,Y)\big\rangle\tag{16}$$

is called the *second fundamental form.*

Here d^2f is the *Hessian*, that is, the second directional derivative

$$d^2f(X,Y)=d^2f\Big(\textstyle\sum_i X^ie_i,\sum_j Y^je_j\Big)=\sum_{i,j}X^iY^j\frac{\partial^2f}{\partial x_i\partial x_j}.\tag{17}$$

By the lemma of Schwarz, second partials commute and so the Hessian $d^2f(X,Y)$ is symmetric in X,Y. Thus b is a symmetric bilinear form. From this property we derive a fact which will be crucial in order to define curvatures:

Theorem 4. *The shape operator S is self-adjoint w.r.t. g:*

$$g(SX,Y)=g(X,SY)\qquad\textit{for all } X,Y\in\mathbb{R}^n.$$

Proof. We take a directional derivative of $0 = \langle \nu, df(Y)\rangle$ to obtain

$$0 = \partial_X \big\langle \nu, df(Y)\big\rangle = \big\langle d\nu(X), df(Y)\big\rangle + \big\langle \nu, df^2(X,Y)\big\rangle.$$

Thus we can relate the second fundamental form to the shape operator:

$$b(X,Y) = \big\langle \nu, d^2f(X,Y)\big\rangle = -\big\langle d\nu(X), df(Y)\big\rangle = g(SX,Y) \qquad \text{for all } X,Y \in \mathbb{R}^n. \tag{18}$$

As pointed out before, the left hand side is symmetric and so is the right hand side. □

Formula (18) explains the choice of sign for S.

With respect to the standard basis $e_1, \dots, e_n$, the second fundamental form has a symmetric matrix

$$b_{ij} := b(e_i, e_j) = -\Big\langle \frac{\partial \nu}{\partial x_i}, \frac{\partial f}{\partial x_j}\Big\rangle = \Big\langle \nu, \frac{\partial^2 f}{\partial x_i \partial x_j}\Big\rangle. \tag{19}$$

Applying (16) to $X := e_i$ and $Y := e_j$, we find

$$b_{ij} = g\big(e_i, S(e_j)\big) = g\big(e_i, \textstyle\sum_\ell S^\ell_j e_\ell\big) = \displaystyle\sum_l g(e_i, e_\ell) S^\ell_j = \sum_l g_{i\ell} S^\ell_j.$$

Thus we arrive at the matrix equation $b = gS$ or $S = g^{-1}b$. Writing $(g^{ij}) := \big((g_{ij})^{-1}\big)$ for the matrix inverse of (g_{ij}), this reads as

$$S^i_j = \sum_k g^{ik} b_{kj}. \tag{20}$$

`5. Lecture, Wednesday 9.11.05`

Remarks. 1. Often the classical two-dimensional notation of the first and second fundamental forms due to Gauss is used:

$$g = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \qquad b = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

2. The product of symmetric matrices need not be symmetric. Thus in general S^i_j is not symmetric. Only when working w.r.t. a g-orthonormal basis will S have a symmetric matrix.

Examples. 1. For the conformally parametrized helicoid (2) we have $\langle df(e_1), df(e_2)\rangle = \langle f_x, f_y\rangle = 0$ by (6). Hence from (13) we obtain

$$b_{11} = \big\langle -d\nu(e_1), df(e_1)\big\rangle = \Big\langle \frac{1}{\cosh^2 x} df(e_2), df(e_1)\Big\rangle = 0, \qquad b_{22} = 0.$$

On the other hand, since $|df(e_2)|^2 = |f_y|^2 = \cosh^2 x$ we find from (14)

$$b_{12} = \big\langle -d\nu(e_1), df(e_2)\big\rangle = \Big\langle \frac{1}{\cosh^2 x} df(e_2), df(e_2)\Big\rangle = 1$$

2. For a graph $f(x) = \big(x, u(x)\big)$ the second fundamental form is

$$b_{ij} = \langle \nu, \partial_{ij} f\rangle = \Big\langle \frac{1}{\sqrt{1+|\nabla u|^2}} \begin{pmatrix} -\nabla u \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \partial_{ij} u \end{pmatrix} \Big\rangle = \frac{\partial_{ij} u}{\sqrt{1+|\nabla u|^2}} \tag{21}$$

In particular, at points p with horizontal tangent plane we have $\nabla u(p) = 0$ and so $b_p(X,Y) = d^2u_p(X,Y)$ or $b_{ij}(p) = \partial_{ij} u(p)$.

3. CURVATURE OF HYPERSURFACES

3.1. **Principal curvatures.** Let us return to our initial problem how to determine the normal curvature of a curve $c := f \circ \gamma$ contained in a hypersurface. Here, γ is a curve in U and we assume c is parameterized by arc length.

In terms of the shape operator S, the normal curvature reads

$$\kappa_{\mathrm{norm}}(c) = \langle (f\circ\gamma)'', \nu\circ\gamma\rangle = \langle d^2 f(\gamma', \gamma'), \nu\circ\gamma\rangle = -\big\langle df(\gamma'), d\nu_\gamma(\gamma')\big\rangle = g(\gamma', S\gamma').$$

Thus, surprisingly, κ_{norm} can be computed without knowing c'':

Proposition 5 (Meusnier). *Let $f \colon U \to \mathbb{R}^{n+1}$ be a hypersurface with Gauss map ν. Then the normal curvature κ_{norm} of each curve c is determined by its tangent $c'(t)$ as well as the data g, S of the surface f.*

Example. For the cylinder (12) we have $Se_1 = e_1$ and $Se_2 = 0$. We represent a unit vector $\gamma' = \cos\alpha\, v_1 + \sin\alpha\, v_2$ in terms of the g-orthonormal basis $v_1 = \frac{e_1}{\|e_1\|}$, $v_2 = e_2$. Then

$$\kappa_{\mathrm{norm}}(\gamma) = g(\gamma', S\gamma') = g\big(\cos\alpha\, v_1 + \sin\alpha\, v_2,\ S(\cos\alpha\, v_1 + \sin\alpha\, v_2)\big) \overset{Sv_1 = v_1,\, Sv_2 = 0}{=} \cos^2\alpha,$$

so that $0 \le \kappa_{\mathrm{norm}}(c) \le 1$.

The following consequence of Theorem 4 is a great simplification:

Theorem 6. *Let (f,ν) be a hypersurface in $\mathbb{R}^{n+1}$ parameterized by U. Then for each $p \in U$ there exists a g-orthonormal basis $v_1, \dots, v_n$ of $\mathbb{R}^n$ consisting of eigenvectors for S_p with eigenvalues $\kappa_1, \dots, \kappa_n$. The eigenvalues are independent of the parameterization chosen.*

To see the last claim, note that two representations S and $\tilde S$ are similar by (11).

Definition. An eigenvector $v \in \mathbb{R}^n \setminus \{0\}$ of S_p is called a *principal curvature direction [Hauptkrümmungsrichtung] in p, its eigenvalue $\kappa(p)$ is* a principal curvature *[Hauptkrümmung].*

Note that we have reduced the problem of determining normal curvtures to a standard problem of linear algebra, which appears here depending on a footpoint: We are given a symmetric bilinear form $b_p(X,Y)$ in the euclidean vector space $(\mathbb{R}^n, g_p)$. We are interested in the principal curvature directions $v_1,\ldots,v_n$. We obtain such a basis from the standard basis $e_1,\ldots,e_n$ by a principal axis transformation.

Along a principal curvature direction, the normal tilts with velocity $-\kappa$; it does not rotate.

Examples. 1. On the cylinder (12) mit $Se_1 = e_1$ bzw. $Se_2 = 0$ both coordinate directions are principal with curvatures 1 and 0, respectively.

2. For the helicoid (5) we have at each point $(x,y) \in \mathbb{R}^2$

$$Se_1 = \frac{1}{\cosh^2 x} e_2, \quad Se_2 = \frac{1}{\cosh^2 x} e_1 \qquad \Rightarrow \qquad S(e_1 \pm e_2) = \pm \frac{1}{\cosh^2 x}(e_1 \pm e_2).$$

Consquently, the two diagonals $e_1 \pm e_2$ are principal curvature directions at each (x,y), and their principal curvatures are $\pm 1/\cosh^2 x$.

3. The cases $\mathbb{R}^n$ and $\mathbb{S}^n_r$ are degenerate: Here each direction is principal, and all principal curvatures agree to be 0 and $\frac{1}{r}$, respectively.

There are two general linear algebra facts which we would like to point out for out special case.

Remarks. 1. Two principal curvature directions for different principal curvatures are g-orthogonal. Indeed, from

$$\kappa_1 g(X_1,X_2) = g(SX_1,X_2) = g(X_1,SX_2) = \kappa_2 g(X_1,X_2),$$

and $\kappa_1 \neq \kappa_2$ we conclude $g(X_1,X_2) = 0$.

2. Let us fix $p \in U$. Consider $X \in \mathbb{R}^n$ critical for the normal curvature $g_p(S_pX,X) =: f(X)$, under the constraint that X be contained in the g-unit sphere

$$\{X \in \mathbb{R}^n \mid h(X) := \|X\|_p^2 = 1\}.$$

By the theorem on Lagrange multipliers, X is critical if and only if a $\lambda \in \mathbb{R}$ exists with $\nabla f(X) = \lambda \nabla h(X)$. Equivalently, componentwise we have $df_X(e_i) = \lambda dh_X(e_i)$ for all i. We can compute this condition:

$$df_X(e_i) = \frac{d}{dt} g\big(S(X+te_i), X+te_i\big)\Big|_{t=0} = g\big(SX,e_i\big) + g\big(Se_i,X\big) = g\big(2SX,e_i\big)$$

Similarly, $dh_X(e_i) = \frac{d}{dt} g(X+te_i, X+te_i)\big|_{t=0} = g(2X,e_i)$. Since both equations hold for all basis vectors $e_1,\ldots,e_n$ the eigenvalue equation $SX = \lambda X$ must hold. Thus a vector X in the g-unit sphere is critical for the normal curvature precisely if it is a principal curvature direction. In particular minimal and maximal normal curvatures are attained at principal curvature directions.

Consequently, for surfaces we have a nice minimax characterization of principal curvatures:

Proposition 7. *For surfaces ($n = 2$) the normal curvature*

$$\kappa_{norm} \colon \mathbb{S}^1 \to \mathbb{R}, \qquad v \mapsto g(Sv,v)$$

is maximal in one principal curvature direction and minimal in a perpendicular direction.

We supply a further proof:

Proof. Let $v_1 \perp v_2$ be two unit principal curvature directions. Then the normal curvature in direction $v_\alpha := \cos\alpha\, v_1 + \sin\alpha\, v_2$ becomes

$$g(Sv_\alpha, v_\alpha) = g(\kappa_1 \cos\alpha\, v_1 + \kappa_2 \sin\alpha\, v_2, \cos\alpha\, v_1 + \sin\alpha\, v_2) = \kappa_1 \cos^2\alpha + \kappa_2 \sin^2\alpha, \tag{22}$$

the so-called *Euler formula*. As a function of α, the normal curvature attains its maximum and minimum at $\pm v_1$ and $\pm v_2$. □

Definition. A regular curve $c = f \circ \gamma$ is a
(i) *curvature line [Krümmungslinie]*, if γ' is a principal curvature direction for each t;
(ii) it is an *asymptote line [Asymptotenlinie]*, if the normal curvature $g(S\gamma',\gamma')$ vanishes for all t.

Examples. 1. A straight line c in a hypersurface is always an asymptote line, since $S\gamma' \perp_g \gamma'$.
2. On the cylinder, the circles are curvature lines, and the straight lines are curvature and asymptotic lines.
3. On $\mathbb{S}^n$ each curve is a curvature line.
4. On $\mathbb{R}^n$ each curve is curvature and asymptotic line.

`6. Lecture, Tuesday 15.11.05`

3.2. Mean and Gauss curvature.

Definition. Let $f \colon U \to \mathbb{R}^{n+1}$ be a hypersurface, and $\kappa_1(p),\ldots,\kappa_n(p)$ be the principal curvatures at $p \in U$ w.r.t. a basis $v_1(p),\ldots,v_n(p)$ of principal curvature directions. The *mean curvature [mittlere Krümmung]* H is the mean, the *Gauss curvature* K is the product of the principal curvatures,

$$H(p) := \frac{1}{n}\operatorname{trace} S_p = \frac{1}{n}\big(\kappa_1(p) + \ldots + \kappa_n(p)\big), \qquad K(p) := \det S_p = \kappa_1(p)\cdot\ldots\cdot\kappa_n(p).$$

Examples. 1. The upper half of the unit cylinder (12) with inner normal has constant principal curvatures 0 und 1. Thus $H \equiv \frac{1}{2}$ and $K \equiv 0$.

2. The helicoid (5) has $\kappa_{1,2}(0,0) = \pm 1/\cosh^2 x$ at $(x,y) \in \mathbb{R}^2$. Thus $H \equiv 0$ and $K(x,y) = -1/\cosh^4 x$.

3. For $\mathbb{S}^n_r$ with inner normal we have $H \equiv \frac{1}{r}$ and $K \equiv (\frac{1}{r})^n$. For $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ we have $H \equiv 0$ and $K \equiv 0$.

From (20) we obtain local representations formulas which are useful to compute H, K.

Theorem 8. *Principal curvature directions are eigenvectors of the matrix $(\sum_k g^{ik} b_{kj})_{ij}$. Mean and Gauss curvature are given by*

$$H = \frac{1}{n}\operatorname{trace}(g^{-1}b) = \sum_{i,j} \frac{1}{n} g^{ij} b_{ij}, \qquad K = \det(g^{-1}b) = \det g^{-1} \det b = \frac{\det b}{\det g}. \tag{23}$$

Remarks. 1. As known from linear algebra $\operatorname{trace} S = \sum_i S^i_i$ and $\det S$ are invariant of the choice of basis. Thus H, K are parametrization independent. We can also deduce this from the fact that the principal curvatures κ_i are invariant of parametrization, see Thm. 6. Moreover, a sign change of the normal induces a sign change for H (what about K?).

2. We have

$$\sum_i \langle d\nu(e_i), df(e_i)\rangle = -\sum_{i,k} \langle df(S^k_i e_k), df(e_i)\rangle = -\sum_i S^k_i g_{ki}.$$

Consider a point $p \in U$, such that the basis (e_i) is g-orthonormal; such a parameterization exists according to Lemma 9 below. Then at the point p we have $g_{ki} = \delta_{ki}$ and so our expression coincides with $\operatorname{trace} S_p$. Thus the mean curvature H is also a

$$\operatorname{div}_M \nu(p) := \sum_{i=1}^n \langle d\nu_p(v_i), df_p(v_i)\rangle = -nH(p),$$

where (v_i) is any basis which is g-orthonormal at p. We call div_M the *Riemannian divergence*. By the first remark, this divergence is parameterization invariant. For this reason we use the index $M = f(U)$ which distinguishes it from the standard divergence $\operatorname{div} \nu = \sum_{i=1}^n \partial_i \nu^i$. The two agree only at points p where f is isometric, that is for $g_p = \delta$.

3. The famous *theorema egregium* by Gauss says that the Gauss curvature is determined by the first fundamental form g, meaning that it is accessible from knowing lengths on a surface alone. In particular, if two surfaces are isometric then their Gauss curvature must coincide!

3.3. Local normal form: principal curvatures as coefficients of the Taylor series.

Locally, surfaces can be written as a graph w.r.t. their tangent plane.

Lemma 9 (Local graph)**.** *Let $(f, \nu)\colon U \to \mathbb{R}^{n+1}$ be a hypersurface and $p \in U$. Then there is a domain $V \subset T_p f$ with $0 \in V$ and a change of variables $\varphi\colon V \to \varphi(V) \subset U$ with $\varphi(0) = p$, such that $\tilde f := f \circ \varphi$ satisfies*

$$\tilde f(X) = P + X + u(X)N \quad \text{for all } X \in V \subset T_p f; \text{ here } P := f(p), N := \nu(p). \tag{24}$$

The height function $u\colon V \to \mathbb{R}$ is smooth with $u(0) = 0$ and $du_0 = 0$.

Proof. Let $\Pi\colon \mathbb{R}^{n+1} \to T_p f$ be orthogonal projection onto the tangent space, that is, $\Pi(x) = x - \langle x, N\rangle N$. The mapping

$$\psi := \Pi \circ (f - P)\colon U \to T_p f$$

projects $f - P$ orthogonally onto the tangent space, so that $\psi(p) = 0$. It has the properties

$$\psi(p) = 0, \qquad d\psi_p = d\Pi \circ df_p \overset{\Pi \text{ linear}}{=} \Pi \circ df_p \overset{df_p(\mathbb{R}^n) = T_p f}{=} df_p. \tag{25}$$

By Lemma 2 the mapping $df_p\colon \mathbb{R}^n \to T_p f$ is an isomorphism; thus also $d\psi_p$ is an isomorphism. The Inverse Mapping Theorem, applied to ψ at p, gives a neighbourhood V of 0 in $T_p f$ together with a local inverse $\varphi\colon V \to U \subset \mathbb{R}^n$, that is, $\psi \circ \varphi = \mathrm{id}_V$.

Let us define $\tilde f\colon V \to \mathbb{R}^{n+1}$ by setting $\tilde f - P = f \circ \varphi$. By orthogonal decomposition we can then define the height function $u\colon V \to \mathbb{R}$ by

$$\tilde f(X) - P = X + u(X)N \quad \in \quad T_p f \oplus N_p f.$$

Clearly, $u(0) = 0$. The proof will be completed by showing $du_0 = 0$. From the last equation, $d\tilde f_0(X) = X + du_0(X)\,N$ for all X; thus also $d\tilde f_0 = \mathrm{id} + du_0\,N$. From

$$d\tilde f_0 = df_p \cdot d\varphi_0 = df_p \cdot (d\psi_p)^{-1} \overset{(25)}{=} \mathrm{id} \tag{26}$$

we conclude $du_0\,N = 0$ and thus $du_0 = 0$. □

We can rewrite (24) in terms of principal curvatures:

Theorem 10. *Let f and $\tilde f$ be as in the Lemma. If $v_1, \ldots, v_n$ is an orthonormal basis of principal curvature directions for $\tilde f$ at 0 with principal curvatures $\kappa_1, \ldots, \kappa_n$ then f has the* local normal form

$$\tilde f(X) = P + X + \Big(\frac{1}{2}\sum_{i=1}^n \kappa_i X^i X^i + O(|X|^3)\Big)N \qquad \text{for all} \quad X = \textstyle\sum_i X^i v_i \in V, \tag{27}$$

where $P := \tilde f(0) = f(p)$ and $N := \tilde\nu(0) = \nu(p)$.

Proof. We expand u in a Taylor series at 0,

$$u(X) = \underbrace{u(0)}_{=0} + \underbrace{du_0}_{=0}(X) + \frac{1}{2} d^2 u_0(X, X) + O(|X|^3).$$

By (24) we have $d^2\tilde f_0(X,Y) = d^2u_0(X,Y)N$; we take the scalar product with $N = \tilde\nu(0)$ to obtain

$$\begin{aligned} d^2u_0(X,X) &= \left\langle d^2\tilde f_0(X,X),\, \tilde\nu(0)\right\rangle \overset{(16)}{=} \tilde g_0(\tilde S X, X) \\ &= \tilde g_0\Big(\sum\nolimits_i \kappa_i X^i v_i,\, \sum\nolimits_j X^j v_j\Big) \overset{v_k \text{ ONB}}{=} \sum_i \kappa_i X^i X^i. \end{aligned}$$

This gives (27). ☐

The *paraboloid [Schmiegparaboloid]*

$$\Big\{ f(p) + X + \Big(\frac12 \sum_{i=1}^n \kappa_i(p) X^i X^i\Big)\nu(p) \mid X = \sum\nolimits_i X^i v_i \in T_p f\Big\}$$

characterizes the behaviour of a hypersurface f at p up to second order. In particular, for dimension $n=2$ we can read off:

Proposition 11. *If $p \in U$ is a point of a surface $f\colon U \to \mathbb{R}^3$ with $K(p) < 0$ then $f(p)$ is a saddle point, that is, in each neighbourhood of $f(p)$ we find pairs of points, whose images lie to the two different sides of $f(p) + T_p f$. On the other hand, if $K(p) > 0$, then f is locally convex, that is, there is a neighbourhood V of $f(p)$, such that $f(q)$ does not meet one side of $f(p) + T_p f$ if $q \in V$.*

Proof. If $K(p) < 0$ then the two principal curvatures have different sign. Then by (27) we find points lying to the side of $T_p f$ into which $\nu(p)$ points by walking in a principal curvature direction with positive principal curvature; similarly walking in the other principal curvature direction we find points of the opposite component of $\mathbb{R}^3 \setminus (f(p) + T_p f)$.

Similarly, $K(p) > 0$ means two principal curvatures of the same sign. There is a neighbourhood of 0 such that the quadratic term in (27) dominates the $O(|X|^3)$ term. On this neighbourhood the surface lies to one side of $f(p) + T_p f$ with the point p the only point lying on $f(p) + T_p f$ itself. ☐

In the case that $K(p) = 0$, the local behaviour of the surface is governed by higher order terms; it can be any of the two mentioned or more complicated.

Example. A torus of revolution has two touching planes perpendicular to the axis of revolution. Each intersects the torus in a circle. We consider the two connected components resulting from a removal of these two circles from the torus. The component to the outside (further away from the axis) is locally convex and so $K \geq 0$ there. On the inner component, however, $K \leq 0$ since each point is a saddle point. By continuity, $K = 0$ on the two circles. As a calculation can show, the two weak inequalities on the components are in fact strict.

Part 2. Minimal surfaces: Basic properties and examples

`7. Lecture, Wednesday 16.11.05`

1. The equations

The definition fundamental for this class is:

Definition. A parameterized surface is a *minimal hypersurface [minimale Hyperfläche]* if $H = 0$ at all points. For the two-dimensional case we say *minimal surface [Minimalfläche]*.

We will see that minimal surfaces are critical for the area only later.

Minimal surfaces were considered before mean curvature was discovered. Indeed, the area minimizing property had originally been used to define them from the 18th century onwards. Only in the 19th century the differential geometry of surfaces was introduced. This gave the previously known equations a clear-cut geometric meaning.

1.1. The equation for the parametric case.

Our starting point is the matrix equation $nH = \operatorname{trace}(g^{-1}b)$, see (23). In dimension 2 the inverse of the metric has the simple representation $g^{-1} = \frac{1}{\det g}\begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11}\end{pmatrix}$, which gives:

Theorem 1. *For $n = 2$ the mean curvature of a surface $f\colon U \to \mathbb{R}^3$ satisfies*

$$(1)\qquad \begin{aligned} H &= \frac{1}{2\det g}(g_{22}b_{11} - 2g_{12}b_{12} + g_{11}b_{22}) \\ &= \frac{|f_y|^2\langle f_{xx},\nu\rangle - 2\langle f_x, f_y\rangle\langle f_{xy},\nu\rangle + |f_x|^2\langle f_{yy},\nu\rangle}{2\big(|f_x|^2|f_y|^2 - \langle f_x, f_y\rangle^2\big)}. \end{aligned}$$

Thus a surface is minimal if and only if $0 \equiv |f_y|^2\langle f_{xx},\nu\rangle - 2\langle f_x, f_y\rangle\langle f_{xy},\nu\rangle + |f_x|^2\langle f_{yy},\nu\rangle$. Assuming a conformal parameterization simplifies the equation considerably:

Theorem 2. *Suppose $f\colon U \to \mathbb{R}^3$ is a two-dimensional surface. Moreover assume the Gauss map is chosen positively oriented, i.e.,* $\det(f_x, f_y, \nu) > 0$. *Then if f is conformal it satisfies the* parametric mean curvature equation

$$(2)\qquad \Delta f = 2H f_x \times f_y \qquad \text{for all } p \in U;$$

here $\Delta = f_{xx} + f_{yy}$ is the standard Laplacian.

A function $u\colon \mathbb{R}^n \to \mathbb{R}$ is called *harmonic [harmonisch]* if $\Delta u \equiv 0$. Thus a conformally parametrized surface is minimal if and only if $\Delta f \equiv 0$, that is, if and only if each of the three component functions is harmonic.

Example. Let us represent an arbitrary plane in $\mathbb{R}^3$. Choose two vectors v, w spanning the plane, such that $|v| = |w| = 1$, $\langle v, w\rangle = 0$. Then $f(x,y) = xv + yw$ is a conformal parameterization of the plane. The fact that the plane is minimal ($H \equiv 0$) is equivalent to the three coordinate functions $f^1(x,y) = xv^1 + yw^1$, etc., being harmonic.

Proof. Conformality means $\lambda := |f_x| = |f_y|$ and $\langle f_x, f_y\rangle = 0$. Using this in (1) gives

$$\frac{1}{2\lambda^2}\langle f_{xx} + f_{yy},\, \nu\rangle = \frac{1}{2\lambda^2}\langle \Delta f, \nu\rangle = H. \tag{3}$$

To derive (2) we claim that the vectors $\Delta f = f_{xx} + f_{yy}$ and ν are parallel. To see this we differentiate the conformality conditions:

$$\frac{\partial}{\partial x}|f_x|^2 = \frac{\partial}{\partial x}|f_y|^2 \quad\Rightarrow\quad \langle f_{xx}, f_x\rangle = \langle f_{xy}, f_y\rangle$$
$$\frac{\partial}{\partial y}\langle f_x, f_y\rangle = 0 \quad\Rightarrow\quad \langle f_{yy}, f_x\rangle = -\langle f_{xy}, f_y\rangle$$

Adding the equalities gives $\langle f_{xx} + f_{yy},\, f_x\rangle = 0$, and similarly $\langle f_{xx} + f_{yy},\, f_y\rangle = 0$. Since f_x, f_y, ν are orthogonal, this proves $\Delta f \parallel \nu$.

Decomposing with respect to the basis f_x, f_y, ν therefore gives

$$\Delta f = \langle \Delta f, \nu\rangle \nu \overset{(3)}{=} 2H\lambda^2\nu.$$

Using the positive orientation of f_x, f_y, ν we prove (2) by

$$\lambda^2\nu = |f_x||f_y|\nu = f_x \times f_y.$$

□

1.2. **Mean curvature of graphs.** Since the equation $nH = \operatorname{trace} S = \sum_i S_i^i$ is difficult to tackle, it is common to work under simplifying assumptions. Conformality is only one such assumption; another is to study graphs.

Theorem 3. *Let $f\colon U \to \mathbb{R}^{n+1}$ be a graph, $f(x) = \big(x, u(x)\big)$, and let ν be its upper normal. Then the mean curvature $H = H(x)$ satisfies*

$$H = \frac{1}{n}\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right). \tag{4}$$

Proof. To calculate $\operatorname{trace} S = \sum_i S_i^i$ we use the defining equation for S,

$$-\partial_i\nu = -d\nu(e_i) = df(Se_i) = df\Big(\sum_k S_i^k e_k\Big) = \sum_k S_i^k df(e_k) = S_i^1\begin{pmatrix} e_1 \\ \partial_1 u\end{pmatrix} + \ldots + S_i^n\begin{pmatrix} e_n \\ \partial_n u\end{pmatrix},$$

valid for $i = 1, \ldots, n$. The i-th component reads $-\partial_i\nu^i = S_i^i$. As stated in I(9) the upper normal ν of a graph is

$$\nu = \left(\frac{-\partial_1 u}{\sqrt{1+|\nabla u|^2}}, \ldots, \frac{-\partial_n u}{\sqrt{1+|\nabla u|^2}}, \frac{1}{\sqrt{1+|\nabla u|^2}}\right).$$

Combining these equations we obtain

$$\partial_i\left(\frac{\partial_i u}{\sqrt{1+|\nabla u|^2}}\right) = S_i^i,$$

and the trace can be written

$$H = \frac{1}{n}\sum_i S_i^i = \frac{1}{n}\sum_i \partial_i\left(\frac{\partial_i u}{\sqrt{1+|\nabla u|^2}}\right) = \frac{1}{n}\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right).$$

□

We stated the divergence form (4) of the equation for a minimal graph. The differentiation can be carried out to give a more standard form for this second order partial differential equation. Let us instead derive this directly from (1). Recall from I(4), I(21) that a two-dimensional graph $f(x,y) = \big(x, y, u(x,y)\big)$ has

$$g = \begin{pmatrix} 1+u_x^2 & u_x u_y \\ u_x u_y & 1+u_y^2\end{pmatrix}, \qquad b = \frac{1}{\sqrt{1+(\nabla u)^2}}\begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy}\end{pmatrix}.$$

Using $\det g = 1 + u_x^2 + u_y^2 = 1 + |\nabla u|^2$ we obtain from (1)

$$H = \frac{1}{2\big(1+|\nabla u|^2\big)}\,\frac{(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy}}{\sqrt{1+|\nabla u|^2}}$$
$$\iff 2H\big(1+|\nabla u|^2\big)^{3/2} = (1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy}.$$

Note that at a point p with horizontal tangent plane, this equation becomes $2H(p) = \Delta u(p)$. In general we obtain:

Theorem 4. *A graph $\big(x, y, u(x,y)\big)$, where $u\colon U \to \mathbb{R}$, represents a minimal surface if and only if u satisfies the* minimal surface equation

$$(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy} = 0 \qquad \textit{for all } (x,y) \in U. \tag{5}$$

The minimal surface equation is a second order partial differential equation, depending on second and first derivatives of u. The function u itself does not enter: If $(x, y, u(x,y))$ is minimal then also $(x, y, u(x,y) + c)$ for $c \in \mathbb{R}$ is minimal. The equation is nonlinear, meaning that $L(u+v) \neq Lu + Lv$, where Lu denotes the left hand side of (5). Nevertheless we will see lateron that it satisfies a maximum principle.

The three most common ways to represent surfaces are parametric, as a graph, and implicit. Ignoring the last, a surface given in form of a graph is sometimes called *non-parametric*, and hence (5) is the *non-parametric minimal surface equation.* Note that, by the implicit function theorem, any surface can locally be written as a graph.

1.3. **The Gauss curvature of minimal surfaces.** The Gauss curvature also plays an important rôle for minimal surfaces.

From the condition $\kappa_1 + \kappa_2 = 0$ we obtain directly:

Theorem 5. *A two-dimensional minimal surface has Gauss curvature $K = \kappa_1\kappa_2 \le 0$. Moreover on a minimal surface the two principal curvatures are equal* (umbilic points *[Nabelpunkte]*) *if and only if the Gauss curvature vanishes.*

This means that at each point a minimal surface looks locally like a saddle.

For any hypersurface, the two maps between n-dimensional Euclidean vector spaces

$$d\nu_p, df_p\colon \big(\mathbb{R}^n, \langle .,.\rangle\big) \to \big(T_pf, \langle .,,.\rangle\big)$$

have a well-defined determinant, measuring the oriented distortion of the volume element at p (we take an orientation of T_pf such that $T_pf, \nu(p)$ is posititively oriented). As f is an immersion, $\det df_p \neq 0$; by a relabelling of the first two coordinates, we can assume that in fact $\det df_p > 0$ for all $p \in U$. The Gauss curvature can be considered the ratio of area distortion for ν compared with f:

$$K(p) = \det S_p = \det\big(-(df_p)^{-1}d\nu_p\big) = (-1)^n\frac{\det d\nu_p}{\det df_p} \tag{6}$$

Gauss used this formula to define K. We conclude:

Corollary 6. *On a minimal surface, the Gauss map is always orientation reversing.*

`8. Lecture, Tuesday 22.11.05` ______________________________

It is convenient to add a related definition:

Definition. Let f be a hypersurface. The quantity $\int_U K\,dS \in [-\infty,\infty]$ is called the *total curvature [Totalkrümmung]* of f.

To give the total curvature a more geometric meaning, we conclude from (6)

$$\det d\nu_p = \pm K(p)\det df_p \qquad\Rightarrow\qquad \det(d\nu_p^T d\nu_p) = K^2(p)\det(df_p^T df_p).$$

We conclude that the area of the Gauss image in $\mathbb{S}^n$ is

$$A_U(\nu) = \int_U \sqrt{\det(d\nu^T d\nu)}\,dx = \int_U |K|\sqrt{\det(df^T df)}\,dx = \int_U |K|\,dS \overset{f \text{ minimal}}{=} -\int_U K\,dS.$$

Note the integrals count the area with multiplicity, i.e., taken as often as the spherical image is attained. Thus the total curvature of a minimal surface is the negative area of the spherical image, taken with multiplicity. For a general, not necessarily minimal, surface $\int K\,dS$ counts the oriented area of the Gauss image with multiplicity.

2. SPECIAL MINIMAL SURFACES

2.1. **Minimal surfaces of revolution: The catenoid.** Let

$$(r,h)\colon I \to \mathbb{R}_+ \times \mathbb{R} \tag{7}$$

be a regular curve. We place it in the (x,z)-plane and rotate about the z-axis. The result is a *surface of revolution [Rotationsfläche]*

$$f\colon I\times\mathbb{R} \to \mathbb{R}^3, \qquad f(t,\varphi) := \big(r(t)\cos\varphi,\, r(t)\sin\varphi,\, h(t)\big). \tag{8}$$

Lemma 7. *For a surface of revolution* (8)*, meridians $t \mapsto f(t,\varphi)$ and latitude circles $\varphi \mapsto f(t,\varphi)$ are curvature lines, with principal curvatures*

$$\kappa_1 = \frac{r'h'' - h'r''}{\sqrt{r'^2 + h'^2}^3} \qquad \text{and} \qquad \kappa_2 = \frac{1}{r}\frac{h'}{\sqrt{r'^2+h'^2}}. \tag{9}$$

Proof. We compute:

$$\partial_1 f = \begin{pmatrix} r'\cos\varphi \\ r'\sin\varphi \\ h' \end{pmatrix} \quad\text{and}\quad \partial_2 f = \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix},$$

and so the first fundamental form g is

$$g = \begin{pmatrix} r'^2 + h'^2 & 0 \\ 0 & r^2 \end{pmatrix} \quad\Rightarrow\quad g^{-1} = \begin{pmatrix} \frac{1}{r'^2+h'^2} & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}. \tag{10}$$

The normal of the curve, $\binom{-h'}{r'}$, normed and rotated about the z-axis, gives the surface normal

$$\nu = \frac{1}{\sqrt{r'^2+h'^2}}\begin{pmatrix} -h'\cos\varphi \\ -h'\sin\varphi \\ r' \end{pmatrix}; \tag{11}$$

it is the inner normal if h is increasing. The second derivatives of f are

$$\partial_{11} f = \begin{pmatrix} r''\cos\varphi \\ r''\sin\varphi \\ h'' \end{pmatrix}, \qquad \partial_{22} f = \begin{pmatrix} -r\cos\varphi \\ -r\sin\varphi \\ 0 \end{pmatrix}, \qquad \partial_{12} f = \partial_{21} f = \begin{pmatrix} -r'\sin\varphi \\ r'\cos\varphi \\ 0 \end{pmatrix}.$$

We obtain

$$b_{11} = \langle \partial_{11} f, \nu\rangle = \frac{r'h'' - h'r''}{\sqrt{r'^2+h'^2}}, \qquad b_{22} = \langle \partial_{22} f, \nu\rangle = \frac{rh'}{\sqrt{r'^2+h'^2}}, \qquad b_{12} = \langle \partial_{12} f, \nu\rangle = 0.$$

Note that both g and b are diagonal, so that $g^{-1}b$ is diagonal as well. Hence the coordinate directions $v_1 = \frac{\partial}{\partial t}$ and $v_2 = \frac{\partial}{\partial\varphi}$ are principal curvature directions, and their principal curvatures are $\kappa_1 = g^{11}b_{11}$ and $\kappa_2 = g^{22}b_{22}$. $\square$

Thus a surface of revolution is minimal, $0 \equiv H = \frac{\kappa_1+\kappa_2}{2}$, if nad only if

$$0 = r(r'h'' - h'r'') + h'(r'^2 + h'^2). \tag{12}$$

We call a surface M *complete [vollständig]* if there is no connected surface (of the same dimension) containing it as a proper subset. For instance, a plane in $\mathbb{R}^3$ is complete, but an open or closed disk (in $\mathbb{R}^2$ or $\mathbb{R}^3$) is not. We say a complete surface is a surface of revolution if it is invariant under rotation (we allow for $r = 0$).

Theorem 8 (Bonnet 1860)**.** *Each complete minimal surface of revolution is either a plane or a* catenoid

$$\begin{aligned} &f\colon \mathbb{R}\times[0,2\pi)\to\mathbb{R}^3, \quad f(t,\varphi) := \big(r(t)\cos\varphi,\, r(t)\sin\varphi,\, t\big) \\ &\textit{with} \quad r(t) := a\cosh\left(\frac{t-t_0}{a}\right) \textit{ for } t_0\in\mathbb{R}, \;\; a>0. \end{aligned} \tag{13}$$

The catenoid was first described by Euler in 1744. The meridian curve cosh is a *catenary* (see problem session): This means that we can obtain the helicoid by rotating the shape of a hanging chain about an axis of a certain distance.

Proof. We may assume that the generating curve (r, h) has arc length representation. Certainly, $h \equiv$ const is a solution of (12). It corresponds to a (horizontal) plane without the point $P := (0, 0, h)$ on the axis; this surface becomes complete by taking the union with P.

If h is not identical to a constant there is a point t_0 such that $h'(t_0) \neq 0$. Then h is locally monotone, and by a reparameterization of our generating curve (7) we can assume that locally $h(t) = t$. In this case the differential equation (12) becomes

$$rr'' = 1 + r'^2, \qquad r > 0. \tag{14}$$

This ODE can be solved by a separation of variables, see [O1, p.64]. Here, we will only check that (13) solves the ODE (14). Indeed, $a\cosh(.)\frac{1}{a}\cosh(.) = 1 + \sinh^2(.)$.

Let us consider the initial values $r(0) > 0$, $r'(0) \in \mathbb{R}$ for (14). Writing the initial values in the form $r'(0) = \sinh(t_0)$ with $t_0 \in \mathbb{R}$ and $r(0) = a\cosh\frac{-t_0}{a}$ with $a > 0$ we see that they are satisfied by the solution family (13). By the uniqueness part of the theorem of Picard-Lindelöf these are all solutions (the system $r' = R$ and $R' = (1 + R^2)/r$ is Lipshitz for $r > \varepsilon > 0$). Moreover, these solutions are defined for all $t \in \mathbb{R}$ and hence complete. In particular, any complete solution of the intitial value problem with $h(t_0) \neq 0$ for some t_0 has in fact $h(t) \neq 0$ for all t.

We have solved all initial value problems for the system (12) and $r'^2 + h'^2 = 1$; thus all solutions are of the claimed type. □

The catenoid is, perhaps, the most important minimal surface: Each embedded annular end of a minimal surface which sits in a half-space is asymptotic to a catenoid; in particular, we will see that an embedded minimal surface with exactly two ends must be the catenoid, that is, there are no embedded minimal surfaces with two catenoidal ends and higher genus.

Let us discuss two more properties of the catenoid. First we consider a boundary value problem. Take a symmetric catenoid, that is, $t_0 = 0$ in (13). A subset of type $f\big([-\frac{d}{2}, \frac{d}{2}], [0, 2\pi)\big)$ of this catenoid is bounded by two circles of radius $\cosh\tau$ which have distance d. Does any such boundary configuration bound a connected subset of some catenoid?

We call two boundary circles *coaxial* if they are contained in parallel planes, have the same radius $r > 0$ and the straight line through their midpoints is perpendicular to the plane; let the distance be $d > 0$. A soap film experiment tells us that for the two coaxial circles to bound a catenoid, they cannot be too far apart, or, the radius cannot be too large. It is sufficient to study the case $r = 1$:

Proposition 9. *Let $f\colon U \to \mathbb{R}^{n+1}$ be a minimal hypersurface and set $f^{(\lambda)} := \lambda f$ where $\lambda > 0$. If f has mean curvature H then $f^{(\lambda)}$ has mean curvature $\frac{1}{\lambda}H$; in particular minimality is preserved by scaling.*

Proof. Note that $df^{(\lambda)} = \lambda df$ and so the normal of $f^{(\lambda)}$ is unchanged. Thus, using obvious notation,

$$S^{(\lambda)} = -\big(df^{(\lambda)}\big)^{-1} d\nu^{(\lambda)} = -\frac{1}{\lambda} df^{-1} d\nu = \frac{1}{\lambda} S.$$

In particular, $H^{(\lambda)} = \frac{1}{\lambda}H$. □

In our catenoid family (13), the parameter a corresponds to scaling.

So let us assume two coaxial unit circles are spaced $d > 0$ apart. By studying the one-parameter family of scaled catenoids the following is proven the problem session:

Proposition 10. *There is a number $d_{\max} = 1.32\ldots$ such that two coaxial unit circles in distance d bound connected subsets of the following minimal surfaces: two different catenoids for $0 < d < d_{\max}$, just one for $d_{\max}$, and none for $d > d_{\max}$.*

`9. Lecture, Wednesday 23.11.05`

Consider the following surfaces of revolution, bounded by the two coaxial unit circles in distance d: The catenoid of smaller area $C_1(d)$ and the catenoid of larger area $C_2(d)$, they exist for each $0 < d < d_{\max}$ and coincide for $d = d_{\max}$. Moreover, let D be two (disconnected) disks; in this context the disks are refered to as the *Goldschmidt solution*. For $d < 1.05\ldots$, the catenoid $C_1(d)$ has smaller area than the disk area 2π. The area of the catenoid $C_2(d)$ is always larger than the disk. The catenoid $C(d_{\max})$ is a regular surface with area $2\pi \cdot 1.19\ldots$; nevertheless, when d is increased further, a soap film will pop. See Nitsche [N, § 515 f] and Oprea [O1, 5.6] for more information.

Let us also calculate the total curvature of the catenoid. By (9) we have for a surface of revolution with $h(t) = t$

$$\kappa_1 = -\frac{r''}{(1+r'^2)^{3/2}}, \qquad \kappa_2 = \frac{1}{r\sqrt{1+r'^2}},$$

that is, for $r(t) = \cosh t$, we find

$$\kappa_1(r,t) = -\frac{\cosh t}{\sqrt{1+\sinh^2 t}^3} = -\frac{1}{\cosh^2(t)}, \qquad \kappa_2(r,t) = \frac{1}{\cosh t\sqrt{1+\sinh^2 t}} = \frac{1}{\cosh^2(t)}.$$

Moreover, by (10),

$$\sqrt{\det g} = \sqrt{(r'^2+1)r^2} = \sqrt{(\sinh^2 t + 1)\cosh^2 t} = \cosh^2 t$$

This gives the total curvature

$$\begin{aligned}
\int_{[0,2\pi]\times\mathbb{R}} K(t,\varphi)\, dS_{(t,\varphi)} &= \int_{[0,2\pi]\times\mathbb{R}} \frac{-1}{\cosh^4 t}\cosh^2 t\, d\varphi\, dt \\
&= -2\pi\int_{-\infty}^{\infty}\frac{1}{\cosh^2 t}\,dt = -2\pi\int_0^{\infty}\frac{1/s}{\cosh^2(\log s)}\,ds \\
&= -2\pi\int_0^{\infty}\frac{1}{\frac{s}{4}(s+\frac{1}{s})^2}\,ds = -4\pi\int_0^{\infty}\frac{2s}{(s^2+1)^2}\,ds \\
&= -4\pi\left[\frac{-1}{s^2+1}\right]_0^{\infty} = -4\pi,
\end{aligned} \tag{15}$$

noting that the substitution $t = \log s$ satisfies $\cosh(\log s) = \frac{1}{2}(e^{\log s} + e^{-\log s}) = \frac{1}{2}(s + \frac{1}{s})$.

Our result means that the Gauss image has the area of the sphere. In fact this is a consequence of the following:

Proposition 11. *The Gauss map of a catenoid with axis direction $e_3 \in \mathbb{S}^2$ is bijective to $\mathbb{S}^2 \setminus \{\pm e_3\}$.*

Proof. By (11), $\nu(t,\varphi) = \frac{1}{\cosh t}\big(-\cos\varphi, -\sin\varphi, \sinh t\big)$. The xy-projection of $\nu(t,\varphi)$ has angle $\varphi + \pi$. The t lines go on the sphere from the south pole ($t \to -\infty$) to the north pole $t \to \infty$; they are 1-1 since the third component $\tanh t$ is strictly monotone. Thus the t-lines have an open semicircle as their Gauss image. It follows that $t \in \mathbb{R}$, $\varphi \in [0, 2\pi)$ parameterizes $\mathbb{S}^2$ except for $\pm(0,0,1)$. $\square$

2.2. **Ruled minimal surfaces: The helicoid.** Let I be an interval and $c, v\colon I \to \mathbb{R}^3$ be two curves. Then the mapping

$$f\colon \mathbb{R}\times I \to \mathbb{R}^3, \qquad f(s,t) = c(t) + sv(t). \tag{16}$$

is called a *ruled surface [Regelfläche]*. From a strict point of view, we should say ruled mapping, since f need not be a surface, i.e. immersion, at each (s,t). The curve c is called the *directrix [Leitkurve]*. The straight lines $s \mapsto f(s,t)$ are the *rulings [Regelgeraden]* of f; they are asymptote lines. Thus a ruled surface has Gauss curvature $K \le 0$ (wherever defined). The notion *ruled* refers to a ruler; the German *Regel* is a mistranslation of the French term *règle* (ruler).

Examples. Many classical surfaces are ruled: So is the circular cylinder or, more generally, a cylinder over an arbitrary curve, and similarly the cone over a circle or over a general curve. Note that cones are immersions only excluding their tip. It is known that the shapes a bent piece of paper in space can attain are locally ruled.

The partial derivatives of f are

$$\partial_1 f(s,t) = v(t), \qquad \partial_2 f = c'(t) + sv'(t). \tag{17}$$

If $c'(t)$ and $v(t)$ are linearly independent for all $t \in I$ then f is an immersion on an open neighborhood U of $\{s = 0\}$ in $\mathbb{R}\times I$. Nevertheless, we will not assume this condition – instead we will derive it from minimality. We call a point (s,t) of a ruled mapping nonsingular if $f(s,t)$ is an immersion in a neighbourhood of (s,t); we call f nonsingular, if it is an immersion for all $s \in \mathbb{R}$.

Discovered by Meusnier in 1776, the helicoid is a ruled surface. For instance, we can let c trace out the z-axis and v rotate in a horizontal plane. This gives an embedding $f\colon \mathbb{R}^2 \to \mathbb{R}^3$,

$$f(s,t) := c(t) + sv(t) = \begin{pmatrix} 0 \\ 0 \\ ht \end{pmatrix} + s\begin{pmatrix} \sin t \\ -\cos t \\ 0 \end{pmatrix} = \begin{pmatrix} s\sin t \\ -s\cos t \\ ht \end{pmatrix} \quad \text{for } h \ne 0, \tag{18}$$

which agrees with our previous representation (2) for $h = 1$. Here, we assumed that v starts on the negative y-axis at $t = 0$. The rotation is anticlockwise for $h > 0$ ("right helicoid") and clockwise for $h < 0$ ("left helicoid").

As follows from I(15) and Proposition 9, the helicoid (18) is minimal. It is unique with these properties:

Theorem 12 (Catalan 1842)**.** *Each complete ruled minimal surface is nonsingular and either the plane or the helicoid.*

`10. Lecture, Tuesday 29.11.05`

To calculate the principal curvatures of an arbitrary ruled surface is somewhat involved. Note that our helicoid representation (18) employs special choices: The rulings are parametrized with unit speed, and they intersect the directrix at a right angle. These choices can be achieved in general (from now on we assume $0 \in I$):

Lemma 13. *For a ruled surface* (16) *a different choice of the s-parameter and of the directrix c results in*

$$|v| \equiv 1, \qquad v \perp v', \qquad v \perp c', \tag{19}$$

and, moreover,

$$|v'(0)| = 1 \quad \Rightarrow \quad \langle c'(0), v'(0)\rangle = 0. \tag{20}$$

Under the assumption (19) *the mean curvature vanishes at a nonsingular point if and only if*

$$\det(\partial_1 f, \partial_2 f, \partial_{22} f) = \det\big(v, c' + sv', c'' + sv''\big) = 0. \tag{21}$$

For the helicoid, (19) is satisfied if and only if we choose any contained helix as its directrix. However, (20) is equivalent to picking the straight z-axis as its directrix (check!).

Proof. First of all, by a linear change in s, we obtain the first assertion in (19). Its derivative is the second.

Let us now alter the choice of the directrix. We replace c with $\tilde c(t) = c(t) - \lambda(t)v(t)$, where

$$\lambda(t) := \langle c'(0), v'(0)\rangle + \int_0^t \langle c'(\tau), v(\tau)\rangle \, d\tau.$$

Let us verify the last assertion in (19) for $\tilde c$:

$$\langle \tilde c', v\rangle = \big\langle c' - \lambda' v - \lambda v', v\big\rangle \overset{v\perp v'}{=} \big\langle c' - \langle c', v\rangle v,\ v\big\rangle \overset{|v|=1}{=} 0 \qquad \forall t \in I.$$

Moreover, the first term of λ was chosen as to give

$$\begin{aligned}\langle \tilde c'(0), v'(0)\rangle \ &= \ \Big\langle c'(0) - \langle c'(0), v(0)\rangle v(0) - \langle c'(0), v'(0)\rangle v'(0),\ v'(0)\Big\rangle \\ &\overset{v\perp v'}{=} \big\langle c'(0), v'(0)\big\rangle \big(1 - |v'(0)|^2\big),\end{aligned}$$

which proves (20). Note that $\tilde c$ is merely a different choice of directrix, which leaves the t-parameter untouched. In the following we write once again c in place of $\tilde c$.

If f is minimal then by (1)

$$0 = g_{22} b_{11} - 2 g_{12} b_{12} + g_{11} b_{22}. \tag{22}$$

But various quantities vanish provided we represent the ruled surface as above. First

$$g_{11} = 1 \quad \text{and} \quad g_{12}(s,t) = \big\langle v(t),\, c'(t) + sv'(t)\big\rangle = 0,$$

and second, upon differentiation of (17),

$$\partial_{11} f(s,t) = 0 \ \Rightarrow \ b_{11} = 0, \qquad \text{and} \qquad \partial_{22} f(s,t) = c''(t) + sv''(t).$$

Thus (22) reduces to

$$0 = b_{22} = \langle \nu, \partial_{22} f\rangle.$$

But since $\nu \parallel \partial_1 f \times \partial_2 f$ this vanishes if and only if (21) holds. □

Proof of Catalan's theorem. We assume the representation of f satisfies the lemma.

If $v' \equiv 0$ then v is constant. But since $c' \perp v$ this implies c is contained in some plane perpendicular to v. But then f is a generalized cylinder, whose principal curvatures are the curvature of c on the one hand and 0 on the other hand. Thus for f to be minimal, c must be a straight line and f a plane.

If f is not a plane, we can assume $v' \neq 0$ at some point; we assume it holds at $t = 0$. Then by reparameterizing t, we achieve in some neighbourhood of $t = 0$

$$|v'|(t) = 1 \quad \Rightarrow \quad \langle v', v''\rangle = 0. \tag{23}$$

We expand the determinant (21) to obtain

$$0 = \det\big(v, c', c''\big) + s\Big(\det(v, v', c'') + \det(v, c', v'')\Big) + s^2 \det(v, v', v''),$$

for each $s \in I$, and t in a neighbourhood of 0. But this is equivalent to the vanishing of the three terms

$$\det\big(v, c', c''\big) = 0, \qquad \det(v, v', c'') + \det(v, c', v'') = 0, \qquad \det(v, v', v'') = 0. \tag{24}$$

The helicoid (18) does satisfy these equations and thus indeed is minimal.

Claim 1: v parametrizes a unit circle in some plane P with unit speed, for t in a neighbourhood of 0. By (19) and (23), $v \perp v'$ and $v'' \perp v'$. Thus the vanishing of the last determinant gives $v \parallel v''$. Hence $(v \times v')' = v' \times v' + v \times v'' = v \times v'' = 0$, meaning that v, v' are contained in a plane P; it is spanned, for instance, by $v(0)$ and $v'(0)$. The claim follows since $|v| = 1$ and $|v'| = 1$.

Claim 2: $c'' = 0$ and $c' \perp P$ hold for t in some neighbourhood of 0. From $v \parallel v''$ we see that the second equation in (24) reduces to $\det(v, v', c'') = 0$ so that on the one hand $c'' \in P$. We conclude

$$c'' = \langle c'', v\rangle v + \langle c'', v'\rangle v'$$

which we insert into the first equation of (24) to obtain

$$\langle c'', v' \rangle \det(v, c', v') = 0.$$

By (19), we have $c' \perp v \perp v'$, furthermore (20) and (23) give $v'(0) \perp c'(0)$. By continuity, on a neighbourhood of $t = 0$, we have $\det(v, c', v') \neq 0$. We conclude that $\langle c'', v' \rangle = 0$ or equivalently $\langle c', v' \rangle' = 0$. But since $\langle c'(0), v'(0) \rangle = 0$ the latter implies $\langle c', v' \rangle \equiv 0$. Using $c' \perp v$, we have $c' \perp P$, and upon differentiation we verify $c'' \perp P$, on the other hand. Our claim is proved.

Our claims 1 and 2 prove that $f(s,t)$ for $|t| < \varepsilon$ is contained in a helicoid (18), where $|h| = |c'|$ and the sign of h is given by the sense of rotation. We claim that the interval J for which our claims 1 and 2 hold, is in fact I. Indeed, by continuity the claims hold on a closed subset of I; but the two conditions we used, (23) and $\det(v, c', v') \neq 0$ hold on an open subinterval of I, and so do the conclusions Claim 1 and 2. Thus the surface is a helicoid for all $(s,t) \in \mathbb{R} \times I$. But by completeness, $I = \mathbb{R}$ for our parametrization, and so f is some helicoid. □

The helicoid (18) has the following properties:

- It is *simply periodic*, that is, any translation by $k(0,0,2\pi h)$, $k \in \mathbb{Z}$, leaves it invariant.
- It is symmetric under rotation by any angle $\varphi \in \mathbb{R}$ followed by a vertical translation by $(0,0,h\varphi)$.
- The helicoid has infinite total curvature, but total curvature -4π in a fundamental domain $f(\mathbb{R}, [0, 2\pi])$.

All but the last property are straightforward. To calculate the total curvature, note that the principal curvatures for a conformal parameterization were computed in I(13) and I(14) to be $\pm 1/\cosh^2(x)$, while $\det g = \cosh^4(x)$. But these agree precisely with the respective values for the catenoid, and so, by (15) a translational fundamental domain has total curvature

$$\int_{[0,2\pi]\times\mathbb{R}} K(t,\varphi)\, dS_{(t,\varphi)} = \int_{[0,2\pi]\times\mathbb{R}} \frac{-1}{\cosh^4 t} \cosh^2 t \, d\varphi \, dt = -4\pi.$$

The simple periodicity implies that the total curvature of the entire helicoid must be infinite.

There is a stunning relationship between catenoid and helicoid. To introduce it, we use the conformal representations.

Theorem 14. *Each surface in the* associated family

$$f_\vartheta \colon \mathbb{R}^2 \to \mathbb{R}^3, \qquad f_\vartheta(x,y) = \cos\vartheta \begin{pmatrix} \cosh x \cos y \\ \cosh x \sin y \\ x \end{pmatrix} + \sin\vartheta \begin{pmatrix} \sinh x \sin y \\ -\sinh x \cos y \\ y \end{pmatrix},$$

where $\vartheta \in \mathbb{R}$, is minmal. Moreover, f_0 parametrizes a catenoid (of waist radius 1) and $f_{\pi/2}$ a helicoid (with vertical period 2π); for $y \in [0, 2\pi)$ the entire catenoid and a period of the helicoid is parameterized. All surfaces in the family are isometric, that is the first fundamental form is independent of ϑ, and the normal is independent of ϑ, too.

We will later see that such a family exists for all minimal surfaces. All our claims can be verified by calculation.

`11. Lecture, Wednesday 29.11.05` ________________________________

2.3. Enneper's surface has an intrinsic rotation. The minimal surface

$$f \colon \mathbb{R}^2 \to \mathbb{R}^3, \qquad f(x,y) = \begin{pmatrix} x - \frac{1}{3}x^3 + xy^2 \\ -y + \frac{1}{3}y^3 - yx^2 \\ x^2 - y^2 \end{pmatrix}$$

was discovered by Enneper in 1864. To see it is minimal, let us calculate

$$f_x = \begin{pmatrix} 1 - x^2 + y^2 \\ -2xy \\ 2x \end{pmatrix}, \quad f_y = \begin{pmatrix} 2xy \\ -1 + y^2 - x^2 \\ -2y \end{pmatrix}.$$

Thus

$$\begin{aligned} |f_x(x,y)|^2 &= \left(1 + x^4 + y^4 - 2x^2 + 2y^2 - 2x^2y^2\right) + 4x^2y^2 + 4x^2 \\ &= 1 + x^4 + y^4 + 2x^2 + 2y^2 + 2x^2y^2 = (1 + x^2 + y^2)^2 = |f_y(x,y)|^2 \end{aligned}$$

and $\langle f_x, f_y \rangle = 0$. We conclude

$$g = \begin{pmatrix} (1+x^2+y^2)^2 & 0 \\ 0 & (1+x^2+y^2)^2 \end{pmatrix} = \begin{pmatrix} (1+r^2)^2 & 0 \\ 0 & (1+r^2)^2 \end{pmatrix}, \tag{25}$$

where $r := \sqrt{x^2 + y^2}$. Thus f is conformal, and hence to check it is minimal, it suffices to assert $\Delta f = (-2x + 2x, 2y - 2y, 2 - 2) = 0$.

To understand the geometry of the Enneper surface, let us first point out that

$$f(\pm x, x) = \Big(\pm (x - \frac{2}{3}x^3), -(x - \frac{2}{3}x^3), 0 \Big),$$

meaning that the surface contains the two horizontal diagonals. On the other hand, $f(x,0) = (x - \frac{1}{3}x^3, 0, x^2)$; at $x = 0$ is asymptotic to the graph of a parabola, while at infinity it is asymptotic to the graph of $(.)^{2/3}$. Since $f(\pm\sqrt{3}, 0) = (0,0,3)$, the Enneper surface is not embedded. In fact, the selfintersections make it hard to visualize (see [DHKW], p.146 for images, or generate them by yourself!).

To analyse Enneper's surface at 0 and infinity, we use the polar coordinate representation

$$f\big(r\cos t, r\sin t\big) = \begin{pmatrix} r\cos t - \frac{1}{3}r^3\cos^3 t + r^3\cos t\sin^2 t \\ -r\sin t + \frac{1}{3}r^3\sin^3 t - r^3\cos^2 t\sin t \\ r^2\cos 2t \end{pmatrix} = \begin{pmatrix} r\cos t - \frac{1}{3}r^3\cos 3t \\ -r\sin t - \frac{1}{3}r^3\sin 3t \\ r^2\cos 2t \end{pmatrix}.$$

To see this, remember $\cos 3t = \operatorname{Re} e^{3it} = \operatorname{Re}(e^{it})^3 = \operatorname{Re}(\cos t + i\sin t)^3 = \cos^3 t - 3\cos t\sin^2 t$ and $\sin 3t = \operatorname{Im} e^{3it} = \operatorname{Im}(e^{it})^3 = \operatorname{Im}(\cos t + i\sin t)^3 = -\sin^3 t + 3\cos^2 t\sin t$.

Consider the circle $\gamma_r(t) = (r\cos t, r\sin t)$ of radius $r > 0$. First, when r is small, we have

$$f\big(\gamma_r(t)\big) = \big(r\cos t, -r\sin t, r^2\cos 2t\big) + O(r^3) \qquad \text{as } r \to 0,$$

meaning that the image circle is asymptotically in the horizontal xy-plane. It is parameterized clockwise. The sign of the z-values alternates four times or every 90°; it is zero only on the diagonals.

Second, to study the image of the circle $\gamma_r(t)$ for large radius r, note that

$$f\big(\gamma_r(t)\big) = \Big(-\frac{1}{3}r^3\cos 3t,\ -\frac{1}{3}r^3\sin 3t,\ r^2\cos 2t\Big) + O(r) \qquad \text{as } r \to \infty.$$

In particular, $|f(\gamma_r(t))| = \frac{1}{3}r^3 + O(r^2)$, meaning that the image "circles" are asymptotically round. Consequently, the radial projection of the image circle onto $\mathbb{S}^2$ reads

$$\frac{f(\gamma_r(t))}{|f(\gamma_r(t))|} = \Big(-\cos 3t + O(\tfrac{1}{r}), -\sin 3t + O(\tfrac{1}{r}),\ \frac{3}{r}\cos 2t + O\big(\tfrac{1}{r^2}\big)\Big) \qquad \text{as } r \to \infty.$$

The third component is small, and so the image of a large circle is again asymptotically a horizontal round circle. However, it runs 3 times round anticlockwise! The sign of the third component indicates to which side of the horizontal xy-plane the surface sits: this time it takes exactly 270° for the image to change sign.

`12. Lecture, Tuesday 6.12.05`

To explain an important property of Enneper's surface, we note that the first fundamental form (25) depends on r alone. In geometric language this yields:

Proposition 15. *Let R_α denote the rotation of $\mathbb{R}^2$ by an angle $\alpha \in \mathbb{R}$. The Enneper surface has an intrinsic rotation, that is f and $f \circ R_\alpha$ are isometric for each $\alpha \in \mathbb{R}$.*

Imagine a plaster model of Enneper's surface as well as a copy of the actual surface made from thin metal; they must represent an embedded portion of the surface, of course. Then the metal copy can be rotated on the surface; while doing so, it will change shape.

Brian Smyth in the 1980's determined all minimal surfaces admitting intrinsic isometries; precisely Enneper's surface, as well as certain analogues with higher dihedral symmetry, have an intrinsic rotation. Bour in 1862 had studied minimal surfaces which are developable to a surface of revolution; I do not know to which level of detail he discussed the surface described by Enneper two years later (see [N], p.57).

Let us mention two more properties:
1. The Enneper surface is algebraic, i.e. there is a a polynomial of 9-th order (see [N, p.77]), such that the Enneper surface is its zero set. Note that any surface can be represented implicitely; the point here is that the function is not trancendental but a polynomial.
2. The Gauss map is injective; it misses only the point $-e_3 \in \mathbb{S}^2$. The total curvature is -4π.

References. [N], p.75–81

2.4. **Scherk's doubly periodic surface.** What are the interesting examples of minimal graphs, $f(x, y) = (x, y, u(x, y))$? As proven by Bernstein, if u is defined on all of $\mathbb{R}^2$, the only such surfaces are planes (u is linear). Nevertheless there are interesting graphs defined over certain subsets of the plane.

Along with four other minimal surfaces, Scherk in 1835 described a graph defined over the open square $Q := \big(-\frac{\pi}{2}, \frac{\pi}{2}\big) \times \big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$ by

$$u\colon Q \to \mathbb{R}, \qquad u(x, y) := \log\cos y - \log\cos x = \log\frac{\cos y}{\cos x}.$$

To prove Scherk's surface is minimal, let us check it satisfies the minimal surface equation for graphs (5),

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0.$$

Indeed, since

$$u_x = \frac{\sin x}{\cos x}, \quad u_y = -\frac{\sin y}{\cos y}, \quad u_{xx} = \frac{1}{\cos^2 x}, \quad u_{yy} = -\frac{1}{\cos^2 y}, \quad u_{xy} = 0$$

we have

$$\frac{1 + \frac{\sin^2 y}{\cos^2 y}}{\cos^2 x} - \frac{1 + \frac{\sin^2 x}{\cos^2 x}}{\cos^2 y} = \frac{\cos^2 y + \sin^2 y - (\sin^2 x + \cos^2 x)}{\cos^2 x\cos^2 y} = 0.$$

The only intersection of the surface with the xy-plane, that is, the only zeros of u, are the two horizontal diagonals $(\pm x, x, 0)$. When (x, y) tends to one of the four (open) boundary edges of the square Q, the function u has limiting values $+\infty$ or $-\infty$. Indeed, if $(x, y) \to (\frac{\pi}{2}, y_0)$, where $|y_0| < \frac{\pi}{2}$, then $\log\cos y - \log\cos x \to \log\cos y_0 + \infty = \infty$.

It remains to discuss limits at the four vertices. We claim that the closure of the graph contains the four vertical lines $(\pm\frac{\pi}{2}, \pm\frac{\pi}{2}) \times \mathbb{R}$. To see that, fix $\lambda > 0$ and note

$$u\Big(\frac{\pi}{2} - \xi, \frac{\pi}{2} - \eta\Big) = \log\frac{\xi - \frac{1}{3!}\xi^3 + \dots}{\eta - \frac{1}{3!}\eta^3 + \dots} \ \to\ \log\lambda \qquad \text{as } \xi = \lambda\eta \searrow 0.$$

But $\log\colon (0,\infty) \to \mathbb{R}$ is surjective, implying that this limit can attain any value in $\mathbb{R}$. This shows that any point on the four vertical lines is a limit point of a sequence of points on the graph.

The surface extends to all (x, y) such that $\cos x, \cos y > 0$, that is, to the black squares of a suitable chequerboard. Let us consider the closure of this surface. This means to join the set of vertical lines over the vertices of this chequerboard. The closure is complete. We will later show that the extended surface is smooth at the vertical lines, and that its mean curvature vanishes. Thus we have constructed a complete minimal surface. This surface is *doubly periodic*, i.e. invariant under the rank 2 lattice $\mathbb{Z}(\pi, \pi, 0) \oplus \mathbb{Z}(-\pi, \pi, 0)$.

References. [N] p.63/64, [DHKW] p.151–159

`13. Lecture, Wednesday 7.12.05`

2.5. **Compact minimal surfaces.** To discuss compact minimal hypersurfaces, we need the notion of a immersed hypersurface. We can define an *immersed hypersurface* to be a connected set $M \subset \mathbb{R}^{n+1}$, such that for each $p \in M$ there is a neighbourhood $V(p)$ with the property that $V(p) \cap M$ is the image of a finite number of parameterized hypersurfaces. For those of you who know about manifolds, we can say that there is an abstract n-manifold Σ and an immersion $f\colon \Sigma \to \mathbb{R}^{n+1}$ whose image is M.

We apply the local normal form to prove the following:

Theorem 16. *Let $M \subset \mathbb{R}^{n+1}$ be an immersed hypersurface which is compact. Then there is a point $P \in M$ with all principal curvatures $\kappa_i(P) > 0$; in particular P has positive Gauss curvature.*

Proof. On the compact subset $M \subset \mathbb{R}^{n+1}$ the continuous function $|x|$ attains a maximum R at some point $P \in M$. Thus

$$\exists P \in \mathbb{S}^n_R \cap M \quad \text{with} \quad M \subset \overline{B_R} := \{x \in \mathbb{R}^{n+1} \mid |x| \le R\}.$$

At P the sphere ∂B_R has the tangent space $P^\perp$. We claim that $P^\perp$ coincides with the tangent plane T_pM for any preimage p of P. If not, there was $V \in T_pM$ with $\langle V, P\rangle \ne 0$. By replacing V with $-V$, if necessary, we can assume $\langle V, P\rangle > 0$. From the local normal form (27) we obtain the representation

$$\tilde f(tV) = P + tV + O(t^2) \quad \Rightarrow \quad |\tilde f(tV)|^2 = R^2 + 2t\langle P, V\rangle + O(t^2)$$

valid for small t. This gives

$$0 \overset{M\subset\overline{B_R}}{\ge} \frac{|\tilde f(tV)|^2 - R^2}{t} = 2\langle P, V\rangle + O(t);$$

the right hand side is positive for small $t > 0$, a contradiction. This proves $T_PM = T_P(\partial B_R) = P^\perp$.

We now apply the normal form (27) once again: Let $\tilde f$ be the normal form of M and $\tilde s$ the normal form of the sphere, taken at the point P with respect to the common normal $\nu(P) := -\frac{P}{|P|}$. Then

$$0 \overset{M\subset\overline{B_R}}{\le} \langle \tilde f - \tilde s,\, \nu(P)\rangle = \frac{1}{2}\sum_{i=1}^n \Big(\kappa_i - \frac{1}{R}\Big) x^i x^i + O\big(|x|^3\big).$$

This representation holds for $x = \sum_{i=1}^n x^i v_i \in T_PM = P^\perp$ in a sufficiently small neighbourhood of 0 (which has both parametrisations defined). Thus we conclude $\kappa_i \ge \frac{1}{R} > 0$ for each $1 \le i \le n$. □

On a minimal surface, $\kappa_1 + \ldots + \kappa_n \equiv 0$, which is impossible if all κ_i are positive. Hence:

Corollary 17. *A minimal hypersurface $M \subset \mathbb{R}^{n+1}$ cannot be compact.*

3. Variations

Lagrange in 1762 introduced what we nowadays call the *calculus of variations [Variationsrechnung]*. He considered the area content of a surface with fixed boundary which is a graph of minimal area. He showed that the graph satisfies the minimal surface equation in divergence form (4)

$$\frac{\partial}{\partial x}\frac{u_x}{\sqrt{1 + u_x^2 + u_y^2}} + \frac{\partial}{\partial y}\frac{u_y}{\sqrt{1 + u_x^2 + u_y^2}} = 0.$$

Only in the 19th century was it fully recognized that the latter condition is precisely the condition that twice the mean curvature vanishes.

The present section introduces the variational ideas. We find it advantageous to use expansions for area rather than derivatives.

3.1. The first and second variation.

Lemma 18. *If f is a two-dimensional surface with Gauss map ν we have*

$$\big\langle d\nu(X), d\nu(Y)\big\rangle = 2Hb(X, Y) - Kg(X, Y) \qquad \text{for all } X, Y \in \mathbb{R}^2. \tag{26}$$

The bilinear form on the left is called the *third fundamental form.*

Proof. Let κ_1, κ_2 be the principal curvatures and define the bilinear form

$$T(X, Y) := \big\langle (d\nu + \kappa_1 df)X, (d\nu + \kappa_2 df)Y\big\rangle.$$

Let X_1, X_2 be two linearly independent principal curvature directions for κ_1, κ_2. Then on the one hand

(27) $$T(X_1, Y) = T(Y, X_2) = 0 \qquad \text{for all } Y \in \mathbb{R}^n.$$

On the other hand we claim T is symmetric. Indeed we can express T in terms of symmetric forms as follows:

$$\begin{aligned} T(X,Y) &= \langle d\nu(X), d\nu(Y)\rangle + \kappa_1\kappa_2\langle df(X), df(Y)\rangle + \kappa_1\langle df(X), d\nu(Y)\rangle + \kappa_2\langle df(Y), d\nu(X)\rangle \\ &= \langle d\nu(X), d\nu(Y)\rangle + Kg(X,Y) - 2Hb(X,Y). \end{aligned}$$

Since T is symmetric and X_1, X_2 is a basis, (27) proves $T \equiv 0$. Plugging this into the last expression for T implies our claim. $\square$

Theorem 19 (First and second variation of area). *Let $f\colon U \to \mathbb{R}^3$ be a surface with Gauss map ν and $u \in C_0^1(U, \mathbb{R})$ be differentiable with compact support V. Then the* normal variation

(28) $$f^t := f + tu\nu\colon U \to \mathbb{R}^3$$

is an immersion for sufficiently small $|t|$, whose area has the following expansion as $t \to 0$,

(29) $$A_V(f^t) = A_V(f) - 2t\int_V uH\,dS + t^2\int_V \frac{1}{2}\|\nabla u\|^2 + u^2K\,dS \; + O(t^3).$$

Here $\|\nabla u\|^2 := \sum_{i,j=1}^2 g^{ij}\partial_i u\,\partial_j u$ and $dS = \sqrt{\det g}\,dx$ is the Riemannian area element of f.

Example. Vary the 2-sphere $S(r)$ through spheres $S(r+t)$ of radius $r+t$. With respect to the outer normal ν we have $u \equiv 1$. Expanding the area content by powers of t we obtain

$$A\big(S(r+t)\big) = 4\pi(r+t)^2 = 4\pi r^2 + t\,8\pi r + t^2 4\pi.$$

Indeed, the coefficient of t gives $2\int_{S(r)} H = 2(4\pi r^2)\frac{-1}{r} = -8\pi r$, while the coefficient of t^2 gives $\int_{S(r)} K = 4\pi r^2 \frac{1}{r^2} = 4\pi$.

Remark. A normal variation seems more restrictive than a general variation $f^t := f + tX$, where $X \in C_0^1(U, \mathbb{R}^n)$ is a vector field with compact support. However, for t small and upon reparametrization, any general variation can be written as a normal variation; this doesn't affect the area.

Proof. Step 1: We calculate the first fundamental form of $f^t(V)$. We have for each i

$$\partial_i f^t = \partial_i f + tu\,\partial_i\nu + t\,\partial_i u\,\nu.$$

Using that ν and df are orthogonal, and $b_{ij} = -\langle\partial_i f, \partial_j\nu\rangle = b_{ji}$ we obtain

$$\begin{aligned} g_{ij}^t := \langle\partial_i f^t, \partial_j f^t\rangle &= g_{ij} - 2tub_{ij} + t^2\Big(u^2\langle\partial_i\nu, \partial_j\nu\rangle + \partial_i u\,\partial_j u\Big) \\ &\overset{(26)}{=} g_{ij} - 2tub_{ij} + t^2\Big(u^2(2Hb_{ij} - Kg_{ij}) + \partial_i u\,\partial_j u\Big). \end{aligned}$$

Step 2: We calculate the determinant of g^t. For that end, we calculate the coefficients of the expansion

$$\det g^t = g_{11}^t g_{22}^t - (g_{12}^t)^2 = (I) + t\,(II) + t^2(III) + O(t^3).$$

Clearly,

$$(I) = g_{11}g_{22} - g_{12}^2 = \det g.$$

The first order terms are

$$(II) = -2u\big(b_{11}g_{22} + b_{22}g_{11} - 2b_{12}g_{21}\big) \overset{(1)}{=} -4uH\det g$$

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Finally, the quadratic terms are

$$\begin{aligned} (III) &= u^2\Big(2H(b_{11}g_{22} + b_{22}g_{11} - 2b_{12}g_{12}) - K(g_{11}g_{22} + g_{22}g_{11} - 2g_{12}^2) + 4(b_{11}b_{22} - b_{12}^2)\Big) \\ &\qquad + g_{22}\,\partial_1 u\,\partial_1 u + g_{11}\,\partial_2 u\,\partial_2 u - 2g_{12}\,\partial_1 u\,\partial_2 u \\ &\overset{(1), I(23)}{=} u^2\big(4H^2\det g - 2K\det g + 4K\det g\big) + \det g\sum_{i,j=1} g^{ij}\,\partial_i u\,\partial_j u. \end{aligned}$$

Collecting all terms we arrive at the desired formula

(30) $$\det g^t = \det g\Big[1 - 4tuH + t^2u^2\big(4H^2 + 2K\big) + t^2\|\nabla u\|^2\Big] + O(t^3).$$

Since $\det g \neq 0$ and u has compact support this formula shows that indeed $\det g^t \neq 0$ for small $|t|$. This verifies f^t is an immersion.

Step 3: We compute the root of the determinant $\sqrt{\det g^t}$. To find the root of $[\dots]$, we use Taylor's formula

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{1}{8}x^2 + O(x^3) \qquad \text{as } x \to 0.$$

Thus (30) gives

$$\begin{aligned} \sqrt{\det g^t} &= \sqrt{\det g}\Big(1 - 2tuH + t^2u^2\big(2H^2 + K\big) + \frac{t^2}{2}\|\nabla u\|^2 - 2t^2u^2H^2 + O(t^3)\Big) \\ &= \sqrt{\det g}\Big(1 - 2tuH + t^2\big(\tfrac{1}{2}\|\nabla u\|^2 + u^2K\big) + O(t^3)\Big) \end{aligned}$$

Step 4: Since $A_V(f^t) = \int_V \sqrt{\det g^t}\,dx$ we obtain the claim by integration. $\square$

Remark. For two reasons we required the variation to have compact support. First, all integrals become finite. Second, all functions being integrated then take a maximum; thus f^t is an immersion for small $|t|$. Given these two properties the theorem applies to U in place of V as well.

For various purposes, parallel surfaces with $u \equiv 1$ are interesting. As a corollary to the variation formula (29) we find that their area behaves as follows.

Corollary 20. *Let $f: U \to \mathbb{R}^3$ be a surface. Then for sufficiently small $|t|$ the* parallel surface *$f^t = f + t\nu$ is an immersion, and, provided U has compact closure, we have*

$$A_U(f^t) = A_U(f) - 2t \int_U H\, dS + \frac{t^2}{2} \int_U K\, dS + O(t^3). \tag{31}$$

3.2. Stationary surfaces and local minima.

Definition. We call a surface $f: U \to \mathbb{R}^{n+1}$ *stationary (for the area content)* if

$$\delta_{u\nu} A_V(f) := \frac{d}{dt} A_V(f + tu\nu)\Big|_{t=0} = 0$$

for all normal variations $u \in C_0^1(U, \mathbb{R})$ with $V := \operatorname{supp} u$.

Theorem 21. *A surface $f: U \to \mathbb{R}^3$ is stationary if and only if $H \equiv 0$.*

In particular, if f has minimal area compared with admissable surfaces $f + tu\nu$ then $H \equiv 0$. This lead to the name *minimal surface* for surfaces with $H \equiv 0$.

Proof. From (29) we find

$$\delta_{u\nu} A_V(f) = -2 \int_V uH\, dS \qquad \text{for all } u \in C_0^\infty(U, \mathbb{R}).$$

The proof then follows using the next lemma. □

Lemma 22 (Fundamental lemma of the calculus of variations)**.** *Suppose $f \in C^0(U, \mathbb{R})$ and*

$$\int_U uf\, dx = 0 \qquad \text{for all } u \in C_0^\infty(U, \mathbb{R}).$$

Then $f \equiv 0$.

Proof. Suppose that $f(x) \neq 0$ for some $x \in U$, say $f(x) > 0$. By continuity of f there is $\varepsilon > 0$ such that, $f(y) > 0$ for all $y \in B_\varepsilon(x)$. Then choose a function u with compact support in $B_\varepsilon(x)$, such that $u(x) > 0$. The existence of such functions is not hard to show. Then $\int uf\, dx > 0$, contradicting the assumption. □

Remark. A soap film spanning a wire frame must be stationary. The variations are physically given by small fluctuations caused by air streams, little motions, etc. On the other hand, the mean curvature H of a soap film can be regarded as a force per area. Indeed, surface tension lets each principal curvature $\kappa_i(p) \neq 0$ result in a force. So the sum over the principal curvatures represents the resulting force at a given point p. If $H(p) \neq 0$, then this force per area caused by the geometry must be balanced by an ambient pressure difference. Thus $H(p)$ is the pressure difference to the two side of the film. The theorem now says that it amounts to the same that a soap film minimizes area, or that the pressure difference $H(p)$ is constant over the film.

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Our variational formula (29) also contains second order information which allows us to decide if a minimal surface is a local minimum of area in the following sense:

Definition. A surface f is called a *local minimum (of area)* if for all $u \in C_0^1(U, \mathbb{R})$ there exists $t_0 > 0$ such that the variation $f^t = f + tu\nu$ satisfies for $V := \operatorname{supp} u$

$$A_V(f) \le A_V(f + tu\nu) \qquad \text{for all } |t| < t_0. \tag{32}$$

In particular, if f is an *absolute minimum* of area, that is

$$A_V(f) \le A_{\tilde V}(\tilde f) \qquad \text{for all } \tilde f \in C^1(U, \mathbb{R}^n) \text{ with } V := \operatorname{supp}(f - \tilde f) \subset\subset U$$

then in particular (32) holds.

From 1-dimensional analysis it is known that for a critical point with $f'(x) = 0$ the condition $f''(x) > 0$ implies local minimality, while a local minimum implies $f''(x) \ge 0$. Similarly, in our infinite dimensional function space $C^1(U, \mathbb{R})$: For a local minimum, the *second variation* must satisfy

$$\delta^2_{u\nu} A_V(f) := \frac{d^2}{dt^2} A_V(f + tu\nu)\Big|_{t=0} \ge 0.$$

If this is in fact positive for all nonzero u then (32) holds. Thus the variation formula (29) gives:

Theorem 23. *Suppose $f: U \to \mathbb{R}^3$ is a minimal surface.*
(i) If f is strictly stable *[strikt stabil], that is, the second variation satisfies*

$$\int_U \frac{1}{2}\|\nabla u\|^2 + u^2 K\, dS \;>\; 0 \qquad \text{for all } 0 \not\equiv u \in C_0^1(U, \mathbb{R}). \tag{33}$$

then f is a local minimum.
(ii) If f is a local minimum, then f is stable, *meaning that* (33) *holds with the weak inequality.*

While $\|\nabla u\|^2 = g\big(\binom{-\partial_1 u}{\partial_2 u}, \binom{-\partial_1 u}{\partial_2 u}\big)$ is non-negative the second term in (33), which involves $K \le 0$, tends to make the inequality false.

Example. Consider the catenoid $f\colon \mathbb{R}\times[0,2\pi) \to \mathbb{R}^3$, $f(r,\varphi) = (\cosh r\cos\varphi, \cosh r\sin\varphi, r)$, see Sect. 2.1. Let us consider the annular portion contained between two coaxial circles of distance $2d \ge 2$. We choose a function $u_d \in C^{0,1}(\mathbb{R}^2,\mathbb{R})$ such that $u_d(r,\varphi) = 1$ for $|r| < d-1$, and $u_d(r,\varphi) = d - |r|$ for $|r| \in [d-1,d]$ and $u_d(r,\varphi) = 0$ for $|r| > d$. Since u_d depends on r alone, for $|r| \in [d-1,d]$ and $g_{11} = 1 + \sinh^2 r = \cosh^2 r$ we have

$$\|\nabla u_d\|^2 = g^{11}\,\partial_1 u_d\,\partial_1 u_d = \frac{1}{\cosh^2 r}(\pm 1)^2 = \frac{1}{\cosh^2 r}$$

Moreover, $dS(r,\varphi) = \cosh^2 r\,dr d\varphi$, and so

$$\int_{\mathbb{R}\times[0,2\pi]} \frac{1}{2}\|\nabla u_d\|^2 dS = \int_0^{2\pi} 2\int_{d-1}^{d} \frac{1}{2\cosh^2 r}\cosh^2 r\,dr d\varphi = 2\pi.$$

Finally, $\int_{\mathbb{R}\times[0,2\pi]} K\,dS = -4\pi$, so that

$$\lim_{d\to\infty} \int_{\mathbb{R}\times[0,2\pi]} \frac{1}{2}\|\nabla u_d\|^2 + u_d^2 K\,dS = 2\pi - 4\pi = -2\pi < 0.$$

Thus (33) and also (32) fail for sufficiently large d. Note that by approximation it is in fact enough to check (33) for a Lipshitz function.

Our calculation confirms that a large annular piece of the catenoid is not a local minimum. Our result is consistent with the soap film experiment: If a given pair of concentric circles bounds two catenoids, then soap film will never produce the (inner) catenoid with a small neck.

3.3. **Leaves of minimal foliations are stable.** For a minimal surface which is not a local minimum, all it takes to check that (32) fails, is to specify a good test function u. However, to verify the local minimality of a minimal surface is much harder. In the following we present a test for local minimality, which we derive independently from the variation formula.

Definition. Suppose $U \subset \mathbb{R}^{n+1}$ is an $n+1$-dimensional domain such that all n-dimensional slices $U_t := \{x \in U, x^{n+1} = t\}$ are domains. Then an immersion $f\colon U \to \mathbb{R}^{n+1}$ is called a *foliation [Blätterung]* with *leaves [Blätter]* $f_t = f|_{U_t}$.

Note that the leaves are immersed hypersurfaces.

Examples. 1. Choose the identity on $\mathbb{R}^{n+1}$ to foliate it by parallel horizontal hyperplanes.

2. Consider the vertical line $L := \{(x,y,z), x^2+y^2 = 0\} \subset \mathbb{R}^3$. Then $\mathbb{R}^3 \setminus L$ is foliated by the set of hyperplanes containing the line L. This foliation looks like the pages of an open book! To describe it, note that conformal polar coordinates $\exp\colon \mathbb{C} \to \mathbb{C}\setminus\{0\}$ map $z = x+iy$ to $(\cos y + i\sin y)e^x$, that is, straight lines $\{y = \text{const}\}$ map to rays of angle y. Thus to define the desired immersion we can set

$$f\colon \mathbb{R}^3 \to \mathbb{R}^3 \setminus L, \qquad (x,y,z) \mapsto (e^{x+iy}, z).$$

3. $\mathbb{R}^{n+1}\setminus\{0\}$ is foliated by the concentric spheres $\{\mathbb{S}^n_t, t > 0\}$; in fact, to be consistent with our setup, we should say $\mathbb{R}^{n+1}$ minus a ray is foliated by spheres. To verify this, take your favourite parameterization f_1 of $\mathbb{S}^n$ minus the north pole by some open set $U \subset \mathbb{R}^n$; Then $f_t(x) := t f_1(x)$ for $(x,t) \in U\times(0,\infty)$ gives the foliation.

4. Let $\big(x, u(x)\big)$ be a graph, where $u\colon U \to \mathbb{R}$. Then a foliation by translated graphs is given by

$$f\colon U\times\mathbb{R} \to \mathbb{R}^{n+1} \qquad \text{with} \qquad f_t(x) := \big(x, u(x)+t\big). \tag{34}$$

We are interested in vector fields normal to a foliation. It is convenient to have them defined on the image; so from now on we assume $f\colon U \to V$ is an embedding, that is a diffeomorphism onto its image $V := f(U)$. Orient the normals ν_t of the surfaces f_t consistently, meaning that $\nu\colon U \to \mathbb{S}^n$ is continuous. Since f is a diffeomorphism, we can define a normal field on V by setting $N(P) := \nu(f^{-1}(P))$.

Consider our last example, namely the foliation with translated graphs, $\big(x, u(x)+t\big)$, where $u\colon U \to \mathbb{R}$. Let ν be the normal of the graph $(x, u(x))$. Then

$$N\colon U\times\mathbb{R} \to \mathbb{S}^n, \qquad N(x^1,\dots,x^n,x^{n+1}) := \nu(x^1,\dots,x^n)$$

is a unit normal field. For a graph (x,u), the normal $\nu = (\nu^1,\dots,\nu^n,\nu^{n+1})$ satisfies

$$nH = \operatorname{div}_n \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = -\operatorname{div}_n(\nu^1,\dots,\nu^n),$$

by (4); here div_n denotes divergence in $\mathbb{R}^n$. Thus if u is minimal, $\operatorname{div}_n(N^1,\dots,N^n) = 0$. Moreover, N does not depend on x^{n+1}. Together we obtain that, surprisingly,

$$\operatorname{div}_{n+1} N = \partial_1 N^1 + \ldots + \partial_n N^n + \partial_{n+1} N^{n+1} = 0$$

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The same continues to hold for any foliation with minimal surfaces:

Lemma 24. *Suppose a diffeomorphism $f\colon U \to \mathbb{R}^{n+1}$ is a* minimal foliation*, that is, each leaf f_t is minimal. Then the normal vector field $N\colon f(U) \to \mathbb{S}^n$ satisfies*

$$\operatorname{div}_{n+1} N(P) = 0 \qquad \textit{for all } P \in f(U).$$

Proof. Let $P = f(p)$ and suppose first that T_pf is a horizontal plane $\{x_{n+1} = \text{const}\}$ so that $N(P) = e_{n+1}$. Also $|N| \equiv 1$, which gives $\langle \partial_{n+1}N, N\rangle = 0$. Evaluating at P we obtain

$$0 = \big\langle \partial_{n+1}N(P), N(P)\big\rangle = \big\langle \partial_{n+1}N(P), e_{n+1}\big\rangle = \partial_{n+1}N^{n+1}(P).$$

Since T_pf is horizontal $f_t(p)$ can be locally represented as a graph. Then for each $1 \le i \le n$ the partial derivative $\partial_i N$ agrees with some directional derivative $\partial_i\nu$ of the surface f_t. Since f_t is minimal, (4) yields $\partial_1 N^1 + \ldots + \partial_n N^n = 0$. Thus $\operatorname{div}_{n+1} N = 0$, as desired.

To recover the same result in the general case, we want to rotate a given foliation such that T_pf becomes a horizontal plane $\{x_{n+1} = \text{const}\}$. We claim that the divergence remains invariant under any rotation. To see this, represent the divergence $\operatorname{div} N(P)$ as an integral over $B_\varepsilon = B_\varepsilon(P)$ using the divergence theorem *[Satz von Gauß]*:

$$\operatorname{div} N(P) = \lim_{\varepsilon\to 0} \frac{1}{V(B_\varepsilon)} \int_{B_\varepsilon} \operatorname{div} N\,dx = \lim_{\varepsilon\to 0} \frac{1}{V(B_\varepsilon)} \int_{\partial B_\varepsilon} \langle N, \nu_{\partial B_\varepsilon}\rangle\,dS$$

For coordinates rotated about P, the integral on the right hand side is unchanged, and so the left hand side remains unchanged. □

Theorem 25. *Let $F\colon U \to V$ be a diffeomorphic foliation with minimal surfaces. Consider a leaf $f = F_t\colon U_t \to V$ and another surface $\tilde f\colon U_t \to V$, such that first $K := \operatorname{supp}(\tilde f - f) \subset U_t$ is compact, and second there is an open set $\Delta \subset V$ with $\partial\Delta = f(U_t) \cup \tilde f(U_t)$. Then*

$$A_K(\tilde f) \ge A_K(f), \tag{35}$$

where equality can only hold if $\tilde f$ reparametrizes f.

In particular, (35) implies f is a local minimum of area.

The converse of the theorem is also true: A neighbourhood of a strictly stable minimal surface can be foliated with minimal surfaces ([N], § 109).

Proof. Let us assume first that $\Delta \subset \mathbb{R}^{n+1}$ has only one connected component. For simplicity we pick Gauss maps ν, $\tilde\nu$ for f, $\tilde f$, such that ν is exterior and $\tilde\nu$ interior to Δ. We apply the divergence theorem to the compact set Δ:

$$0 \overset{\operatorname{div} N=0}{=} \int_\Delta \operatorname{div} N\,dx = \int_{\partial\Delta} \langle N, \nu_{\partial\Delta}\rangle\,dS = \int_{f(K)} \langle N, \nu\rangle\,dS - \int_{\tilde f(K)} \langle N, \tilde\nu\rangle\,dS$$

Here, as in the following, strictly speaking we mean by ν the mapping $\nu \circ f^{-1}$ etc., so that everything becomes defined on the image.

Now we use that $\nu = N|_{f(K)}$, but $\langle N, \tilde\nu\rangle \le |N||\tilde\nu| \le 1$ so that

$$A_K(f) = \int_{f(K)} \langle N, \nu\rangle\,dS = \int_{\tilde f(K)} \langle N, \tilde\nu\rangle\,dS \le \int_{\tilde f(K)} dS = A_{\tilde K}(\tilde f).$$

Equality can hold only if $\tilde\nu$ agrees with N, that is for $\tilde f$ a reparameterization of f.

In general Δ is the union of the connected components. Then the above reasoning applies to each component, and a summation over all components leads to the same conclusion. □

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Examples. 1. Consider a standard catenoid $M = M_1$ together with its tangent cone C. Let $A \subset M$ be the annular component of $M \setminus C$ which has compact closure. Suppose that A is parametrized by

$$f_1\colon (-r_0, r_0) \times [0, 2\pi) \to \mathbb{R}^3 \qquad \text{with } f_1(r,\varphi) = (\cosh r\cos\varphi, \cosh r\sin\varphi, r),$$

see Sect. 2.1. It can be checked that the scaled catenoidal annuli

$$f\colon (0,\infty) \times (-r_0, r_0) \times [0, 2\pi) \to \mathbb{R}^3, \qquad f(t,r,\varphi) := (t\cosh r\cos\varphi, t\cosh r\sin\varphi, tr),$$

give a foliation of the exterior of the cone, that is, the doubly connected component of $\mathbb{R}^3 \setminus C$. Thus by the theorem, any compact subset of A is a local minimum, in agreement with the soap bubble experiment.

2. Rotate the helicoid M about its vertical axis A by $R_\varphi \in \mathsf{SO}(3)$, for φ in a neighbourhood of 0. To see this defines a foliation, we need to check that the rotation direction $X(p) := \frac{d}{d\varphi}R_\varphi(p)$ is not a tangent direction, as long as p is in $M \setminus A$. But $X(p)$ is a nonzero horizontal vector perpendicular to the rulings, while the tangent space at p is spanned by the rulings and the non-horizontal tangent vector to the helix through p. Thus "half a helicoid" is a local minimum. Note that in fact this gives a foliation of $\mathbb{R}^3 \setminus A$.

3. For the Scherk surface, the vertical translations give a minimal foliation, showing that any compact subset of the Scherk surface is a local minimum of area.

In fact, any graph leads to a foliation with translated graphs (34), and we obtain in general:

Corollary 26. *Any compact subset of a minimal graph is a local minimum of area.*

As we know, an arbitrary minimal surface need not be a local minimum of area. However, sufficiently small pieces are:

Theorem 27. *Given a minimal hypersurface $f\colon U \to \mathbb{R}^{n+1}$ and $p \in U$, there is a neighbourhood of $f(p)$ which is a local minimum of area.*

The statement implies that any minimal surface can locally be modelled with soap film.

Proof. By the implicit mapping theorem, some neighbourhood of $f(p)$ can be represented as a graph. The claim then follows from the Corollary. □

Remark. The results of this section apply, if correctly phrased, also to foliations which are not diffeomorphisms (embeddings).

References. [DHKW] Sect. 2.7, 2.8; Grosse-Brauckmann, Pacific Journal 175 (1996)

4. Maximum principles

The maximum principle is perhaps the most important tool to analyse minimal surfaces. We introduce it here gradually, starting with harmonic functions, then for linear elliptic equations, and finally for minimal surfaces.

4.1. **Harmonic functions.** Let us look first at the model case for an elliptic equation. This is the Laplace equation $\Delta u = 0$, whose solutions $u \in C^2$ are called *harmonic*.

Examples of harmonic functions include: 1. Constant and linear functions, 2. Real or imaginary parts of holomorphic functions, 3. functions like $x^2 - y^2$ are harmonic, 4. coordinate functions of minimal surfaces.

A crucial property of harmonic functions is the (weak) *maximum principle*:

Proposition 28. *Let U be a bounded domain and $u \in C^2(U) \cap C^0(\overline{U})$ be harmonic. Then*

$$\sup_{\partial U} u = \sup_U u \qquad \textit{and} \qquad \inf_{\partial U} u = \inf_U u.$$

Note that for a continuous function $\sup_{\overline{U}} u = \sup_U u$. The boundedness of Ω is essential (find a counterexample for Ω unbounded!).

Proof. Step 1: Consider for $\varepsilon > 0$ the auxiliary function

$$v(x) := u(x) + \varepsilon |x|^2.$$

Then $\Delta v = \Delta u + 2n\varepsilon > 0$ for all $x \in U$.

Step 2: On the compact set $\overline{U}$, the continuous function v takes a maximum at $y = y(\varepsilon) \in \overline{U}$. Suppose that y is an interior point of U. Then for each i the restriction $v(y + te_i)$ has a maximum and so $\partial_{ii} v(y) \le 0$. Summing, we find $\Delta v(y) \le 0$, in contradiction to $\Delta v > 0$. Consequently, $y \in \partial U$.

Step 3: Since $\varepsilon |x|^2$ is non-negative we have

$$\sup_U u \le \sup_U \left(u + \varepsilon |x|^2\right) \le \sup_{\partial U} \left(u + \varepsilon |x|^2\right) \qquad \text{for all } \varepsilon > 0.$$

But $\varepsilon |x|^2$ is bounded on U, and so the claim follows. Similarly for the minimum. □

Another way to derive the maximum principle is by using a surprising property of harmonic functions: Each value $u(x)$ is the mean of the function u over any ball $\overline{B_r(x)} \subset U$, that is, $u(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy$, see [GT, Thm.2.1]. this property does not generalize to more general equations, so we do not discuss it any further here (see, however, Sect. 5.1).

Remark. For the maximum principle to hold it is suffcicient to assume that $\Delta u \ge 0$, for the minimum principle that $\Delta u \le 0$.

The most important application of the maximum principle is to show the uniqueness of the Dirichlet problem to Poisson's equation:

Proposition 29. *Let U be a bounded domain and $f, g \in C^0(U, \mathbb{R})$. Then any two solutions $u_1, u_2 \in C^2(U, \mathbb{R}) \cap C^0(U, \mathbb{R})$ to the* Poisson equation

$$\Delta u(x) = f(x) \textit{ for all } x \in U \qquad \textit{with } u_1(x) = u_2(x) \textit{ for all } x \in \partial U,$$

coincide.

Proof. The function $v := u_1 - u_2$ satisfies $\Delta v \equiv 0$ and has boundary values $v = u_1 - u_2 = g - g = 0$. Thus by Proposition 28 we have

$$\sup_{\partial U} v = \sup_U v = 0 = \inf_U v = \inf_{\partial U} v,$$

so that $0 = v = u_1 - u_2$ on U. □

It is a much more difficult task to construct such solutions. For special domains U, like the ball or a half-space, there are explicit formulas. In general, however, some abstract existence schemes must be used ("Perron process", see [GT] Sect. 2.8).

Let us conclude this section with an example of a more general equation which does *not* satisfy a maximum principle: Consider the eigenvalue problem

$$\Delta u + \lambda u = 0 \quad \text{ for } \lambda > 0,$$

where u has zero boundary values. If we let U be the cube $(-\frac{\pi}{2}, \frac{\pi}{2})^n$ then the functions

$$f(x) := \cos(k_1 x_1) \cdot \ldots \cdot \cos(k_n x_n) \qquad \text{with } k_1, \ldots, k_n \in \mathbb{N}$$

have zero boundary values and satisfy $\Delta u + k_1^2 \cdots k_n^2 u = 0$. But $u \equiv 0$ solves the equation as well. Hence for λ a product of squared natural numbers there are at least two solutions. Functions satisfying $\Delta u = \lambda u$ are called *eigenfunctions* of the Laplace operator.

References. [GT], Sect. 2

`18. Lecture, Tuesday 3.1.06` ———

4.2. **The maximum principle for linear elliptic equations.** On our way to the minimal surface equation we introduce more general equations which behaves much alike the Laplace equation.

Definition. A *linear partial differential operator of second order* is given by

$$Lu(x) := \sum_{i,j=1}^{n} a^{ij}(x)\,\partial_{ij}u(x) + \sum_{k=1}^{n} b^k(x)\,\partial_k u(x) = \operatorname{trace}(Ad^2u) + \langle b, \nabla u\rangle \tag{36}$$

where a^{ij}, b^k are functions on U, such that the matrix $A = (a^{ij})$ is symmetric and $|b|$ is bounded on U. The operator L is called *uniformly elliptic* if there exists $\lambda > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j > \lambda|\xi|^2 \qquad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\} \text{ and } x \in U. \tag{37}$$

The ellipticity condition (37) means that the lowest eigenvalue of A is bounded below uniformly on U. The symmetry assumption on A can always be achieved (how?).

Note that linearity means $L[u+v] = L[u] + L[v]$.

Examples. 1. If A is the identity matrix and $b = 0$ then L is the Laplacian.
2. A special case is that A is constant and symmetric. Then (37) is equivalent to A being positive definite.
3. Later, we will encounter operators in *divergence form*, that is,

$$Lu(x) = \sum_{i,j} \partial_i\big(\tilde a^{ij}\partial_j u\big) = \operatorname{div}\big(\tilde A(x)\nabla u(x)\big),$$

where $\tilde A$ is a symmetric matrix of functions in $C^1(\overline{U})$. Indeed, differentiating we find

$$Lu = \sum_{i,j} \tilde a^{ij}\partial_{ij}u + \sum_j \Big(\sum_i \partial_i \tilde a^{ij}\Big)\partial_j u.$$

Thus if $\tilde a^{ij}$ satisfies the ellipticity condition (37) and if we set $b^k := \sum_i \partial_i \tilde a^{ik}$ then L as in (36) is elliptic.

Lemma 30 (Weak maximum principle)**.** *Let U be bounded, $u \in C^2(U,\mathbb{R}) \cap C^0(\overline{U},\mathbb{R})$, and L be uniformly elliptic. If $Lu \geq 0$ for all $x \in U$ then*

$$\sup_U u = \sup_{\partial U} u \tag{38}$$

Similarly, if $Lu \leq 0$ then $\inf_U u = \inf_{\partial U} u$.

Proof. Step 1: We consider an auxiliary function $v := u + \varepsilon e^{\gamma x_1}$ where $\varepsilon > 0$. Moreover $\gamma > 0$ is chosen large enough so that the last inequality in

$$Lv \overset{L\text{ linear}}{=} Lu + \varepsilon L(e^{\gamma x_1}) \overset{Lu\geq 0}{\geq} \varepsilon\left(a^{11}\gamma^2 + b^1\gamma\right)e^{\gamma x_1} \overset{L\text{ ell.}}{\geq} \varepsilon\left(\lambda\gamma^2 - |b^1|\gamma\right)e^{\gamma x_1} \overset{|b|\text{ bdd.}}{>} 0 \tag{39}$$

holds.

Step 2: On the compact set $\overline{U}$, the function v takes a maximum $y \in \overline{U}$. Suppose y is an interior point of U. Then $\nabla u(y) = 0$ so that $Lv(y) = \operatorname{trace}\big(A(y)d^2v(y)\big)$ and $d^2v(y)$ is negative semidefinite. We claim that the product of $d^2v(y)$ with the positive definite matrix $A(y)$ is again negative semidefinite; this gives for the trace $Lv(y) \leq 0$.

For any matrices M, N, we have $\operatorname{trace}(MN) = \operatorname{trace}(NM)$. Thus as long as we are only interested in the trace of a matrix product we may multiply the matrices in any order. In particular, for each $T \in \mathsf{GL}(n)$,

$$\operatorname{trace}\left(A\,d^2v\right) = \operatorname{trace}\left((T^{-1}AT)(T^{-1}d^2v\,T)\right) = \operatorname{trace}\left(\widehat{A}\,\widehat{d^2v}\right).$$

Since A is symmetric there is a matrix $T \in O(n)$ such that $\widehat{A} := T^{-1}AT$ is diagonal. Note that

$$\xi^t\big(T^{-1}MT\big)\xi = (\xi^t T^t)M(T\xi) = (T\xi)^t M(T\xi),$$

so that $\widehat{d^2v}(y) = Td^2v(y)T^{-1}$ is still negative semidefinite. In particular all diagonal entries must have the sign $\widehat{d^2v}(y)_{ii} \leq 0$. Similarly $\widehat{A}$ is positive definite, meaning $\widehat{a}^{ii} > 0$ for all i. If follows that

$$Lv(y) = \sum_{i=1}^{n} \widehat{a}^{ii}(y)\widehat{d^2v(y)}_{ii} \leq 0,$$

which establishes our claim. The contradiction to (39) proves that $y \in \partial U$.

Step 3: We have $\sup_U u \leq \sup_U(u + \varepsilon e^{\gamma x_1}) = \sup_{\partial U}(u + \varepsilon e^{\gamma x_1})$. Since this holds for each $\varepsilon > 0$ and $e^{\gamma x_1}$ is bounded on U, the claim follows. $\square$

As for harmonic equations, the uniqueness of solutions for elliptic equations with prescribed boundary values is again immediate:

Theorem 31. *Let U be bounded and $f \in C^0(U,\mathbb{R})$. If $u_1, u_2 \in C^2(U,\mathbb{R}) \cap C^0(\overline{U},\mathbb{R})$ are two solutions of the elliptic boundary value problem $Lu = f$ with the same boundary values $u_1|_{\partial U} = u_2|_{\partial U}$ then $u_1 = u_2$.*

Proof. The function $u_1 - u_2$ has zero boundary values, satisfies $L(u_1 - u_2) = 0$, and so the maximum principle gives $u_1 - u_2 \equiv 0$. $\square$

For the following lemma, the boundary of the domain needs to be suitably good: A domain U satisfies an *interior sphere condition* at $p \in \partial U$, if there is a ball $B_r(q) \subset U$ with $p \in \partial B_r(q)$. An interior normal ν to U at p is the inner normal of B_r at p. The local normal form of a hypersurface implies that a C^2 boundary satisfies such a condition: We need to choose the radius smaller than any inverse principal curvature of the boundary, $r < 1/\kappa_i(x)$.

Lemma 32 (Hopf boundary point lemma). *Suppose U satisfies an interior sphere condition at $p \in \partial U$ with interior normal ν. Suppose that $u \in C^2(U,\mathbb{R}) \cap C^0(\overline{U},\mathbb{R})$ satisfies $Lu \geq 0$ where L is uniformly elliptic and A bounded. Moreover suppose $u(p)$ is a strict boundary maximum, $u(x) < 0 = u(p)$ for all $x \in U$. Then the normal derivative, provided it exists, satisfies*

$$\frac{\partial u}{\partial \nu}(p) < 0.$$

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Proof. Let $B = B_r(y) \subset U$ be a ball with $p \in \partial B$. Without loss of generality, $y = 0$. Then for each $c > 0$ the auxiliary function

$$\varphi(x) := e^{-c|x|^2} - e^{-cr^2}$$

is positive on the annulus $R = B_r(0) - B_{r/2}(0)$; it vanishes at p. Moreover, we have $\partial_i\varphi(x) = -2cx_i e^{-c|x|^2}$ and hence

$$L\varphi(x) = e^{-c|x|^2}\Big(4c^2\sum_{i,j} a^{ij}x_ix_j - 2c\sum_i(a^{ii} + b^ix_i)\Big) \geq 2ce^{-c|x|^2}\Big[2c\lambda|x|^2 - \sum_i a^{ii} - |b||x|\Big].$$

Since $A, b, |x|$ are bounded on R, and $|x|^2 > r^2/4$, we can choose c large enough to have $[\dots] \geq 0$ on R. We conclude $L\varphi \geq 0$.

u is negative in the interior, so there is $\varepsilon > 0$ with $u + \varepsilon\varphi \leq 0$ on $\partial B_{r/2}$; on ∂B_r this is ≤ 0 anyway. Since $L(u + \varepsilon\varphi) \geq 0$ the weak maximum principle (38) applies to give $u + \varepsilon\varphi \leq 0$ on R, while $(u + \varepsilon\varphi)(p) = 0$. Thus the normal derivative of this function at p is ≤ 0 which implies

$$\frac{\partial u}{\partial \nu}(p) \leq -\varepsilon\frac{\partial \varphi}{\partial \nu}(p) < 0.$$

For the last inequality we used the fact that for the rotationally symmetric function φ we can compute the normal derivate $\frac{\partial\varphi}{\partial\nu}(p)$ as $\langle\partial_1\varphi(re_1), -e_1\rangle < 0$. ☐

At a general boundary with corners, however, there may exist normal derivatives which do not vanish: Consider the harmonic function $u(x,y) = xy$ on the quadrant $(0,\infty)\times(-\infty,0)$. At the origin, the boundary of this domain has a corner. Since $\nabla u(0,0) = 0$ any directional derivative at the origin vanishes.

We can now conclude:

Theorem 33 (Strong maximum principle). *Let L be uniformly elliptic and $Lu \geq 0$ on U. If u achieves an interior maximum then u is constant.*

Proof. Suppose, contrary to the statement, that u achieves a maximum $m \geq 0$ at an interior point but that u is non-constant. Then the set $U^< := \{x \in U, u(x) < m\}$ is non-empty. Pick a point $p \in U^<$ be that is closer to $\partial U^<$ than to ∂U, and consider the largest open ball $B(p) \subset U^<$. Then $u(x) = m$ for some point $x \in \partial B(p)$ while $u < m$ on $B(p)$. But by the Hopf boundary point lemma, $\nabla u(x) \neq 0$, contradicting the fact that m is the maximum of u. ☐

References. [GT], Sect. 3.1 and 3.2

4.3. **The maximum principle for minimal surfaces.** To formulate a maximum principle for minimal graphs $\big(x, u(x)\big)$, our starting point is the minimal surface equation in its divergence form (4): Let

$$Qu := \sum_{i=1}^n \partial_i\Big(\frac{\partial_i u}{\sqrt{1+|\nabla u|^2}}\Big) - nH, \tag{40}$$

then a surface $\big(x, u(x)\big)$ of mean curvature $H = H(x)$ satisfies $Qu = 0$.

The equation $Qu = 0$ is nonlinear ($Qu = Qv = 0$ does not imply $Q(u+v) = 0$), but still there is a maximum principle for minimal graphs $(x, u(x))$:

Theorem 34. *Let U be bounded and $u, v \in C^2(U,\mathbb{R}) \cap C^0(\overline{U},\mathbb{R})$ describe two graphs of mean curvature $H \in C^0(U,\mathbb{R})$ with respect to the upper normal. Then we have:*
(i) Interior maximum principle: *If $u \leq v$ and $u(p) = v(p)$ at some interior point $p \in U$, then $u \equiv v$.*
(ii) Boundary maximum principle: *Let ∂U be smooth and $u, v \in C^2(U,\mathbb{R}) \cap C^0(\overline{U},\mathbb{R})$. Suppose that the boundary satisfies an interior sphere condition at $p \in \partial U$ and that u and v have a normal derivative at $p \in \partial U$. Moreover, let $u \leq v$ for all interior points $x \in U$ and $u(p) = v(p)$ for some $p \in \partial U$. Suppose that $\frac{\partial(u-v)}{\partial\nu}(p) = 0$. Then $u \equiv v$.*

Example. Once again, the boundary smoothness is necessary for (ii) to hold: The Enneper surface is graph in a neighbourhood of the origin O. For U a quadrant the Enneper surface over $O \cap U$ takes 0 boundary values. Nevertheless any "normal" derivative of the Enneper graph at 0 vanishes.

Proof. As we have seen in the uniqueness proofs for Laplace's or elliptic equations, it is useful to consider the difference $u - v$.

We set

$$a^i(p) := \frac{p_i}{\sqrt{1+|p|^2}}, \qquad 1 \leq i \leq n.$$

For each $x \in U$ the graphs u, v have mean curvature $H(x)$. Hence we have

$$\begin{aligned}
0 = Qu - Qv &= \sum_i \partial_i a^i(\nabla u) - \partial_i a^i(\nabla v) = \sum_i \partial_i a^i\Big(t\nabla u + (1-t)\nabla v\Big)\Big|_{t=0}^{t=1} \\
&= \int_0^1 \frac{d}{dt} \sum_i \partial_i a^i \Big(t\nabla u + (1-t)\nabla v\Big)\, dt \\
&= \sum_{i,j} \int_0^1 \partial_i \left(\frac{\partial a^i}{\partial p_j}\Big(t\nabla u + (1-t)\nabla v\Big)\big(\partial_j u - \partial_j v\big)\right)\, dt \\
&= \sum_{i,j} \partial_i \left(\left[\int_0^1 \frac{\partial a^i}{\partial p_j}\Big(t\nabla u + (1-t)\nabla v\Big) dt\right] \big(\partial_j u - \partial_j v\big)\right)
\end{aligned}$$

Thus setting $[\ldots] =: a^{ij}(x)$ the difference $w := u - v$ satisfies the equation

$$Lw := \sum_{i,j} \partial_i\big(a^{ij}(x)\partial_j w\big) = \sum_{i,j} \partial_i\big(a^{ij}(x)\partial_j(u-v)\big) = Qu - Qv = 0,$$

which is a linear partial differential equation in divergence form. We have achieved linearity of this equation at the expense of having the coefficients a^{ij} depend on u, v. This, however, will not cause any difficulties since, for the present proof, these two functions are given once and for all.

We claim that L is elliptic on any compact subset $K \subset\subset U$. To see this, set $P(t,x) := t\nabla u(x) + (1-t)\nabla v(x)$. Then

$$\frac{\partial a^i}{\partial p_j}(P) = \frac{\delta_{ij}}{\sqrt{1+|P|^2}} - \frac{P_i P_j}{\sqrt{1+|P|^2}^3}, \qquad 1 \le i, j \le n.$$

For any $\xi \in \mathbb{R}^n \setminus 0$ let us now first apply the Schwarz inequality and then use the fact that $|P(t,x)|$ takes a maximum over the compact set $\{x \in K, t \in [0,1]\}$ to define a positive number $\lambda = \lambda(K, u, v)$ such that the following holds:

$$\sum_{i,j} \frac{\partial a^i}{\partial p_j}(P)\xi_i\xi_j = \frac{1}{\sqrt{1+|P|^2}^3}\Big((1+|P|^2)|\xi|^2 - \langle P, \xi\rangle^2\Big) \ge \frac{1}{\sqrt{1+|P(t,x)|^2}^3}|\xi|^2 > \lambda|\xi|^2$$

Integration of this equation with respect to t verifies the ellipticity condition for L,

$$\sum_{i,j} a^{ij}(x)\xi_i\xi_j = \sum_{i,j}\Big[\int_0^1 \frac{\partial a^i}{\partial p_j}(P(t,x))\, dt\Big]\xi_i\xi_j > \int_0^1 \lambda|\xi|^2\, dt = \lambda|\xi|^2.$$

Let us prove (i). Then by the strong maximum principle, Thm. 33, $w = u - v \equiv 0$ for any compact set $K \subset U$ which contains p; this implies $u = v$ on all of U.

Likewise, if the normal derivatives of u and v at p exist, then $|P|^2$ is bounded on an interior ball B tangent to p. Hence L is also elliptic on B, and the Hopf Boundary Point Lemma 32 gives (ii). □

`20. Lecture, Tuesday 10.1.06`

4.4. **Convex hull property and uniqueness of minimal graphs.** We discuss interesting applications of the maximum principle.

Theorem 35. (i) *A bounded minimal hypersurface M is contained in the convex hull of its boundary values, $M \subset \operatorname{conv}(\partial M)$.*
(ii) *If M touches the boundary $\partial \operatorname{conv}(\partial M)$ of its convex hull at a point interior to M, then M is contained in a plane.*

Proof. (ii) Suppose there is a hyperplane H which is tangent to M at an interior point but does not meet ∂M. We can represent M locally as a graph over H. Then by the maximum principle Thm. 34(i) locally M must agree with H. But the set of points for which M agrees with H is closed (consider sequences) and open (again by the maximum principle). Since M is connected all of M must be a subset of H.

(i) If not the closed set $\overline{M}$ would intersect some plane H at an interior point of M, that is H would not intersect ∂M. The maximum principle Thm. 34(i) leads to a contradiction. □

In particular, a complete minimal surface (with no boundary) cannot be compact – this provides a further proof of Corollary 17.

We have not yet addressed the existence question for minimal graphs with prescribed boundary values (the so-called Dirichlet problem). It is, however, possible to discuss the uniqueness problem:

Theorem 36. *Let U be bounded and $u \in C^2(U,\mathbb{R}) \cap C^0(\overline{U},\mathbb{R})$. Suppose $f(x) = \big(x, u(x)\big)$ is a minimal graph.*
(i) *Then $\big(x, u(x)\big)$ is the unique minimal graph with these boundary values.*
(ii) *If U is convex, then f is unique among bounded minimal surfaces whose boundary agrees with the graph.*

Proof. The maximum principle says that two minimal surfaces cannot intersect at interior points, but they can intersect if in one surface the intersection point is on the boundary. Thus the strategy of the following indirect proofs is to create an interior intersection.

(i) Suppose $u \not\equiv v$ are two solutions with the same boundary values. Then $u - v$ attains a maximum $m \ge 0$; by exchanging u with v we can assume $m > 0$. Then, since $u = v$ on ∂U, the maximum is attained at an interior point $p \in U$. Thus $u \le v + m$ with strict inequality at some point, but $u(p) = v(p) + m$. The interior maximum principle Thm. 34(i) proves $u \equiv v + m$. But since $u = v$ along ∂U this is a contradiction.

(ii) Take a bounded minimal surface $\tilde f\colon V \to \mathbb{R}^3$, attaining the same boundary values. The set $\overline{U}\times\mathbb{R}$ is convex, and so $\tilde f$ takes image in this set by the convex hull property. In fact, by Thm. 35(ii), $\tilde f(V)\subset U\times\mathbb{R}$.

Now consider the foliation with translated graphs, $f(t,x) = \big(x, u(x)+t\big)$. Since $\tilde f$ is bounded, there exist $t_{\max} := \sup\{t\in\mathbb{R}, f_t(U)\cap\tilde f(V)\neq\emptyset\}$ and $t_{\min} := \inf\{\ldots\}$. If $\tilde f$ is not a reparametrisation of f, then one of $t_{\max}, t_{\min}$ is nonzero, say $t_{\max}$.

The two surfaces $\tilde f$ and $f(t_{\max}, .)$ are compact, hence they intersect; the intersection must take place in $U\times\mathbb{R}$. Thus $\tilde f$ and $f(t_{\max}, .)$ intersect at a point which is interior in both surfaces. Then either $\tilde f$ reparametrizes f. Else we find a point $p\in V$ such that $\tilde f(p) = f(q)$, but in any neighbourhood of p the surface $\tilde f$ does not reparameterize f. The normals of $\tilde f$ at p and f at q agree (otherwise $\tilde f$ would contain points to both sides of f). Thus $\tilde f$ can be represented as a graph in a neighbourhood of p. The maximum principle shows this case is impossible. $\square$

In case (ii) existence for arbitrary boundary values will follow from the solution of the Plateau problem in the next part, together with the present statement. However, in the general case (i), existence does not hold for arbitrary boundary values.

Example. We want to show that convexity of the domain is necessary for uniqueness in part (ii) to hold. For our counterexample the domain $U\subset\mathbb{R}^2$ is an annulus, $U = D_R\setminus D_r$ with $R>0$; the radius $r<R$ will be defined later. Consider a catenoid M with waist radius $\rho<R$. Then M contains a circle of radius R and will define the graph over the annulus U. Arbitrary blow downs of the catenoid, $M_\lambda := \lambda M$ for $0<\lambda<1$ will also contain a circle of radius R. Moreover, if we translate M_λ so that this circle agress with the original circle of radius r in M then M and M_λ will intersect at another circle of some radius $r\in(\rho,R)$. Since the translation is by a nonzero amount, the portion of M bounde by the two circles will not contain its waist and is graph. On the other hand, the portion of M_λ bounded by the two circles contains its waist and is not graph (this also follows from part (i)!).

`21. Lecture, Wednesday 11.1.06` ______________________

4.5. **Doubly connected minimal surfaces.** In Prop. 10 we saw that two coaxial unit circles of distance larger than $d_{\max} = 1.32\ldots$ do not bound a piece of a catenoid. The following statement is stronger and says that the circles do not bound any connected minimal surface.

Let us denote by C the (double) cone tangent to the catenoid; then $\mathbb{R}^3\setminus C$ has three connected components. We let U_1 and U_2 be the two components which are simply connected, and U_3 be the componant which is not simply connected.

Theorem 37. *Let M be a bounded connected minimal surface whose boundary is contained in $U_1\cup U_2$. Then M as well as its boundary is in fact contained in only one of the sets U_1 or U_2.*

Note, however, that disconnected minimal surfaces may exist. An example is the Goldschmidt solution bounding two concentric circles, one in U_1, the other in U_2.

Proof. We will prove that M does not meet the set $\overline{U_3}$. Thus the connected surface M must be contained in one of the connected components of $\mathbb{R}^3\setminus\overline{U_3}$, which gives the desired fact.

As a first step we show that M does not meet the set $\overline{U_3}\setminus\{0\} = \bigcup_{\lambda>0} M_\lambda$, where $\{M_\lambda, \lambda>0\}$ is the family of scaled catenoids. Moreover, let $A_\lambda\subset M_\lambda$ be the bounded annular component of $M_\lambda\setminus C$. Consider the infimum

$$0\le\Lambda := \inf\{\lambda>0, M\cap M_\lambda=\emptyset\}$$

Since M is bounded, for some λ we have $M\cap M_\lambda=\emptyset$, implying that $\Lambda<\infty$. Moreover, the sets A_λ foliate U_3 and $\overline{A}_\lambda$ is compact. Thus $\Lambda>0$ if and only if M contains a point $0\neq p\in(\overline{A}_\lambda\cap M)\in\overline{U}_3$. But this intersection is one-sided, hence the maximum principle gives that both surfaces coincide in a neighbourhood, where both surfaces are graph over the common tangent plane at p. Now the subset of points in M, for which $M\subset M_\Lambda$ is relatively closed and, by the maximum principle, open; thus if p exists then $\emptyset\neq\partial M\subset M_\lambda$, which contradicts our assumption. Consequently $\Lambda=0$ and $M\subset(\mathbb{R}^3\setminus\overline{U}_3\cup\{0\})$ as desired.

But the immersed surface M cannot contain the origin: Whatever the tangent plane at $0\in M$ is, the surface will penetrate into U_3. $\square$

Part 3. Solution of Plateau's problem

1. INTRODUCTION

1.1. **History.** The Belgian physicist Plateau popularized soap film experiments in the 19th century. He studied the physics of soap films although he has not seen many of the experiments: he turned blind in 1843. In his treatise of 1873 he postulated that every wire frame bounds at least one soap film; this is nowadays called the *Plateau problem.* It seems Lagrange also had considered the problem in the 18th century.

In the present course, the exposition of the Plateau problem provides the first general existence statement. The Plateau solution will also allow us to construct many interesting minimal surfaces, beyond boundary value problems.

If the soap films of nature tell us that the Plateau problem always has a solution, why should we make an effort to prove the existence mathematically? Like for any mathematical conjecture, there are various reasons. First, whatever the number of successful experiments may be, perhaps there is still a case not yet considered? Second, a mathematical theorem makes a vague conjecture precise; this in turn leads to a much deeper understanding of the problem. Often only this way the appropriate assumptions emerge, which must be imposed on the problem.

Complex analytic techniques were successfully applied to minimal surfaces in the 19th century: The Weierstrass represention, which we will discuss in the next part, represents any minimal surface as an integral in terms of holomorphic functions. A different kind of integral representation was known before for many partial differential equations: Representations of the solutions in terms of boundary or initial values. We will discuss the Poisson formula as an example in this course. From the point of view of the late 19th century it seemed very reasonable to expect a similar representation for a minimal surface in terms of its boundary values; it would solve the Plateau problem and could perhaps be derived from the Weierstrass representation formulas.

However, these efforts failed, and left the solution to a later time which employed new analytic techniques. It became a major mathematical breakthrough in the sense that more implicit modern methods for existence were used. The American mathematician Douglas solved the problem from 1928 to 1931. Later Courant gave improvements. Independently from Douglas and about the same time, Radó gave a different proof. See [DHKW p.277/78] for a short discussion of the approach. For his ground-breaking work, Jesse Douglas was awarded one of first two Fields medals in 1936, the Nobel prize of mathematicians, together with Ahlfors. (See [HT,p.129ff] and [N, p.247-258] for more on the history). In the present part, we give the Douglas/Courant approach, mainly in the form given in Jost's book [J, Sect. 4.4].

1.2. **Jordan curves and soap films.** To model the wire frame we define:

Definition. A *Jordan curve* is a continuous injective map $\Gamma\colon \mathbb{S}^1 \to \mathbb{R}^n$.

Using the general topological theorem that an injective continuous mapping of a compact space is a homeomorphism, it is equivalent to say:

Proposition 1. *A Jordan curve is a homeomorphism onto its image.*

Proof. To prove that, we need to show continuity of Γ^{-1}. A convergent sequence of points on the image can uniquely written as $\Gamma(x_n) \to \Gamma(x)$. But $\mathbb{S}^1$ is compact, so x_n has a subsequence x_m converging to some x_0. By continuity $\Gamma(x_0) = \lim \Gamma(x_m) = \Gamma(x)$. By injectiviy $x = x_0$. This shows that any converging subsequence of x_n converges to x, hence x_n converges to x. □

Let us now make the mathematical problem precise: For a given Jordan curve, we seek a parameterized minimal surface (immersion) $f\colon \overline{D} \to \mathbb{R}^3$, where the boundary values $f(\partial D)$ are a bijective parameterization of the Jordan curve Γ.

Soap film experiments tell us that the surfaces bounded by a Jordan curve may have different topological types, they may be oriented or not. Our formulation of the problem where the domain of f is a disk applies only to the simplest case, Plateau solutions of *disk-type*.

By working with immersed surfaces, which can self-intersect, we do not exactly model a soap film. Soap film cannot intersect, but instead it allows for the singular sets which once again Plateau described: Triple lines, where three sheets of the surface come together at 120 degrees, and quadruple points, where four sheets meet at the angles of a tetrahedron. If we were interested to model soap films we would need more general objects than the surfaces of differential geometry. The flat chains modulo 2 of geometric measure theory are an appropriate model (see Frank Morgan's book [M]).

In the case that f is embedding, however, the physical and our mathematical problem agree. It is known that embeddedness must hold provided the Jordan curve is contained on the boundary of a convex set.

1.3. **Area and energy.** For a given Jordan curve Γ, we want to solve a variational problem: We seek an immersed surface f_0 in the class of admissible mappings

$$(1)\qquad \mathcal{C} := \{f \in C^0(\overline{D}, \mathbb{R}^n) \cap C^1(D, \mathbb{R}^n), f|_{\partial D} \text{ is a bijective parameterization of } \Gamma\}.$$

such that it gives the area functional

$$A_D(f) = \int_D \sqrt{|f_x|^2|f_y|^2 - \langle f_x, f_y\rangle^2}\, dxdy$$

a minimum over $\mathcal{C}$. An obvious strategy to construct f_0 is to consider a minimizing sequence (f_n) such that $A(f_n) \to \inf_{f\in\mathcal{C}} A(f)$. Then we hope for $f_0 := \lim f_n$ to exist and to be the solution to the Plateau problem. This approach fails for various reasons.

The main problem is that the minimizing sequence can degenerate so that the limit is not defined or not a surface which attains the given boundary values. Indeed, for any nice smooth f the area functional $A(f)$ will be changed as little as we please by growing little spokes, that is, thickened hairs out of the surface. It is obvious that this way the images $f_n(U)$ can approximate a dense open set. This means that the limit f (taken in any reasonable sense) will in no way be a surface. The same problem cannot arise for curves (why?).

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A better integral to consider is the *Dirichlet integral* or *energy* of $f\colon U^m \to \mathbb{R}^n$,

$$E_U(f) := \frac{1}{2}\int_U |df(x)|^2 dx = \frac{1}{2}\int_U \sum_{i=1}^m |\partial_i f(x)|^2\, dx = \frac{1}{2}\int_U \sum_{1\le i\le m, 1\le j\le n} (\partial_i f^j(x))^2\, dx$$

This integral measures the total amount of streching and so favours small area together with a good parametrization. More precisely:

Lemma 2. *For any $f \in C^1(U^2, \mathbb{R}^n)$ we have*

$$E_U(f) \ge A_U(f).$$

Equality holds if and only if f is a conformal map.

Proof. The inequality between arithmetic and geometric mean gives

$$\frac{1}{2}(a+b) \ge \sqrt{ab} \ge \sqrt{ab-c} \qquad \text{for all } a, b \ge 0 \text{ and } 0 \le c \le ab,$$

with equality precisely for $a = b$ and $c = 0$. (A proof of the first inequality follows from $0 \le (a-b)^2 \Leftrightarrow 2ab \le a^2 + b^2 \Leftrightarrow 4ab \le (a+b)^2$.)

Since

$$E(f) = \frac{1}{2}\int |f_x|^2 + |f_y|^2\, dxdy \ge \int \sqrt{|f_x|^2|f_y|^2 - \langle f_x, f_y\rangle^2}\, dxdy = A(f),$$

the result follows. ∎

By so-called uniformization, any surface can be reparameterized conformally. Thus the two variational problems of minimizing energy and area have the same solutions. However, by minimizing energy we get a good parameterization for free. For this reason minimizing energy is superior to minimizing area.

1.4. **Jordan curves bounding finite area disks.** We must impose restrictions on the bounding Jordan curve as not every such curve bounds a disk of finite area:

Example. For $n \in \mathbb{N}$, let $p_n := (1/\sqrt{n}, 0, 0)$, $q_n := (0, 1/\sqrt{n}, 0)$, and consider a rectangular spiral in the xy-plane running through the points $p_1, q_1, -p_1, -q_2; p_2, q_2, -p_2, -q_2; \ldots$. Extend this curve continuously with, say, a circular arc in the xz-plane running from the origin back to p_1. The result is a closed Jordan curve Γ. We need two facts: First, we can estimate the area of the region U_n between the n-th and $(n+1)$-st arc roughly by

$$A(U_n) \ge \frac{2}{\sqrt{n}}\Big(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\Big) = \frac{2}{\sqrt{n}}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}\sqrt{n+1}} = \frac{2}{\sqrt{n}}\frac{1}{\sqrt{n}\sqrt{n+1}(\sqrt{n+1}+\sqrt{n})} \overset{\sqrt{n+1}\le 2\sqrt{n}}{\ge} \frac{1}{3n^2}.$$

Second, any map f bounding Γ has a projection π to the xy-plane that covers almost every point between the n-th and $(n+1)$-st arc at least n times. We will not argue this fact rigorously. Since a projection decreases area, we can conclude that indeed

$$A(f) \ge \pi(A(f)) \ge \sum_{n=1}^\infty n\pi(A(U_n)) \ge \sum_{n=1}^\infty \frac{n}{3n^2} \ge \frac{1}{3}\sum_{n=1}^\infty \frac{1}{n} = \infty.$$

See Nitsche's book [N] for a complete reasoning.

For Jordan curves which are *piecewise* C^1 this problem cannot arise. This means there is a finite partition of $\mathbb{S}^1$ such that the restrictions of type $\Gamma|_{[t_i,t_{i+1}]}$ are continuously differentiable.

Lemma 3. *A closed curve Γ which is piecewise C^1 bounds a disk of finite energy (and area).*

Proof. Let c be the curve and define in polar coordinates $f(r,\varphi) := rc(\varphi)$, which parameterizes the cone of c with tip 0. For a C^1-curve both $|c|$ and $|c'|$ are bounded by some constant C. Expressing the Dirichlet integral in polar coordinates, we get, as desired:

$$E(f) = \int_0^{2\pi}\int_0^1 \Big(\Big|\frac{\partial f}{\partial r}\Big|^2 + \Big|\frac{1}{r}\frac{\partial f}{\partial\varphi}\Big|^2\Big) r\, dr d\varphi = \int_0^{2\pi}\int_0^1 \big(|c(\varphi)|^2 + |c'(\varphi)|^2\big) r\, dr d\varphi \le 4\pi C^2$$

∎

1.5. **Rigorous formulation of Plateau's problem.** We can now formulate the Plateau problem of disk-type.

Theorem 4. *Let $\Gamma \subset \mathbb{R}^n$ be a closed Jordan curve which is piecewise C^1, and D be the open unit disk in $\mathbb{R}^2$. Then there exists a mapping $h \in C^0(\overline{D}, \mathbb{R}^n) \cap C^2(D, \mathbb{R}^n)$ with the properties: h assumes the finite infimum of energy over $\mathcal{C}$, h is harmonic and conformal on D, and h maps ∂D bijectively onto Γ.*

Note that by Lemma 3 the infimum of energy is finite. If h is an immersion then h is a minimal surface by Thm.(2)2. Note that the immersion property for a conformal map is equivalent to h_x or $h_y \neq 0$.

Since for $n = 1$ there is no Jordan curve, let us always assume $n \geq 2$ in the secquel of this part.

The proof we give below relies on certain properties of harmonic functions. After introducing those, we will prove Theorem 4 in Section 3 as a result of the Theorems 24, 25, and 28.

From the statement alone, it is not clear that h also minimizes area in $\mathcal{C}$, so that h is an absolute minimizer of area. This follows from uniformization: A sequence minimizing A can be reparameterized conformally and so, by 2, the resulting sequence will be minimizing for E as well.

Let us discuss a few questions being raised by the theorem.

1. Is h an immersion, that is, a surface in the sense of differential geometry? The mapping h may have *branch points*, that is points p such that $dh(p) = 0$. This is known to be true for any interior point. At smooth boundary points, we also expect that f is an immersion, in the sense that there is an extension to an immersion. To establish this property has turned out to be amazingly difficult. Only partial results are known then, for instance if the Jordan curve is analytic, then f is an immersion including the boundary. In general, a satisfactory answer to the problem is still open.

2. Is h an embedding? There are certainly knotted Jordan curves which for topological reasons will not allow for embedded solutions. On the other hand, sufficient conditions for h to be an embedding are known. From Thm. 36 follows that Jordan curves which are graphs with convex projection must bound a Plateau solution which is graph as well, and hence embedded. A more general condition for h to be an embedding is that Γ is contained in the boundary of a convex set.

3. The theorem gives us just one minimal surface (in fact, an absolute minimizer of area). Are there further minimal surfaces? In general there are, see for instance [DHKW, p.222/223]. Different techniques are needed to exhibit minimal surfaces which are not local minimizers (see, e.g.: Struwe, Variational methods).

There are various generalizations of the theorem: It holds for rectifiable Jordan curves; it generalizes to multiply connected domains if certain conditions are met; it holds in different ambient spaces (manifolds). Moreover, functionals other than area can similarly be minimized to lead to existence statements for, e.g., surfaces of constant or prescribed mean curvature.

`23. Lecture, Wednesday 18.1.06` ________________

2. HARMONIC AND CONFORMAL MAPS

In order to solve Plateau's problem we will deal with the following tasks:
- Solve the Dirichlet problem for harmonic functions on a disk,
- take limits of harmonic functions,
- control the Dirichlet energy of their limit.

The present section collects these auxiliary results.

References. For results on harmonic maps see [GT], Sect 2, or any other book on elliptic partial differential equations. For conformal maps, we will cover the results essential for minimal surfaces; for broader introductions see Ahlfors [A], III.2 and 4 from the point of view of complex analysis and [Sy1], IV §7 for the point of view of real analysis.

2.1. **The divergence theorem.** Let us recall the divergence theorem on $\mathbb{R}^n$. Suppose U is bounded and has C^1-boundary ∂U, whose exterior normal we denote by ν. Then any vector field $X \in C^1(\overline{U})$ satisfies

$$\int_U \operatorname{div} X \, dx = \int_{\partial U} \langle X, \nu \rangle \, dS, \tag{2}$$

where dS stands for the $(n-1)$-dimensional area element of ∂U.

The divergence theorem will also appear in the following versions:

1. If $u \in C^2(\overline{U})$, then choosing $X := \nabla u$ gives

$$\int_U \Delta u \, dx = \int_{\partial U} \langle \nabla u, \nu \rangle \, dS = \int_{\partial U} \frac{\partial u}{\partial \nu} \, dS \tag{3}$$

2. Setting $X = (0, \dots, u, \dots, 0) = ue_k$ we obtain $\int_U \partial_k u \, dx = \int_{\partial U} u\nu_k \, dS$. We can assemble these components to form a vector valued integrand:

$$\int_U \nabla u \, dx = \int_{\partial U} u\nu \, dS \tag{4}$$

3. If $X = v\nabla u$ where $v|_{\partial U}$ vanishes then, by divergence theorem and product rule,

$$0 = \int_{\partial U} \langle v\nabla u, \nu\rangle dS = \int_U \operatorname{div}(v\nabla u)\, dx = \int_U \langle \nabla v, \nabla u\rangle\, dx + \int_U v\Delta u\, dx. \tag{5}$$

2.2. **Variational characterization and Dirichlet's principle.** *Harmonic maps* $h \in C^2(U, \mathbb{R}^m)$ are defined by the fact that each component has vanishing Laplacian,

$$0 = \Delta h = \operatorname{div} \nabla h = \sum_{i=1}^n \partial_{ii} h.$$

As for minimal surfaces, there is a variational characterization:

Proposition 5. *A map $h \in C^2(U, \mathbb{R}^m)$ is harmonic if and only if its Dirichlet integral $E(h)$ has vanishing first variation,*

$$\frac{d}{dt} E(h + tf)\Big|_{t=0} = 0 \qquad \text{for all } f \in C_0^\infty(U, \mathbb{R}^n).$$

Proof. The Jacobians satisfy $|d(h + tf)|^2 = |dh|^2 + 2t\langle dh, df\rangle + t^2|df|^2$. Thus

$$\frac{d}{dt} E(h + tf)\Big|_{t=0} = \int_U \langle dh, df\rangle\, dx \overset{(5)}{=} -\int_U (\Delta h) f\, dx = 0.$$ □

Quite unlike the area functional, the Dirichlet integral is quadratic in the gradient and hence presents a nice convex functional. For this reason its minima, the harmonic functions, are unique:

Proposition 6 (Dirichlet's principle). *Let $U \subset \mathbb{R}^n$ be a bounded domain, $\partial U \in C^1$. Suppose $h \in C^2(U, \mathbb{R}^n) \cap C^0(\overline{U}, \mathbb{R}^n)$ is harmonic, with finite Dirichlet integral, $E_U(h) < \infty$. Then*

$$E(f) \geq E(h)$$

for all $f \in C^2(U, \mathbb{R}^n) \cap C^0(\overline{U}, \mathbb{R}^n)$ with the same boundary values, $f|_{\partial U} = h|_{\partial U}$.

Proof. Let us first consider the case $n = 1$. The function $f - h$ has zero boundary values, and so by (5)

$$E(f) = E\big(h + (f - h)\big) = \frac{1}{2}\int |\nabla h|^2 + 2\big\langle \nabla h, \nabla(f - h)\big\rangle + |\nabla(f - h)|^2\, dx$$
$$\overset{(5)}{=} \frac{1}{2}\int |\nabla h|^2 + |\nabla(f - h)|^2\, dx \geq E(h).$$

In the vector-valued case, the same holds by summation over all components (or writing d in place of ∇ in the previous proof). □

2.3. **Mean value formula.** Let us denote the volume of the n-dimensional unit ball by $V(B^n) =: \omega_n$. The divergence theorem shows the bounding unit sphere has volume $V(S^{n-1}) = n\omega_n$. The following property exemplifies how *harmonic* our functions are:

Theorem 7. *Let $u \in C^2(U, \mathbb{R})$ be harmonic. Then for any ball $B = B_R(y)$ with $\overline{B} \subset\subset U$*

$$u(y) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B} u(x)\, dS_x \qquad \text{and} \qquad u(y) = \frac{1}{\omega_n R^n}\int_B u(x)\, dx. \tag{6}$$

Proof. We take $B_\rho = B_\rho(y)$ with $\rho < R$. On the boudnary of the ball we consider polar coordinates $\rho = |x - y|$ and $\omega = \frac{x-y}{\rho}$. The divergence theorem gives

$$0 \overset{(3)}{=} \int_{\partial B_\rho} \frac{\partial u}{\partial \nu} dS = \int_{\partial B_\rho} \frac{\partial u}{\partial \nu}(y + \rho\omega)\, dS = \rho^{n-1}\int_{|\omega|=1} \frac{\partial u}{\partial \nu}(y + \rho\omega)\, d\omega$$
$$= \rho^{n-1}\frac{\partial}{\partial \rho}\int_{|\omega|=1} u(y + \rho\omega)\, d\omega = \rho^{n-1}\frac{\partial}{\partial\rho}\Big(\frac{1}{\rho^{n-1}}\int_{\partial B_\rho} u\, dS\Big).$$

Thus for any $\rho \in (0, R)$,

$$\rho \mapsto \frac{1}{\rho^{n-1}}\int_{\partial B_\rho} u\, dS = \text{const}\,.$$

But u is continuous and so

$$\lim_{\rho\to 0} \frac{1}{n\omega_n \rho^{n-1}}\int_{\partial B_\rho} u\, dS = u(y).$$

Thus we obtain the first mean value equality. The solid version follows from integration:

$$\int_{B_R} u(x)\, dx = \int_0^R \Big(\int_{\partial B_\rho} u(x)\, dS\Big) d\rho = \int_0^R n\omega_n\rho^{n-1} u(y) d\rho = \omega_n R^n u(y)$$ □

One consequence of the mean value inequality is the maximum principle. Another are *a priori estimates*:

Theorem 8. *Let $u \in C^\infty(U, \mathbb{R})$ be harmonic, $K \subset\subset U$ be compact, and α be a multi-index. Then there exists a constant $C = C\big(n, \alpha, \operatorname{dist}(K, \partial U)\big)$ such that*

$$\sup_K |\partial^\alpha u| \leq C \sup_U |u|. \tag{7}$$

Proof. If u is harmonic, then $\Delta\nabla u = \nabla\Delta u = 0$, so that ∇u is harmonic. For $y \in K$ and $R = \frac{1}{2}\operatorname{dist}(K, \partial U)$ we apply the mean value equality (6) over $B = B_R(y)$:

$$\nabla u(y) = \frac{1}{\omega_n R^n}\int_B \nabla u\, dx \overset{(4)}{=} \frac{1}{\omega_n R^n}\int_{\partial B} u\nu\, dx$$

Thus, using $|\int f| \leq \int |f|$,

$$|\nabla u(y)| \leq \frac{n}{R}\frac{1}{n\omega_n R^{n-1}}\int_{\partial B} |u|\, dx \leq \frac{n}{R}\sup_{\partial B} |u|.$$

This reasoning can be iterated on an increasing sequence of nested balls (with $R \to 0$ suitably). $\square$

`24. Lecture, Tuesday 24.1.06`

The main use of uniform estimates is to establish the convergence of a sequence of harmonic functions. Let us recall first:

Theorem 9 (Arzelà-Ascoli). *For $K \subset\subset \mathbb{R}^n$ suppose the sequence $u_k \in C^0(K)$ is*
(i) uniformly bounded, that is, $\sup_{k\in\mathbb{N}} \|u_k\|_{C^0(K)} < \infty$, and
(ii) equicontinuous [gleichgradig stetig], that is, for all $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that

$$|x - y| < \delta \Rightarrow |u_k(x) - u_k(y)| \le \varepsilon \quad \text{for all } k \in \mathbb{N}. \tag{8}$$

Then some subsequence of u_k converges uniformly on K.

Proof. For $\varepsilon > 0$ pick $j \in \mathbb{N}$ such that $\frac{1}{j} < \delta(\varepsilon/3)$.

The set K is compact. Hence we can take finitely many points $\{K_j\}$ such that $K \subset \bigcup_{z\in K_j} B_{1/j}(z)$. The boundedness of u_k implies we can find a subsequence of u_k, also denoted u_k, which converges pointwise at all points $z \in K_j$. This subsequence u_k is Cauchy and so there is $N = N(\varepsilon)$, such that

$$z \in K_j \quad \Rightarrow \quad |u_k(z) - u_\ell(z)| \le \frac{\varepsilon}{3} \qquad \text{for all } k, \ell \ge N.$$

Let $x \in K$. Then there is $z \in K_j$ with $|x - z| < \delta$. Applying (8) with $\varepsilon/3$ twice and using our previous estimate we find that for all $k, \ell \ge N(\varepsilon)$:

$$|u_k(x) - u_\ell(x)| \le |u_k(x) - u_k(z)| + |u_k(z) - u_\ell(z)| + |u_\ell(z) - u_\ell(x)| < \varepsilon$$ $\square$

We obtain a compactness result for harmonic functions:

Theorem 10. *Any bounded sequence of harmonic functions $u_k \in C^\infty(U, \mathbb{R})$ contains a subsequence that converges uniformly on each compact subset $K \subset\subset U$ to a harmonic function; all derivatives converge as well.*

We will later see (Thm. 16) that it is sufficient to assume $u_k \in C^2$.

Proof. By (7) first derivatives are uniformly bounded over K and hence (u_k) is equicontinuous. By the Arzela-Ascoli theorem this means we obtain a converging subsequence. Moreover, by (7) the second and third derivatives are bounded, and so for a (further) subsequence of (u_k) the second derivatives also converge. Thus the limit function is harmonic. We can iterate the argument for higher derivatives. $\square$

The theorem, as well as all other results of the present section, generalize to the harmonic map case (check!).

2.4. **Conformality in dimension two.** The present and next section collect prerequisites for an important representation formula for harmonic maps, the Poisson formula.

In dimension two, conformal and holomorphic maps are closely related:

Proposition 11. *Let $U, V \subset \mathbb{C}$. A map $h \in C^1(V, U)$ is holomorphic if and only if it is conformal,*

$$|\partial_1 h| = |\partial_2 h| \qquad \text{and} \qquad \langle \partial_1 h, \partial_2 h\rangle = 0$$

and orientation preserving, $\det dh \ge 0$.

Proof. By definition, a holomorphic map h satisfies the *Cauchy-Riemann equations*

$$\partial_1 h_1 = \partial_2 h_2, \qquad \partial_2 h_1 = -\partial_1 h_2. \tag{9}$$

These equations show that the map h is conformal. To see a holomorphic map is orientation preserving, compute

$$\det dh = \det \begin{pmatrix} \partial_1 h_1 & \partial_2 h_1 \\ \partial_1 h_2 & \partial_2 h_2 \end{pmatrix} \overset{(9)}{=} \det \begin{pmatrix} \partial_1 h_1 & \partial_2 h_1 \\ -\partial_2 h_1 & \partial_1 h_1 \end{pmatrix} = \partial_1 h_1^2 + \partial_2 h_1^2 \ge 0. \tag{10}$$

The converse direction follows from the fact that given a pair $v = \partial_1 h$ and $w = \partial_2 h$ of vectors in $\mathbb{R}^2$ which is orthogonal and of the same length, then $w = \pm Jv$, where J is positively oriented 90-degree rotation; the positive orientation implies $w = Jv$. $\square$

From the proof it is clear that there are also orientation reversing conformal maps. Namely a map is *anti-holomorphic* if

$$\partial_1 h_1 = -\partial_2 h_2, \quad \partial_2 h_1 = \partial_1 h_2 \qquad \Rightarrow \qquad \det dh \le 0. \tag{11}$$

The simplest example is $\overline{z}$. More generally, $f(\overline{z})$ is antiholomorphic iff $f(z)$ is holomorphic (check!). Note that holomorphic as well as anti-holomorphic functions are harmonic maps: $\Delta h_1 = \partial_{11} h_1 + \partial_{22} h_1 = \partial_1\partial_2 h_2 - \partial_2\partial_1 h_2 = 0$ in the holomorphic case, and $\Delta h_1 = -\partial_1\partial_2 h_2 + \partial_2\partial_1 h_2 = 0$ in the anti-holomorphic case; similarly for the second component.

For later use, we want to study composition properties for conformal maps.

Lemma 12. *Let $U, V \subset \mathbb{R}^2$. Suppose $h \in C^2(U, \mathbb{R})$ is harmonic and $f \in C^2(V, U)$ conformal. Then $h \circ f$ is harmonic.*

Proof. Let $i = 1, 2$. By the chain rule,

$$\partial_i(h \circ f) = (\partial_1 h \circ f)\partial_i f_1 + (\partial_2 h \circ f)\partial_i f_2. \tag{12}$$

Differentiating once more we obtain

$$\partial_{ii}(h \circ f) = \partial_{11} h(\partial_i f_1)^2 + 2\partial_{12} h\, \partial_i f_1 \partial_i f_2 + \partial_{22} h(\partial_i f_2)^2 + \partial_1 h\, \partial_{ii} f_1 + \partial_2 h\, \partial_{ii} f_2,$$

where all derivatives of h on the right hand side are to be evaluated at f. Writing $I_i + II_i + III_i + IV_i + V_i$ for the right hand side, we see that $II_1 + II_2 = 0$ due the Cauchy-Riemann equations (9). Also by (9), we see that $I_1 + III_2 = \Delta h(\partial_1 f_1)^2$ vanishes, similiary $I_1 + III_1$. Finally, $IV_1 + IV_2$ vanishes since f is harmonic, and similarly so $V_1 + V_2$. The same arguments are also valid in the antiholomorphic case. Thus, as desired, $\Delta(h \circ f) = 0$. □

`25. Lecture, Wednesday 25.1.06`

For later use (Thm. 21) we state that the energy is invariant under conformal maps:

Lemma 13. *Let $f \in C^1(V, U)$ be conformal and $h \in C^1(U, \mathbb{R}^n)$. Then $E_V(h \circ f) = E_U(h)$.*

Proof. From (12) we get

$$|\partial_i(h \circ f)|^2 = |\partial_1 h|^2(\partial_i f_1)^2 + |\partial_2 h|^2(\partial_i f_2)^2 + 2\langle \partial_1 h, \partial_2 h\rangle \partial_i f_1 \partial_i f_2, \qquad i = 1, 2,$$

where on the right hand side $\partial_i h$ is always evaluated at f. This gives

$$\begin{aligned} &|\partial_1(h \circ f)|^2 + |\partial_2(h \circ f)|^2 \\ &\quad = |\partial_1 h|^2\big(\partial_1 f_1^2 + \partial_2 f_1^2\big) + |\partial_2 h|^2\big(\partial_1 f_2^2 + \partial_2 f_2^2\big) + 2\langle \partial_1 h, \partial_2 h\rangle\big(\partial_1 f_1 \partial_1 f_2 + \partial_2 f_1 \partial_2 f_2\big) \\ &\quad \overset{(10)(11)}{=} |\partial_1 h|^2 |\det df| + |\partial_2 h|^2 |\det df|, \end{aligned}$$

where in the last step we used $\langle \partial_1 f, \partial_2 f\rangle = 0$ as well.

Using this, we now obtain the desired result upon integration, invoking a change of variables:

$$\begin{aligned} E_V(h \circ f) &= \frac{1}{2}\int_V |\partial_1(h\circ f)|^2 + |\partial_2(h\circ f)|^2\, dx_V = \frac{1}{2}\int_V \big(|\partial_1 h \circ f|^2 + |\partial_2 h \circ f|^2\big)|\det df|\, dx_V \\ &= \frac{1}{2}\int_U \big(|\partial_1 h|^2 + |\partial_2 h|^2\big) dx_U = E_U(h) \end{aligned}$$

□

The significance for the minimization of the Dirichlet integral is that we are allowed to reparameterize conformally without changing its energy. By a good choice of conformal maps we will secure convergence of a minimizing sequence.

2.5. **The conformal group of the disk.** The previous lemma says energy is invariant under conformal maps. For the Plateau problem (of disk-type!) we will consider the case that the domain U is the disk D. We need explicit knowledge on the conformal maps from D to D.

Definition. Let $D \subset \mathbb{R}^2$ be the open unit disk. We call

$$\operatorname{Aut}(D) := \Big\{\omega(z) = \omega_{a,\varphi}(z) := e^{i\varphi}\frac{a - z}{1 - \overline{a}z},\ \varphi \in \mathbb{R},\ a \in D\Big\}$$

the *conformal automorphisms* of the unit disk or the group of *Möbius transformations*.

We leave it as an exercise to check $\operatorname{Aut}(D)$ is closed under composition. Note that the identity is $\omega_{0,\pi}(z) = z$, that $\omega_{0,\varphi+\pi}$ is rotation by an angle φ, and $\omega_{a,0}(0) = a$, $\omega_{0,0}(a) = 0$. It is also straightforward to calculate the inverse. We substantiate the remaining claims contained in the definition:

Lemma 14. *Any $\omega \in \operatorname{Aut}(D)$ maps D to D conformally, and ∂D to ∂D.*

Proof. First $|\overline{a}z| \le |a||z| < 1$ implies that the denominator of ω does not vanish on D.

Next, the product rule $\overline{zw} = \overline{z}\,\overline{w}$ gives

$$|\omega(z)|^2 = \omega(z)\overline{\omega(z)} = \frac{(a - z)(\overline{a} - \overline{z})}{(1 - \overline{a}z)(1 - a\overline{z})} = \frac{|a|^2 + |z|^2 - a\overline{z} - \overline{a}z}{1 + |a|^2|z|^2 - a\overline{z} - \overline{a}z}.$$

For $|z| = 1$ numerator and denominator agree so that $|\omega(z)| = 1$. For $z \in D$ we have $0 < (1 - |a|^2)(1 - |z|^2)$ so that $|a|^2 + |z|^2 < 1 + |a|^2|z|^2$. This shows that the (real) numerator is less than the (real) denominator. Thus $\omega(z) \in D$. Standard differentiation rules show that ω is holomorphic, and so conformality follows from Prop. 11. □

We can calculate

$$\omega'(z) = e^{i\varphi}\frac{-(1 - \overline{a}z) + (a - z)\overline{a}}{(1 - \overline{a}z)^2} = e^{i\varphi}\frac{|a|^2 - 1}{(1 - \overline{a}z)^2} \ne 0 \qquad \text{for all } z \in \overline{D}.$$

Hence ω is a diffeomorphism onto its image. Moreover, since ω is invertible (check!) we see that $\omega\colon D \to D$ is a diffeomorphism, and likewise its extension $\omega\colon \overline{D} \to \overline{D}$.

Remark. Any conformal map from D to D is in the group $\operatorname{Aut}(D)$. This follows with the help of a lemma by Schwarz, see [Sy1], pp. 262–264.

2.6. **Poisson's formula.** The *Dirichlet problem* for a domain U is to find a solution $u \in C^2(U, \mathbb{R}) \cap C^0(\overline{U}, \mathbb{R})$ of

$$\Delta u = 0 \text{ in } U \qquad \text{with } u|_{\partial U} = \varphi,$$

where φ is prescribed. For U bounded we proved the uniqueness of solutions to this problem in Prop. 29. In the special case that U is a ball, we now give an existence statement.

Note that the Dirichlet problem has fixed prescribed boundary values while the Plateau problem has free boundary parameterization. Since the Dirichlet problem does not present this extra freedom, it is easier to solve than the Plateau problem.

Theorem 15. *Let φ be a continuous function on ∂B_R where $B_R = B_R(0)$. Then*

$$u(x) := \begin{cases} \dfrac{R^2-|x|^2}{n\omega_n R}\displaystyle\int_{\partial B_R}\frac{\varphi(y)}{|x-y|^n}\,dS_y & \text{for } x\in B_R,\\ \varphi(x) & \text{for } x\in\partial B_R,\end{cases} \tag{13}$$

belongs to $C^2(B_R,\mathbb{R})\cap C^0(\overline{B}_R,\mathbb{R})$ and is harmonic in B_R.

It is convenient to define the *Poisson kernel*

$$K(x,y) := \frac{R^2-|x|^2}{n\omega_n R|x-y|^n}, \qquad x\in B_R, \quad y\in\partial B_R;$$

the Poisson integral then becomes the *convolution [Faltung]* $u(x) = \int_{\partial B} K(x,y)\varphi(y)\,dS_y$.

Proof. (i) Since φ is independent of x, and K is differentiable in $x\in B_R$, it follows that $u(x)$ is differentiable infinitely often on B_R, in particular $u\in C^2(B_R,\mathbb{R})$.

(ii) For $u\in C^2(B_R,\mathbb{R})$ we claim

If $u\in C^0(\overline{B}_R,\mathbb{R})$ and $\Delta u = 0$ on B_R then $u(x) = \displaystyle\int_{\partial B} K(x,y)\varphi(y)\,dS_y$ for all $x\in B_R$.

Lateron we will only need the case $n=2$ and $R=1$. We deal with this case only and derive the Poisson integral from the mean value formula. Consider for $|x|<1$

$$\omega_x(z) := \frac{x-z}{1-\overline{x}z} \quad \in \mathrm{Aut}(D).$$

Since $u\in C^2(D)$, Lemma 12 gives that the function $u\circ\omega_x$ is harmonic. In particular, it has the mean value property (6) on B_r for all $0<r<1$:

$$u(x) = u\big(\omega_x(0)\big) = \frac{1}{2\pi}\int_{\partial B_r}(u\circ\omega_x)(z)dS_z$$

But u is continuous on $\overline{B}_1$ with boundary values φ. Hence as the limit $r\to 1$ we obtain

$$u(x) = \frac{1}{2\pi}\int_{\partial B_1}(\varphi\circ\omega_x)(z)dS_z.$$

Let us now change variables of the curve integral. Note first that ω_x is its own inverse:

$$\omega_x\big(\omega_x(z)\big) = \frac{x-\frac{x-z}{1-\overline{x}z}}{1-\overline{x}\frac{x-z}{1-\overline{x}z}} = \frac{x-x\overline{x}z-x+z}{1-\overline{x}z-\overline{x}x+\overline{x}z} = \frac{(1-|x|^2)z}{1-|x|^2} = z$$

`26. Lecture, Tuesday 31.1.06`

Thus

$$\varphi(x) = \frac{1}{2\pi}\int_{\partial B_1}(\varphi\circ\omega_x)(z)dS_z = \frac{1}{2\pi}\int_{\partial B_1}\varphi(y)|\omega_x'(y)|dS_y.$$

But

$$\omega_x'(y) = \frac{-(1-\overline{x}y)+(x-y)\overline{x}}{(1-\overline{x}y)^2} = \frac{|x|^2-1}{(1-\overline{x}y)^2}.$$

Now for $y\in\mathbb{S}^1$ we have

$$|1-\overline{x}y|^2 = (1-\overline{x}y)(1-x\overline{y}) = 1-\overline{x}y-x\overline{y}+|x|^2 = (y-x)(\overline{y}-\overline{x}) = |y-x|^2$$

and so

$$|\omega_x'(y)| = \frac{1-|x|^2}{|y-x|^2}.$$

This proves step (ii):

$$u(x) = \frac{1}{2\pi}\int_{\partial B_1}\varphi(y)|\omega_x'(y)|dS_y = \frac{1}{2\pi}\int_{\partial B_1}\varphi(y)\frac{1-|x|^2}{|y-x|^2}dS_y$$

(iii) We claim K has the properties of a convolution kernel, namely

$$\int_{\partial B} K(x,y)dS_y = 1 \qquad\text{and}\qquad \Delta K(x,y) = 0 \ \text{ for } x\in B_R,\ y\in\partial B_R, \tag{14}$$

where $\Delta = \sum\frac{\partial^2}{\partial x_i\partial x_i}$. Once again, we provide the proof only for $n=2$. Apply step (ii) to the harmonic function $u\equiv 1$; the result is the first assertion. To prove the second, consider the following expansion of the Poisson kernel:

$$2\pi RK(x,y) = \frac{R^2-|x|^2}{|y-x|^2} = \frac{y\overline{y}-y\overline{x}+y\overline{x}-x\overline{x}}{(y-x)(\overline{y}-\overline{x})} = \frac{y}{y-x}+\frac{\overline{x}}{\overline{y}-\overline{x}} \tag{15}$$

The right hand side is the sum of the holomorphic function $x\mapsto\frac{y}{y-x}$ and the antiholomorphic function $x\mapsto\frac{\overline{x}}{\overline{y}-\overline{x}}$. Each of these terms is harmonic, hence the sum $x\mapsto 2\pi RK(x,y)$ is harmonic as well.

(iii) We show that $u(x)$ is harmonic. This is straightforward from (14):

$$\Delta u(x) = \int_{\partial B}\big(\Delta K(x,y)\big)\varphi(y)\,dS_y = 0.$$

(iv) It remains to show that u is continuous at ∂B. Let $x_0\in\partial B$ and $\varepsilon>0$ be arbitrary. Since φ is continuous, there is $\delta>0$ such that $|\varphi(x)-\varphi(x_0)|<\varepsilon$ for $|x-x_0|<\delta$; moreover

$|\varphi| \le M$ on ∂B. Then for $|x - x_0| < \delta/2$ we have

$$\begin{aligned}|u(x) - u(x_0)| &= \Big| \int_{\partial B} K(x,y)\big(\varphi(y) - \varphi(x_0)\big)\, dS_y \Big| \\ &\le \int_{y \in \partial B, |y-x_0| \le \delta} K(x,y)\big|\varphi(y) - \varphi(x_0)\big|\, dS_y + \int_{y \in \partial B, |y-x_0|>\delta} K(x,y)\big|\varphi(y) - \varphi(x_0)\big|\, dS_y \\ &\le \varepsilon + nR^{n-2}\frac{(R^2 - |x|^2)}{(\delta/2)^n} 2M,\end{aligned}$$

where for the first summand we used that K has integral 1. We can now require that $|x - x_0|$ is sufficiently small for the second summand to become less than ε. Thus u is continuous at x_0. $\square$

Remarks. 1. When $x = 0$, the Poisson integral reduces to

$$u(0) = \frac{R^2}{n\omega_n R}\int_{\partial B(0)} \frac{\varphi(y)}{|y|^n} dS_y = \frac{1}{n\omega_n R^{n-1}}\int_{\partial B(0)} \varphi(y)\, dS_y,$$

which is precisely the mean value formula. In general, the Poisson can be considered an uncentered version of the mean value formula; consider the kernel $K(x, y)$ a weight on the boundary distribution.

2. Does the Poisson formula extend to more general domains U than balls to give a solution of the Dirichlet problem on U? Only for very special domains, like annuli or half-spaces, similarly explicit formulas are known. The standard approach to the general case, the so-called Perron process, makes use of Poisson's formula (see [GT], Par.2.8).

3. By the maximum principle (or claim (ii)), the Poisson integral represents the unique harmonic function with boundary values φ.

4. In the 19th century, integral kernel representation formulas were seeked and obtained for many partial differential equations. The modern point of view, which prevailed in the 20th century is to establish solutions in terms of more abstract existence theorems. We will see an example in the proof of the Plateau problem, which establishes existence of a minimal surface without representing it by an explicit formula.

5. For the complete proof in dimension $n > 2$ see [J], p.86–89, or [GT], Sect. 2.4 and 2.5.

2.7. **Analyticity of harmonic maps and minimal surfaces.** Cauchy's integral formula expresses a holomorphic function as an integral. One of the virtues of this formula is that it shows that holomorphic functions are differentiable infinitely often. We derive the same fact for harmonic functions from the Poisson formula.

Theorem 16. *Let $u \in C^2(U, \mathbb{R})$ be harmonic. Then $u \in C^\infty(U, \mathbb{R})$ and in fact u is real analytic.*

Note that a function is called *(real) analytic*, if each point of the domain has a neighbourhood where the function can be represented as a power series. One example of smooth functions which are not analytic are the bump functions used for partitions of unity.

Proof. Since U is open, there is a ball $B = B_R(x)$ whose closure is contained in U. In B_R, the function u is represented by the Poisson integral (13) in terms of its values on ∂B_R. As in part (i) of the proof of the Poisson formula, the fact that $u \in C^\infty(B_R, \mathbb{R})$ follows directly from differentiating the Poisson integral.

We want to show that u can be represented by a power series in the ball $B_R(x)$ which, after translation, becomes $B_R = B_R(0)$. Using a geometric series expansion for (15), we find

$$\begin{aligned}2\pi R K(x,y) &= \frac{y}{y-x} + \frac{\overline{x}}{\overline{y} - \overline{x}} = \frac{1}{1 - \frac{x}{y}} + \frac{\overline{x}}{\overline{y}}\frac{1}{1 - \frac{\overline{x}}{\overline{y}}} \\ &= \sum_{k=0}^\infty \Big(\frac{x}{y}\Big)^k + \sum_{k=1}^\infty \Big(\frac{\overline{x}}{\overline{y}}\Big)^k = 1 + 2\,\mathrm{Re}\Big(\sum_{k=1}^\infty y^{-k}x^k\Big);\end{aligned}$$

this is valid for $|x| < |y| = R$. Consequently,

$$\begin{aligned}u(x) &= \int_{\partial B_R} K(x,y)\varphi(y)\, dS_y = \frac{1}{2\pi R}\int_{\partial B_R}\Big(1 + 2\,\mathrm{Re}\Big(\sum_{k=1}^\infty y^{-k}x^k\Big)\Big)\varphi(y)\, dS_y \\ &= \frac{1}{2\pi R}\int_{\partial B_R}\varphi(y)\, dS_y + \frac{1}{\pi R}\,\mathrm{Re}\sum_{k=1}^\infty \Big(\int_{\partial B_R}\frac{\varphi(y)}{y^k}\, dS_y\Big)x^k.\end{aligned}$$

This is indeed a power series for $u(x)$: Note that x^k is a polynomial of order k in $\mathrm{Re}\, x$ and $\mathrm{Im}\, x$, and so is $\mathrm{Re}(x^k)$. Hence with suitable coefficients a_{ij} we can write $u(x)$ in the form $\sum_{k=0}^\infty \sum_{i+j=k} a_{ij}(\mathrm{Re}\, x)^i(\mathrm{Im}\, x)^j$. $\square$

Let us draw two conclusions for minimal surfaces, independently of the Plateau problem:

Corollary 17. *A minimal surface $f : U^2 \to \mathbb{R}^n$ in conformal parameterization is real analytic.*

Proof. Since $\Delta f = 0$ holds, each component is real analytic by the previous theorem. Thus f is real analytic. $\square$

The Corollary means that if we force a soap film to change somewhere then it will change everywhere – changes cannot be local, but must always be global. More precisely: If two complete (connected) minimal surfaces agree on a nonempty open set, then they agree everywhere.

References. Many books on elliptic partial differential equations contain the proof for the n-dimensional Poisson formula, see e.g. [GT], Sect. 2.4 and 2.5.

`27. Lecture, Wednesday 1.2.06`

3. SOLUTION TO PLATEAU'S PROBLEM

We want to solve Plateau's problem by a series of statements. As formulated in Thm. 4 we seek a solution in the class $\mathcal{C}$ from (1).

Let $(f_k) \in \mathcal{C}$ be a *minimizing sequence*, that is,

$$E(f_k) \to \inf\{E(f), f \in \mathcal{C}\} \qquad \text{as } k \to \infty.$$

We want to take a limit $f = \lim f_k$; we call $f \in \mathcal{C}$ with $E(f) = \inf\{E(h), h \in \mathcal{C}\}$ a *minimizer*.

There are two different tasks: First, we need to make sure that f_k converges at all. In order to achieve convergence we will alter the minimizing sequence. Second, once convergence is obvious, we must prove that the limiting map represents a surface in $\mathcal{C}$, that is, a minimal surface whose boundary attains the desired Jordan curve Γ topologically.

3.1. The three-point-condition.

The first problem we face is that a minimizing sequence need not converge. For an example, consider the conformal automorphisms

$$h_k(z) := \frac{z - \frac{1-k}{k}}{1 - \frac{1-k}{k} z} = \frac{zk - 1 + k}{k - z + kz} = \frac{k(z+1) - 1}{k(z+1) - z} \quad \in \mathrm{Aut}(D).$$

For $k \to \infty$ we have $h_k(z) \to 1$ for $z \in \overline{D} \setminus 1$, while $h_k(-1) = -1$. Since automorphisms are bijective, $E(h_k) = A(h_k) = |D| = \pi$. Let us extend h_k to a surface in $\mathbb{R}^3$ with boundary $\Gamma = \mathbb{S}^1$. Then h_k does not converge to the minimum of area, but instead to a constant. What fails with the example is that the parameterization degrades as $k \to \infty$. Spelled out more formally, we can say that the 3-dimensional group $\mathrm{Aut}(D)$ is non-compact and contains Cauchy sequences converging to functions not in $\mathrm{Aut}(D)$. Indeed, we observe that for h_k our parameter $a = \frac{1-k}{k}$ converge to the non-admissable value $a = -1 \notin D$.

It goes without saying that the same problem will arise if we compose any well behaved minimizing sequence f_k with our conformal maps h_k. Conversely, given an arbitrary minimzing sequence f_k, we will now choose conformal automorphisms in a way as to secure convergence.

Proposition 18. *For any two sets of triples $0 \le \vartheta_1, \vartheta_2, \vartheta_3 < 2\pi$ and $0 \le \alpha_1, \alpha_2, \alpha_3 < 2\pi$ there exists a unique Möbius transformation $\omega \in \mathrm{Aut}(D)$ of the unit disk such that*

$$\omega\big(e^{i\vartheta_j}\big) = e^{i\alpha_j} \qquad \text{for } j = 1, 2, 3.$$

The proof becomes simpler if we work with a model different from the disk D, namely the upper half plane $H = \{(x, y) \in \mathbb{C}, y > 0\}$. In the following sense, D and H are conformally equivalent:

Lemma 19. *The map*

$$\eta\colon \mathbb{C} \setminus \{-i\} \to \mathbb{C} \setminus \{1\}, \qquad \eta(z) := \frac{z - i}{z + i}.$$

is a conformal diffeomorphism. It maps bijectively the following sets onto oneanother: (i) H to D, (ii) $\overline{H}$ to $\overline{D} \setminus \{1\}$, and (iii) the real axis to the unit circle without the point 1.

Proof. The inverse w of η,

$$w = \eta(z) = \frac{z - i}{z + i} \;\Leftrightarrow\; zw + iw = z - i \;\Leftrightarrow\; i(1+w) = z(1-w) \;\Leftrightarrow\; z = i\frac{1+w}{1-w} =: \zeta(w),$$

is defined for all $w \neq 1$. We conclude $\eta \circ \zeta = \mathrm{id}_D$ and η is bijective. Moreover, since w and η are differentiable, η is a diffeomorphism.

Let us now show $\eta(H) \subset D$. This follows from $|z - i| < |z + i|$ for $z \in H$, which is obvious geometrically. Indeed, $\mathrm{Re}(z - i) = \mathrm{Re}(z + i)$ while $\mathrm{Im}(z - i) < \mathrm{Im}(z + i)$ for $z \in H$. On the other hand, for $w \in D$

$$\begin{aligned} 2\,\mathrm{Im}\big(\zeta(w)\big) &= \frac{1}{i}\big(\zeta(w) - \overline{\zeta(w)}\big) = \frac{1+w}{1-w} + \frac{1+\overline{w}}{1-\overline{w}} = \frac{(1+w)(1-\overline{w}) + (1+\overline{w})(1-w)}{(1-w)(1-\overline{w})} \\ &= \frac{2 - 2|w|^2}{|1-w|^2} > 0. \end{aligned}$$

This gives (i).

Similarly (ii) follows from $\eta(\overline{H}) \subset \overline{D}$, and $w \in D \setminus \{1\}$ implies $\mathrm{Im}\big(\zeta(w)\big) \ge 0$, so that $\zeta\colon D \setminus \{1\} \to \overline{H}$. Likewise for (iii). $\square$

The complex plane can be compactified with just one point ∞. A sequence z_k is said to *converge to* ∞, if $|z_k| \to \infty$. For example, the complex sequence $k(-1)^k$ converges to ∞ (!). A function $f\colon \mathbb{C} \cup \infty$ is called differentiable at ∞ if $f(1/z)$ is differentiable at $z = 0$. A good model of the compactification of $\mathbb{C}$ is the 2-sphere $\mathbb{S}^2$: By stereographic projection, $\mathbb{C}$ corresponds to $\mathbb{S}^2$ minus a pole (even conformally). Using the compactification, we can summarize our previous lemma by saying that $\eta\colon \overline{H} \cup \infty \to \overline{D}$ is a diffeomorphism.

An analogue of Proposition 18 for H reads:

Lemma 20. *For every triple of real numbers $a < b < c$ there is a conformal diffeomorphism*

$$f\colon \overline{H} \cup \{\infty\} \to \overline{H} \cup \{\infty\},$$

such that $f(a) = 0$, $f(b) = 1$, $f(c) = \infty$.

Proof. Let
$$f\colon H \to H, \qquad f(z) := \frac{z-a}{z-c}\cdot\frac{b-c}{b-a}.$$
Then f is holomorphic and thus conformal. Moreover, f maps H to H since $\frac{b-c}{b-a} < 0$ and
$$2i\,\mathrm{Im}\,\frac{z-a}{z-c} = \frac{z-a}{z-c} - \frac{\overline{z}-a}{\overline{z}-c} = \frac{(z-a)(\overline{z}-c)-(\overline{z}-a)(z-c)}{|z-c|^2} = \frac{(a-c)(z-\overline{z})}{|z-c|^2}$$
$$= 2\,\mathrm{Im}\,z\frac{a-c}{|z-c|^2} < 0.$$
On $\partial H \cup \infty$ we have equality, and so f maps the extended real line to itself; the particular values are immediate. To see f is invertible, compute $f^{-1}(w) = \frac{a-wb}{1-w}$. □

Via the detour to H we can now prove the proposition:

Proof of the proposition. Set $a := \eta^{-1}(e^{i\vartheta_1})$, $b := \eta^{-1}\big(e^{i\vartheta_2}\big)$, $c := \eta^{-1}\big(e^{i\vartheta_3}\big)$. Then, the mapping $f\circ\eta^{-1}$, where f was constructed in the previous lemma, maps the triple of points $e^{i\vartheta_1}$, $e^{i\vartheta_2}$, $e^{i\vartheta_3}$ to the triple 0, 1, ∞ in $\partial H \cup \infty$.

Since the mappings are diffeomorphisms, any triple $e^{i\vartheta_1}$, $e^{i\vartheta_2}$, $e^{i\vartheta_3}$ can be mapped conformally to any other triple $e^{i\alpha_1}$, $e^{i\alpha_2}$, $e^{i\alpha_3}$. □

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We can now control the boundary values of a minimizing sequence:

Theorem 21. *Let $(f_k) \in \mathcal{C}$ be a minimizing sequence, $p_1, p_2, p_3 \in \Gamma$ be pairwise distinct points, and $0 \le \vartheta_1, \vartheta_2, \vartheta_3 < 2\pi$. Then each f_k can be reparameterized without changing its energy to give a minimizing sequence $(\tilde f_k) \in \mathcal{C}$, which satisfies the* 3-point-condition

(16) $$\tilde f_k(e^{i\vartheta_j}) = p_j, \qquad \text{for } j = 1,2,3 \text{ and all } k \in \mathbb{N}.$$

We could fix ϑ_j once and for all to be third roots of unity.

Proof. Since $f_k\colon \mathbb{S}^1 \to \Gamma$ is bijective there exist $\alpha_1, \alpha_2, \alpha_3$ distinct such that $f_k(e^{i\alpha_j}) = p_j$, $j = 1,2,3$. By composing f_k with a rotation and a reflection if necessary, we may assume $0 \le \alpha_1 < \alpha_2 < \alpha_3 < 2\pi$; this leaves $E(f_k)$ unchanged.

By Proposition 18 there is a Möbius transformation ω_k with $\omega_k(e^{i\vartheta_j}) = e^{i\alpha_j}$ for $j = 1,2,3$. Then $\tilde f_k := f_k \circ \omega_k$ satisfies the 3-point-condition. Moreover, ω_k is conformal and hence $E(f_k\circ\omega_k) = E(f_k)$ by Lemma 13. In particular, the sequence $\tilde f_k = f_k \circ \omega_n$ is again minimizing. □

3.2. **Courant-Lebesgue-Lemma and equicontinuity.** As an intermediate step to establishing the convergence of a minimizing sequence, we will show in the present section that a minimizing sequence which satisfies the 3-point-condition has equicontinuous boundary values. We show first:

Lemma 22 (Courant-Lebesgue)**.** *Let $f \in C^1(D,\mathbb{R}^n) \cap C^0(\overline{D},\mathbb{R}^n)$ with $E(f) \le C$. Pick $0 < \delta < 1$ and $z_0 \in \overline{D}$. Then there exists $r = r(z_0, f) \in (\delta, \sqrt{\delta})$ such that for all $z_1, z_2 \in \overline{D}$ with $|z_1 - z_0| = r = |z_2 - z_0|$*
$$|f(z_1) - f(z_2)|^2 \le \frac{8\pi C}{\log(1/\delta)}.$$

Proof. Take polar coordinates (ρ,φ) centered at z_0. Integrating over $S_r = S_r(z_0) = \{z \in \mathbb{C}, |z - z_0| = r\}$, and using Hölder's inequality we get, on the one hand,

(17) $$|f(z_1) - f(z_2)|^2 \le \left(\int_{S_r\cap D} |f_\varphi(r,\varphi)|\,d\varphi\right)^2 \le 2\pi \int_{S_r\cap D} |f_\varphi(r,\varphi)|^2\,d\varphi.$$

The Dirichlet integral expressed in polar coordinates is
$$E(f) = \frac12 \int \Big(|f_\rho|^2 + \frac{1}{\rho^2}|f_\varphi|^2\Big)\rho\,d\varphi d\rho.$$

Therefore, the generalized mean value theorem of integration gives for some $r \in (\delta, \sqrt{\delta})$, on the other hand,

(18) $$2E(f) \ge \int_\delta^{\sqrt\delta} \frac1\rho \Big(\int_{S_\rho\cap D} |f_\varphi|^2 d\varphi\Big) d\rho = \int_\delta^{\sqrt\delta} \frac1\rho d\rho \int_{S_r\cap D} |f_\varphi|^2 d\varphi.$$

Combining (17) with (18) we obtain the desired inequality
$$|f(z_1) - f(z_2)|^2 \le \frac{4\pi E(f)}{\log\sqrt\delta - \log\delta} \le \frac{4\pi C}{\log\frac{1}{\sqrt\delta}} = \frac{8\pi C}{\log\frac1\delta}.$$
□

Remember that equicontinuity of a sequence of functions (f_k) means that δ in the definition of continuity does not depend on the index k.

Theorem 23. *A minimizing sequence $(f_k) \in \mathcal{C}$ which satisfies the 3-point condition* (16) *has equicontinuous boundary restriction $\varphi_k := f_k|_{\partial D}$.*

Proof. Let $\varepsilon > 0$. We need to show that there exists $\delta > 0$ such that for all $z_0 \in \partial D$ and $z \in \partial D$ with $|z - z_0| < \delta$

(19) $$\big|\varphi_k(z) - \varphi_k(z_0)\big| < \varepsilon \qquad \text{for all } k \in \mathbb{N}.$$

Let us first state a fact for our given Jordan curve Γ. We claim there is $0 < \lambda = \lambda(\varepsilon, \Gamma) \le \varepsilon$ with the following property: Any pair of points $q_1, q_2 \in \Gamma$ with ambient distance $|q_1 - q_2| < \lambda$ decomposes Γ into two subarcs Γ_1, Γ_2, such that one of them, say Γ_1, has diameter less than ε. This means that $|x - y| < \varepsilon$ for all $x, y \in \Gamma_1$. We suppose that ε is chosen less than all pairwise distances of the image triple p_1, p_2, p_3 of the 3-point-condition. Then the arc $\overline{\Gamma}_1$ will contain at most one of these points.

By Lemma 3 the infimum of energy $\inf\{E(f), f \in \mathcal{C}\}$ is finite. Thus for (f_k) minimizing $E(f)$ in $\mathcal{C}$ we can assume that $E(f_k) \le C$ uniformly where, for instance, C is twice the infimal energy. As $\log \frac{1}{\delta} \to \infty$ for $\delta \to 0$, we can choose $\delta = \delta(\varepsilon) \in (0, 1)$ small enough to ensure

$$\frac{8\pi C}{\log(1/\delta)} < \lambda^2(\varepsilon). \tag{20}$$

We now apply the Courant-Lebesgue-Lemma. For each $z_0 \in \partial D$ and for each $k \in \mathbb{N}$ it yields a radius $r = r(z_0, \delta, k) \in (\delta, \sqrt{\delta})$ such that the two points $z_1 \neq z_2$ on ∂D in distance r from z_0 satisfy

$$\left|\varphi_k(z_1) - \varphi_k(z_2)\right| < \lambda \le \varepsilon. \tag{21}$$

The points z_1, z_2 decompose $\partial D = \mathbb{S}^1$ into the two semiarcs

$$S_1 = S_1(z_0, \delta, k) := \{z \in \partial D, |z - z_0| \le r\}, \qquad S_2 := \partial D \setminus S_1.$$

By decreasing δ (and hence r) if necessary, we can assume $\sqrt{\delta}$ is less than all pairwise distances of the triple $\vartheta_1, \vartheta_2, \vartheta_3$ from the 3-point condition (16). This ensures that S_1 contains at most one of the points $e^{i\vartheta_j}$.

We would like to say that the entire arc S_1 (not only its endpoints z_1, z_2) maps under φ_k into some ε-ball about z_0. By (21), we can apply the above fact about Jordan curves to the pair of points $q_i := \varphi_k(z_i)$, $i = 1, 2$. This gives us two subarcs $\Gamma_1, \Gamma_2(z_0, k, \delta)$, from which Γ_1 is the short arc, in the sense that its diameter is less than ε.

We use the 3-point-condition to assert that the arc S_1 gets mapped to the short arc $\overline{\Gamma}_1$, not to the long arc $\overline{\Gamma}_2$. Indeed, the arc S_1 contains at most one point of the triple, and so does its image under the bijective map φ_k. Thus S_1 cannot be mapped to Γ_2 which contains at least two points of the triple.

We conclude that $\varphi_k(S_1) = \overline{\Gamma}_1$ has diameter at most ε. In particular $|\varphi_k(z) - \varphi_k(z_0)| < \varepsilon$ for all $z \in S_1$. On the first sight this statement seems not uniform in k, as our arc S_1 depends on r and hence on k. But always $r(z_0, k, \delta) > \delta$, where the right hand side depends on ε alone. Thus for all k the arc $S_1(z_0, \delta, k)$ contains all points $z \in \partial D$ with $|z - z_0| < \delta$, and in particular (19) holds, as desired. $\square$

`29. Lecture, Wednesday 8.2.06`

3.3. **Construction of a minimizer.** The present section contains the core of the proof of Plateau's problem.

Let us illustrate the problem we face. Clearly, we want our minimizing sequence to have a limit in C^1, so that the area of the limit can be computed. However, we cannot expect a uniformly convergent sequence of C^1 functions to converge to a function which is again differentiable – what would an example for the case of functions from $\mathbb{R}$ to $\mathbb{R}$ be? To overcome this problem, the key idea of the proof is to replace an arbitrary minimizing sequence with harmonic maps. Then, as asserted in Thm. 10, the limit exists and is differentiable in the interior. The Poisson formula allows us to replace any continuous map with a harmonic map which has the same boundary values.

To state the theorem, we must also deal with a minor technical problem. When passing to a limit, the injectivity of our boundary parameterization can be lost. (The same is well-known for convergent sequences of strictly monotone functions: their limit in general is merely monotone.) To allow for such a behaviour, we will preliminarily establish the solution first in a more general class than $\mathcal{C}$. We call a map $\varphi\colon \mathbb{S}^1 \to \Gamma$ *monotone* if φ is continuous and surjective, and for each point in Γ the preimage is a closed connected arc (including the particular case of a point). We define

$$\mathcal{C}' := \{\varphi \in C^1(D, \mathbb{R}^n) \cap C^0(\overline{D}, \mathbb{R}^n), \varphi \text{ maps } \partial D \text{ monotonically to } \Gamma\}.$$

Theorem 24. *There is a minimizing sequence of harmonic maps $h_k \in \mathcal{C}$ which converges uniformly to a harmonic map $h \in \mathcal{C}'$ with $E(h) = \inf\{E(f), f \in \mathcal{C}\}$.*

Proof. We consider an arbitrary minimizing sequence (f_k). By Thm. 21, we can assume each f_k satisfies the 3-point-condition. Thus Thm. 23 gives that the sequence (f_k) has equicontinuous boundary restriction $\varphi_k = f_k|_{\partial D}$. By the Arzelà-Ascoli Theorem 9, the boundary restrictions converge to a monotone limit $\varphi\colon \partial D \to \mathbb{R}^n$.

Let us now consider the Dirichlet problem for the disk

$$\Delta h_k(x) = 0 \text{ for } x \in D, \qquad h_k(x) = \varphi_k(x) \text{ for } x \in \partial D.$$

Applying the Poisson integral formula (13) componentwise, we obtain solutions $h_k \in C^2(D, \mathbb{R}^n) \cap C^0(\overline{D}, \mathbb{R}^n)$ which are in the class $\mathcal{C}$.

The maps h_k are harmonic. Dirichlet's principle, Prop. 6, implies that h_k has minimal energy among maps with the same boundary values,

$$E(h_k) \le E(f_k) \to \inf\{E(f), f \in \mathcal{C}\}.$$

In particular, (h_k) is once again a minimizing sequence.

The weak maximum and minimum principle for harmonic functions, Prop. 1.28, implies a maximum principle for the modulus of harmonic functions (see Problems) and then a maximum principle for the modulus of harmonic maps. This proves

$$\sup_D |h_k - h_\ell| \le \sup_{\partial D} |\varphi_k - \varphi_\ell| \to 0 \qquad \text{as } k, \ell \to \infty.$$

Hence $h_k\colon \overline{D} \to \mathbb{R}^n$ converges uniformly and has a continuous limit $h\colon \overline{D} \to \mathbb{R}^n$. In particular, $h|_{\partial D} = \varphi$.

But the compactness result, Theorem 10, proves that the limiting map h is harmonic in D (invoke Theorem 16 to see the h_k are smooth). □

It remains to show that the limiting map h is conformal and assumes the boundary condition.

3.4. **The minimizer is conformal.** If we minimize the Dirichlet energy for fixed boundary, then we obtain just a harmonic map (Dirichlet's principle, Prop. 6). The map h which will solve the Plateau problem, is a minimizer of the Dirichlet energy among maps which are subject to the free boundary condition of parameterizing Γ injectively. The latter class is larger and so we get some additional property of the minimizer:

Theorem 25. *The limiting map h of Thm. 24 is conformal.*

To see this, we will compare h with maps which change the parameterization, including the boundary parameterization. To formulate this variationally, we compare h with maps $h \circ \sigma_t$, where $\sigma_t\colon \overline{D} \to \overline{D}$ is a family of diffeomorphisms of D where $\sigma_0 = \mathrm{id}$. Note that these variations act on the domain, not on the target. If σ_t is not constant on the boundary, the boundary parameterization gets changed.

Proposition 26. *Suppose $U \subset \mathbb{R}^2$ and $\sigma\colon (-\varepsilon, \varepsilon) \times \overline{U} \to \overline{U}$ is a smooth family of diffeomorphisms σ_t such that $\sigma_0 = \mathrm{id}_{\overline{U}}$. Then any $f \in C^1(U, \mathbb{R}^n)$ with finite energy has* first inner variation

$$\frac{d}{dt} E_U(f \circ \sigma_t)\Big|_{t=0} = \int_U \frac{1}{2}\big(\xi_x - \eta_y\big)\big(|f_x|^2 - |f_y|^2\big) + \big(\xi_y + \eta_x\big)\langle f_x, f_y\rangle \, dz, \tag{22}$$

where ξ, η are the components of the smooth variation vector field

$$X(z) = \big(\xi(z), \eta(z)\big) := \frac{d}{dt}\sigma_t(z)\Big|_{t=0}. \tag{23}$$

Given any vector field X on U there is in fact a flow $\sigma_t\colon U \to \mathbb{R}^2$ such that $\sigma_0 = \mathrm{id}$ and (23) hold. Indeed, for each point $x \in U$, the ordinary differential equation (23) can be solved.

Note that if f is conformal, then (22) vanishes. In order to show the converse, we will use that at the minimizer h of E, the inner variation (22) vanishes for arbitrary X.

Proof. For a change of variables we set $w = \sigma_t^{-1}(z)$. Then

$$E_U(f \circ \sigma_t) = \frac{1}{2}\int_U |d(f \circ \sigma_t)(w)|^2 \, dw \overset{\text{chain rule}}{=} \frac{1}{2}\int_U |df(\sigma_t(w))(d\sigma_t)(w)|^2 \, dw$$
$$\overset{\text{ch. of var.}}{=} \frac{1}{2}\int_{\sigma_t(U)=U} |df(z) d\sigma_t(\sigma_t^{-1}(z))|^2 \, |\det d\sigma_t^{-1}(z)| \, dz$$

Note that these integrals exist since $d\sigma_t$ and $d\sigma_t^{-1}$ are bounded on U.

Since σ and σ^{-1} has bounded t-derivatives on $\overline{U}$, we can differentiate under the integral sign with respect to t. To do so, we use a product rule of type $(g^2h)' = 2gg'h + g^2h'$ as well as $d\sigma_0 = \mathrm{id}$ to obtain

$$\frac{d}{dt}E_U(f \circ \sigma_t)\Big|_{t=0} = \int_U \Big\langle df(z), df(z)\frac{d}{dt}\big(d\sigma_t(\sigma_t^{-1}(z))\big)\Big|_{t=0}\Big\rangle + \frac{1}{2}|df(z)|^2 \frac{d}{dt}\det d\sigma_t^{-1}(z)\Big|_{t=0} \, dz.$$

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To calculate the t-derivatives of σ, let us apply the chain rule (think of $d\sigma_t(\sigma_t^{-1})$ as a function $d\sigma(t, v)$ where $v = \sigma_t^{-1}(z)$):

$$\frac{d}{dt}\big(d\sigma_t(\sigma_t^{-1}(z))\big)\Big|_{t=0} = \frac{d}{dt}d\sigma_t\Big|_{t=0}(z) + d^2\sigma_0 \frac{d}{dt}\Big[\sigma_t^{-1}(z)\Big]_{t=0} = d\Big[\frac{d}{dt}\sigma_t(z)\Big]_{t=0} = dX(z).$$

Here, we used that $\sigma_0 = \mathrm{id}$, so that its second partials $d^2\sigma_0$ vanish. Moreover, for the first term we used that partial derivatives and the t-derivative of σ commute; indeed,

$$\frac{d}{dt}d\sigma_t(z) \cdot v\Big|_{t=0} = \frac{d}{dt}\frac{d}{ds}\sigma_t(z + sv)\Big|_{s,t=0} = \frac{d}{ds}\frac{d}{dt}\sigma_t(z + sv)\Big|_{s,t=0}$$
$$= \frac{d}{ds}X(z + sv)\Big|_{s=0} = dX(z) \cdot v.$$

To calculate the derivative of the determinant, let us note that if $A(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$ and $A(0) = E$ then

$$(\det A)'(0) = (ad - bc)'(0) = (a'd + ad' - b'c - bc')(0) = (a' + d')(0) = \operatorname{trace} A'(0).$$

Thus the change of the volume element of $\sigma_t = \mathrm{id} + tX + O(t^2)$ is precisely the divergence of X:

$$\frac{d}{dt}\det(d\sigma_t(z))\Big|_{t=0} = \operatorname{trace}\frac{d}{dt}d\sigma_t\Big|_{t=0} = \operatorname{trace} d\Big[\frac{d}{dt}\sigma_t\Big]_{t=0} = \sum \frac{\partial}{\partial x_i}X^i = \operatorname{div} X.$$

Similarly, due to the identity $\sigma_t \circ \sigma_t^{-1} = \mathrm{id}$ we have the expansion $\sigma_t^{-1} = \mathrm{id} - tX + O(t^2)$. Thus $\frac{d}{dt}\sigma_t^{-1}|_{t=0} = -X$ and $\frac{d}{dt}\det(d\sigma_t^{-1}(z))|_{t=0} = -\operatorname{div} X$.

The result of these calculations is the first inner variation formula

$$\frac{d}{dt}E_U(f\circ\sigma_t)\Big|_{t=0}=\int_U\big\langle df(z),df(z)dX(z)\big\rangle-\frac{1}{2}|df(z)|^2\operatorname{div}X\;dz. \tag{24}$$

It holds regardless of dimension.

Let us now show (24) agress with (22) for dimension two. Writing out the matrices with column vectors, we see the first term of (24) gives

$$\begin{aligned}\big\langle df(z),\,df(z)dX(z)\big\rangle&=\Big\langle(f_x,f_y),\;(f_x,f_y)\begin{pmatrix}\xi_x&\xi_y\\\eta_x&\eta_y\end{pmatrix}\Big\rangle\\&=\Big\langle(f_x,f_y),\;\big(\xi_xf_x+\eta_xf_y,\;\xi_yf_x+\eta_yf_y\big)\Big\rangle=\xi_x|f_x|^2+\eta_x\langle f_x,f_y\rangle+\xi_y\langle f_x,f_y\rangle+\eta_y|f_y|^2,\end{aligned}$$

while the second contributes $-\frac{1}{2}(\xi_x+\eta_y)(|f_x|^2+|f_y|^2)$. Summing these two terms and integrating we obtain

$$\frac{d}{dt}E_U(f\circ\sigma_t)\Big|_{t=0}=\int_U\frac{\xi_x}{2}\big(|f_x|^2-|f_y|^2\big)+\xi_y\langle f_x,f_y\rangle+\eta_x\langle f_x,f_y\rangle-\frac{\eta_y}{2}\big(|f_x|^2-|f_y|^2\big)\;dz$$

which gives the desired result. □

Proof of the theorem. Let h be the harmonic map from Thm. 24. We set

$$\psi\colon D\to\mathbb{C},\qquad\psi:=\frac{1}{2}\big(|h_x|^2-|h_y|^2\big)-i\langle h_x,h_y\rangle \tag{25}$$

and want to show $\psi\equiv0$.

The map h minimizes the energy E in the class $\mathcal{C}'$. Thus for any smooth family of diffeomorphisms $\sigma_t\colon\overline{D}\to\overline{D}$, the map $t\mapsto E(h\circ\sigma_t)$ attains a minimum at $t=0$. In particular,

$$\begin{aligned}0=\frac{d}{dt}E_D(h\circ\sigma_t)\Big|_{t=0}&=\int_D\big(\xi_x-\eta_y\big)\operatorname{Re}\psi-\big(\xi_y+\eta_x\big)\operatorname{Im}\psi\;dz\\&=\operatorname{Re}\int_D\big(\xi_x-\eta_y+i(\xi_y+\eta_x)\big)\psi\,dz=2\operatorname{Re}\int_D(\partial_{\bar z}X)\psi\,dz,\end{aligned}$$

where we employed the shorthand notation

$$\partial_{\bar z}f:=\frac{1}{2}(\partial_xf+i\,\partial_yf)=\frac{1}{2}(\partial_x\operatorname{Re}f-\partial_y\operatorname{Im}f)+\frac{i}{2}(\partial_x\operatorname{Im}f+\partial_y\operatorname{Re}f) \tag{26}$$

for $f=\operatorname{Re}f+i\operatorname{Im}f$. Note that the relation $\partial_{\bar z}f=0$ is equivalent to the validity of the Cauchy-Riemann equations for f.

As a first step, we show ψ is holomorphic. The operator $\partial_{\bar z}f$ relates to a divergence,

$$2\operatorname{Re}\partial_{\bar z}f=\partial_x\operatorname{Re}f-\partial_y\operatorname{Im}f=\big(\partial_x\operatorname{Re}+\partial_y\operatorname{Im}\big)\overline{f}=\operatorname{div}\overline{f}.$$

Thus

$$\operatorname{div}(\overline{fg})=2\operatorname{Re}\partial_{\bar z}(fg)=2\operatorname{Re}\big((\partial_{\bar z}f)g+f\partial_{\bar z}g\big)$$

and so we can partially integrate as follows:

$$0=2\operatorname{Re}\int_D(\partial_{\bar z}X)\psi\,dz=\int_D\operatorname{div}(\overline{X\psi})\,dz-2\operatorname{Re}\int_DX\partial_{\bar z}\psi\,dz. \tag{27}$$

Now suppose X has compact support in D. Then clearly $\sigma_t(z):=z+tX$ is a diffeomorphism of $\overline{D}$ for small $|t|$, and so, by the divergence theorem, (27) reduces to $0=\operatorname{Re}\int_DX(\partial_{\bar z}\psi)\,dz$. Now we apply the fundamental lemma of the calculus of variations, Lemma 22. Choosing X purely real yields $\operatorname{Re}\partial_{\bar z}\psi(z)=0$ for all $z\in D$, while X purely imaginary yields $\operatorname{Im}\partial_{\bar z}\psi(z)=0$. This proves ψ is holomorphic on D.

In a second step, we use this fact to show $\psi\equiv0$. We apply the divergence theorem. Since ψ is defined on the open disk only, we work with disks D_r of smaller radius $r<1$. Using the identity $\langle\overline{w},z\rangle=\operatorname{Re}w\operatorname{Re}z-\operatorname{Im}w\operatorname{Im}z=\operatorname{Re}(wz)$ we find

$$\begin{aligned}0&\overset{(27),\,\partial_{\bar z}\psi=0}{=}\int_D\operatorname{div}(\overline{X\psi})\,dz=\lim_{r\nearrow1}\int_{D_r}\operatorname{div}(\overline{X\psi})\,dz\overset{\text{div.thm.}}{=}\lim_{r\nearrow1}\int_{\partial D_r}\Big\langle\overline{X\psi},\frac{z}{|z|}\Big\rangle\,ds\\&=\lim_{r\nearrow1}\int_{\partial D_r}\operatorname{Re}\Big(X\psi\,\frac{z}{|z|}\Big)\,ds=\lim_{r\nearrow1}\operatorname{Re}\int_{\partial D_r}X\frac{z}{|z|}\psi\,ds.\end{aligned} \tag{28}$$

Up to this point, perhaps, the proof appears not too inspiring, but now comes the beautiful turn. Let us consider a variation σ_t which rotates each z by an angle $t\alpha(z)$,

$$\sigma_t(z):=ze^{it\alpha(z)},$$

for $\alpha\colon\overline{D}\to\mathbb{R}$ to be specified later. Note that $\sigma_t\colon\overline{D}\to\overline{D}$ is a diffeomorphism for small $|t|$ such that $\sigma_0(z)=z$, and

$$X(z):=\frac{\partial}{\partial t}\sigma_t(z)\Big|_{t=0}=z\frac{d}{dt}e^{it\alpha(z)}\Big|_{t=0}=iz\alpha(z).$$

Inserting this into (28) and then introducing polar coordinates $z=re^{i\vartheta}$ we obtain

$$\begin{aligned}0&=-\lim_{r\nearrow1}\int_{\partial D_r}\operatorname{Re}\Big(X\frac{z}{|z|}\psi\Big)\,ds=\lim_{r\nearrow1}\int_{\partial D_r}\frac{\alpha(z)}{|z|}\operatorname{Im}\big(z^2\psi(z)\big)\,ds\\&=\lim_{r\nearrow1}\int_0^{2\pi}\frac{1}{r}\alpha(re^{i\vartheta})\operatorname{Im}\big((re^{i\vartheta})^2\psi(re^{i\vartheta})\big)r\,d\vartheta.\end{aligned} \tag{29}$$

The function $z^2\psi(z)$ is holomorphic on D and thus has a power series representation which converges on D,

$$z^2\psi(z)=\sum_{k=0}^\infty(a_k+ib_k)z^k=\sum_{k=0}^\infty(a_k+ib_k)(re^{i\vartheta})^k=\sum_{k=0}^\infty(a_k+ib_k)r^k\big(\cos(k\vartheta)+i\sin(k\vartheta)\big).$$

The left hand side vanishes at $z=0$ and so $a_0+ib_0=0$. The imaginary part is

$$\operatorname{Im}(z^2\psi(z))=\sum_{k=1}^\infty a_kr^k\sin(k\vartheta)+b_kr^k\cos(k\vartheta).$$

Let us recall the fact that the functions $\{\sin(k\vartheta), \cos(k\vartheta), k \in \mathbb{N}\}$ are orthogonal with respect to the inner product $\langle f, g\rangle := \int_0^{2\pi} f(\vartheta)g(\vartheta)\,d\vartheta$, but each function has a nonzero norm $\langle f, f\rangle$. Thus if for $k \in \mathbb{N}$ we choose $\alpha(z) = r^k \cos k\vartheta = \mathrm{Re}(z^k)$ then (29) gives $b_k = 0$, while for $\alpha(z) = r^k \sin k\vartheta = \mathrm{Im}(z^k)$ we obtain $a_k = 0$. This proves $z^2\psi(z) \equiv 0$, that is, $\psi(z) \equiv 0$ on D, as desired. □

Remark. Our proof indicates a close relationship between Fourier series and power series. Suppose $h\colon \overline{D} \to \mathbb{C}$ is holomorphic. Then $f(\vartheta) := h(e^{i\vartheta})$ is a continuous function $\mathbb{R} \to \mathbb{C}$, with period 2π. Restricting the power series expansion $h(z) = \sum_{k\in\mathbb{Z}} c_k z^k$ to the boundary we find $f(\vartheta) = \sum c_k e^{ik\vartheta}$, which gives us the Fourier series of f.

`31. Lecture, Wednesday 15.2.06`

3.5. **The minimizer is injective on the boundary.** Let us first consider harmonic functions once again.

Theorem 27 (Schwarz reflection principle)**.** *Let U^+ be an open domain of the halfspace $\{x_n > 0\}$ whose boundary contains an open subset T of the hyperplane $\{x_n = 0\}$. Suppose $u \in C^2(U^+, \mathbb{R}) \cap C^0(U^+ \cup T, \mathbb{R})$ is harmonic and $u|_T = 0$. Denote by $U^- := \{x \in \mathbb{R}^n, (x_1, \dots, -x_n) \in U^+\}$ the reflection of U^+ across T. Then the extension of u by odd reflection,*

$$\tilde{u}(x_1, \dots, x_n) := \begin{cases} u(x_1, \dots, x_n) & x_n \geq 0, \\ -u(x_1, \dots, -x_n) & x_n < 0, \end{cases}$$

is in $C^2(U^+ \cup T \cup U^-)$ and harmonic.

The harmonicity assumption creates the unexpected regularity along T: The function $u(x) := \sqrt[3]{x}$, is smooth on $(0, \infty)$, and extends continuously with $u(0) := 0$. Its odd extension $\tilde{u}(x) := \sqrt[3]{x}$ is, however, not differentiable at 0.

Proof. Let us denote reflection by $\sigma(x) := (x_1, \dots, -x_n)$.

It is easy to check by direct calculation that $\Delta\tilde{u}(x) = 0$ for all $x \in U^-$. Indeed,

$$\frac{\partial}{\partial x_i^2}\tilde{u}(x) = \frac{\partial}{\partial x_i^2}\big(-u(x_1, \dots, -x_n)\big) = -(\partial_{ii}u)\big(\sigma(x)\big)$$

for all i (including $i = n$), and so odd reflection preserves harmonicity,

$$\Delta\tilde{u}(x) = -(\Delta u)\big(\sigma(x)\big) = 0. \tag{30}$$

We now apply the Poisson formula in conjunction with the maximum principle to show that at $x_0 \in T$ the extension $\tilde{u}$ is smooth and has $\Delta\tilde{u}(x_0) = 0$. For some $r > 0$ we have $B_r(x_0) \subset U^+ \cup T \cup U^-$. We set $B_r := B_r(x_0)$ and $B_r^\pm := B_r \cap \{\pm x_n > 0\}$. Without loss

of generality, $x_0 = 0$. Set $\varphi\colon \partial B_r \to \mathbb{R}$ equal to $\tilde{u}$. Extend these boundary values by the Poisson formula to a harmonic function $h(x) := \frac{R^2 - |x|^2}{n\omega_n R^n}\int_{\partial B_r} \frac{\varphi(y)}{|x-y|^n}\,dS_y$. We claim h is odd, that is, the two functions $h(x)$ and $\tilde{h}(x) := -h(\sigma(x))$ agree on B_r. But h and $\tilde{h}$ are C^2, harmonic by (30), and have the same boundary values. Hence they agree by the maximum principle.

In particular, $h(x) = 0$ for all $x \in T \cap B_r$. But this means that the two functions $h(x)$ and $\tilde{u}(x)$ agree on ∂B_r^+. But both functions are harmonic. Hence once again they agree on the interior B_r^+. The same holds for B_r^-, while on T the functions agree anyway. But h is C^2 and harmonic on B_r, and so is $\tilde{u}$. □

Let us show that Schwarz reflection also applies to D. By Lemma 19, the map $\eta\colon \overline{H} \to \overline{D} \setminus \{1\}$ is a conformal diffeomorphism. Now suppose β is some arc contained in the unit circle ∂D and $u\colon D \to \mathbb{R}$ is harmonic with zero boundary values on β. By a rotation we can assume that $1 \notin \beta$. By Lemma 12, the map $v := u \circ \eta$ is harmonic; moreover, it has zero boundary values on the arc $\eta^{-1}(\beta)$ contained in the real axis. By standard Schwarz reflection v extends across this arc in an odd way to some harmonic mapping $\tilde{v}$. Writing $\tilde{u} := \tilde{v} \circ \eta^{-1}$ we obtain the desired extension accross β.

Let us apply this to the Plateau problem. Remember that in general in a limit, strict monotonicity may degrade to monotonicity. Nevertheless we can show that $h \in \mathcal{C}$:

Theorem 28. *The limiting map h of Thm. 24 has a restriction to ∂D which is injective and hence topological.*

Proof. To show that h is injective on ∂D assume on the contrary there is an open arc $\gamma \subset \partial D$ which maps to a point. For simplicity we assume this point is 0. By the Schwarz reflection principle we can reflect a neighbourhood of β in D across β to extend each component of h harmonically. The extended map is C^2 on an open set O which includes the arc β.

As asserted in Theorem 25 the map h is conformal on D. We claim that the extension by reflection is also conformal, including the arc β. Since h is C^1 along β, the conformality relations $|h_x| = |h_y|$ and $\langle h_x, h_y\rangle = 0$ extend continuously from D to β. Moreover, conformality is clearly preserved by reflection and thus holds on all of O.

Let us now derive the desired contradiction from conformality along β. If β is non-empty then, in polar coordinates, $|h_\varphi| = 0$ and, by conformality, $|h_r| = 0$ on β. Thus $h'(z) = 0$ on β. It follows that the power series of h' vanishes, and since h is analytic, it it equal to a constant on O and hence on all of D. But h is continuous on ∂D, thus also constant there. This contradicts the 3-point condition for h, and so β must be empty. □

The previous theorem, together with Theorems 24 and 25, completes the proof of the Plateau problem as stated in Theorem 4.

`Summer 06: 33. Lecture, Wednesday 19.4.06` ______________________

3.6. **Summary of the proof.** The proof employs a so-called direct method, meaning we minimize a functional by taking the limit of a minimizing sequence. It is hopeless to do this for area, so we work with energy instead.

Consider a sequence $f_k \in \mathcal{C}$ tending to the infimum of energy over the class $\mathcal{C}$. Suppose for a moment that it would be convergent to some mapping $f \in \mathcal{C}$. As a minimum of energy, f is harmonic. Moreover, due to the free boundary condition which constitutes the class $\mathcal{C}$, the mapping f is also conformal. Thus f is minimal (assuming $n = 3$).

In order to prove the desired convergence $f_k \to f$, we need to alter the f_k. Using Poisson's formula componentwise we may replace them with harmonic maps. Thereby we will not increase energy. The gain lies in the fact that a sequence of harmonic maps converges (with all derivatives) to some map f. But in general this does not yield $f \in \mathbb{C}$. To guarantee the boundary condition, we need to require the 3-point condition for each f_k.

4. Properties of the Plateau solution

4.1. **Riemann mapping theorem.** Let us cite the Riemann mapping theorem (see, for instance, [A], p.229ff):

Let $U \subset \mathbb{C}$ be a non-empty domain which is simply connected and not equal to $\mathbb{C}$. Then there exists a biholomorphic function $f : D \to U$.

By defintion, a *biholomorphic* function is holomorphic and bijective with holomorphic inverse.

Remarks. 1. For the excluded case $U = \mathbb{C}$, the map f^{-1} contradicts Liouville's theorem.

2. The theorem comprises the nontrivial topological statement that there exists a homeomorphism at all. The mapping f is not unique. As for the Plateau problem, it becomes unique once three points or $f(z_0) = 0$ and $f'(z_0) > 0$ are prescribed.

Our proof of the Plateau problem was given for arbitrary dimensions $n \geq 2$. In particular it works for $n = 2$. So consider a Jordan curve $\Gamma \subset \mathbb{R}^2$. By the Jordan curve theorem there is a bounded domain U such that $\partial U = \Gamma$. Let us now state the special case of the Riemann mapping theorem which is a corollary to the Plateau problem:

Theorem 29. *Let $\Gamma = \partial U$ be a closed Jordan curve which is piecewise C^1 where U is a bounded simply connected domain. Then there exists a biholomorphic map $f : D \to U$ whose boundary restriction $f|_{\partial D}$ parameterizes Γ homeomorphically.*

Note that for $n \geq 3$ the Plateau solution is in general not injective.

Proof. The solution to the Plateau problem gives us a harmonic and conformal map $f : D \to \mathbb{C}$ which continuously extends to the boundary ∂D. By Prop. 11 the map f is holomorphic or anti-holomorphic. Thus f becomes holomorphic after composing it with a reflection of D, if necessary. It remains to show that f takes image in U and is biholomorphic.

Since f is non-constant, the zero set $Z \subset D$ of the holomorphic map f' is discrete, and so is its image set $f(Z)$. Denote with $X_k \subset \mathbb{C} \setminus \{f(Z) \cup \Gamma\}$ the subset of points with k preimages under f. Since $f' \neq 0$ at all points in $f^{-1}(X_k)$ we can apply the inverse mapping theorem to see the equation $f(z) = q$ has the same number k of preimages for all q in some neighbourhood of a given point $p \in X_k$. This shows the openness of X_k. The complement of X_k is the union of the open sets with number of preimages different from k, hence also open in $\mathbb{C} \setminus \{f(Z) \cup \Gamma\}$. Consequently X_k is also relatively closed.

It follows that the number of preimages is constant on each connected component of $\mathbb{C} \setminus \{\Gamma \cup f(Z)\}$. But $f(Z)$ is discrete and so the number of components is equal to the number of components of $\mathbb{C} \setminus \Gamma$, which in turn is equal to 2 by the Jordan curve theorem. We claim that $\{\mathbb{C} \setminus \overline{U}\} \setminus f(Z)$ has $k = 0$ preimages while for $U \setminus f(Z)$ there is precisely $k = 1$ preimage.

The image of $\overline{D}$ under the continuous map f is compact. Thus at infinity of $\mathbb{C}$ the number of preimages is 0. This proves the first assertion.

The second assertion is a topological fact, and can be best formulated in terms of a mapping degree (see, e.g., [K, p.50/51]). We give an elementary argument. To begin with, the number of preimages of $U \setminus f(Z)$ must be $k \geq 1$. Indeed, by the previous reasoning, a path $D \setminus Z$ must map into U (if it maps to a point of Γ then there would also be images in $\mathbb{C} \setminus \overline{U}$). Thus there is at least 1 preimage for points in U.

To show that $k = 1$, let us first claim that Γ has no preimages in the open pointed disk $D \setminus Z$. Indeed, were there a preimage in the set $D \setminus Z$, the implicit function theorem would give preimages for a small neighbourhood, in particular in the complement of $\overline{U}$ which is impossible. Thus all but finitely many points of Γ have precisely one preimage under $f : \overline{D} \to \mathbb{C}$.

Then pick four distinct points $F \subset \Gamma$. Assume they are in anticlockwise order, and consider two curves c_1, c_2 in $\overline{U}$, whose endpoints connect every other point of F. Moreover, since U is connected and open we can assume that c_1, c_2 are in $U \setminus f(Z)$ except for the endpoints, and that they intersect in precisely one point $p \in U \setminus f(Z)$.

By assumption p has k preimages, and the preimage $c_1^{-1}(U)$ consists of k paths, which must be pairwise disjoint except for their common endpoints; similarly so for $c_2^{-1}(U)$. To imagine the preimage set, think of a grid of two times k paths, one set running from -1 to 1, being crossed by the other set running from i to $-i$. But then the preimage of p consists of k^2 preimages. Thus $k^2 = k$ which proves $k = 1$.

Let us finally show that f is biholomorphic, that is, $Z = \emptyset$. Suppose that $f'(z_0) = 0$ for some $z_0 \in D$; for simplicity assume $z_0 = 0$. Then Taylor's formula gives $f(z) - f(0) = az^k + O(z)^{k+1}$ with $a \neq 0$ and $k \geq 2$. But in a small neighbourhood of 0 this equation has k distinct roots z, meaning there are $k \geq 2$ preimages in a neighbourhood of z_0. This contradicts the previous argument. □

Remarks. 1. The biholomorphic map f becomes unique provided we impose a three-point condition. Indeed, if not, we had a conformal map from D to D which fixes three points. As pointed out in the final remark of Sect. 2.5 (without proof) this Möbius transformation must be the identity.
2. We can call two open subsets of $\mathbb{C}$, or, more generally, of an open manifold, *conformally equivalent*, if there is a conformal diffeomorphism between them. The Riemann mapping theorem says that all proper subsets of $\mathbb{C}$ belong to the same equivalence class; moreover $\mathbb{C}$ and $\mathbb{S}^2$ are further simply connected spaces which are not conformally equivalent. For multiply connected sets or spaces, there is a manifold of different classes, however. The problem to find a conformal map from a given set to some standard representative set or manifold is called the *uniformization problem*, see [FK].

`34. Lecture, Tuesday 25.4.06`

4.2. **Henneberg's surface is non-orientable and has branch points.** We discuss an example of a minimal surface with some pathological behaviour. The surface

$$f\colon \mathbb{R}^2 \to \mathbb{R}^3, \qquad f(x,y) = \begin{pmatrix} \cosh 2x \cos 2y \\ -\sinh x \sin y - \frac{1}{3}\sinh 3x \sin 3y \\ -\sinh x \cos y + \frac{1}{3}\sinh 3x \cos 3y \end{pmatrix}$$

was discovered by Henneberg in 1875.

The surface is conformally parameterized. To see this, calculate

$$f_x = \begin{pmatrix} 2\sinh 2x \cos 2y \\ -\cosh x \sin y - \cosh 3x \sin 3y \\ -\cosh x \cos y + \cosh 3x \cos 3y \end{pmatrix}, \qquad f_y = \begin{pmatrix} -2\cosh 2x \sin 2y \\ -\sinh x \cos y - \sinh 3x \cos 3y \\ \sinh x \sin y - \sinh 3x \sin 3y \end{pmatrix}.$$

Thus, making use of the addition theorems, we obtain

$$\begin{aligned} \langle f_x, f_y\rangle &= -4\sinh 2x \cosh 2x \sin 2x \cos 2y \\ &\qquad + (\sinh x \cosh 3x + \sinh 3x \cosh x)(\sin 3y \cos y + \sin y \cos 3y) \\ &= -\sinh 4x \sin 4y + \sinh 4x \sin 4y = 0. \end{aligned}$$

A longer calculation, making use of the addition theorems for the hyperbolic trigonometric functions, gives

$$|f_x(x,y)|^2 = |f_y(x,y)|^2 = 4\cosh^2 x(\sinh^2 2x + \sin^2 2y),$$

so that f is conformal. Let

$$S := \{(0,y) : y = \frac{1}{2}k\pi, k \in \mathbb{Z}\}$$

be the singular set of the so-called branch points where $f_x, f_y = 0$. Note that f is an immersion precisely on $\mathbb{R}^2 \setminus S$.

Now

$$\begin{aligned} \Delta f = {}& 4\cosh 2x \cos 2y - \sinh x \sin y - 3\sinh 3x \sin 3y - \sinh x \cos y + 3\sinh 3x \cos 3y \\ & - 4\cosh 2x \cos 2y + \sinh x \sin y + 3\sinh 3x \sin 3y + \sinh x \cos y - 3\sinh 3x \cos 3y = 0. \end{aligned}$$

We conclude f is a conformal harmonic map on $\mathbb{R}^2$, hence minimal on $\mathbb{R}^2 \setminus S$.

The Gauss map of f is

$$\nu(x,y) = \frac{1}{\cosh x}\begin{pmatrix} \sinh x \\ \sin y \\ -\cos y \end{pmatrix},$$

which means it extends continuously to the singular points $(x,y) \in S$. Indeed, clearly $|\nu|^2 = 1$ and

$$\begin{aligned} &\cosh x\,\langle \nu, f_x\rangle \\ &\quad = 2\sinh 2x \sinh x \cos 2y - \cosh x(\sin^2 y - \cos^2 y) - \cosh 3x(\sin 3y \sin y + \cos 3y \cos y) \\ &\quad = \cos 2y(2\sinh 2x \sinh x + \cosh x - \cosh 3x) \\ &\quad = \frac{1}{2}\cos 2y\big((e^{2x} - e^{-2x})(e^x - e^{-x}) + e^x + e^{-x} - e^{3x} - e^{-3x}\big) = 0 \end{aligned}$$

as well as

$$\begin{aligned} &-\cosh x\,\langle \nu, f_y\rangle \\ &\quad = 2\cosh 2x \sinh x \sin 2y + 2\sinh x \cos y \sin y + \sinh 3x(\cos 3y \sin y - \sin 3y \cos y) \\ &\quad = \sin 2y(2\cosh 2x \sinh x + \sinh x - \sinh 3x) \\ &\quad = \frac{1}{2}\sin 2y\big((e^{2x} + e^{-2x})(e^x - e^{-x}) + e^x - e^{-x} - e^{3x} + e^{-3x}\big) = 0. \end{aligned}$$

We would like to discuss two properties of Henneberg's surface. First, we study the local behaviour at the branch point $(0,0) \in S$. Using expansions such as $\cosh t = 1 + \frac{1}{2}t^2 + O(t^4)$

and $\cos t = 1 - \frac{1}{2}t^2 + O(t^4)$ we obtain

$$f(x,y) = \begin{pmatrix} \big(1+\frac{1}{2}(2x)^2+O(x^4)\big)\big(1-\frac{1}{2}(2y)^2+O(y^4)\big) \\ -xy - \frac{1}{3}3x3y + O((x,y)^4) \\ -(x+\frac{1}{6}x^3)(1-\frac{1}{2}y^2) + \frac{1}{3}(3x+\frac{1}{6}(3x)^3)(1-\frac{1}{2}9y^2) + O((x,y)^4) \end{pmatrix}$$

$$= \begin{pmatrix} 1+2x^2-2y^2 \\ -4xy \\ -4xy^2+\frac{4}{3}x^3 \end{pmatrix} + O((x,y)^4)$$

To anlayse the mapping more closely we use polar coordinates. Then

$$f(r\cos\varphi, r\sin\varphi) = \begin{pmatrix} 1+2r^2(\cos^2\varphi-\sin^2\varphi) \\ -4r^2\cos\varphi\sin\varphi \\ -4r^3(\cos^2\varphi\sin\varphi-\frac{1}{3}\sin^3\varphi) \end{pmatrix} + O(r^4) = \begin{pmatrix} 1+2r^2\cos 2\varphi \\ -2r^2\sin 2\varphi \\ -\frac{4}{3}r^3\sin 3\varphi \end{pmatrix} + O(r^4).$$

Now consider the image of a circle $\varphi \in [0, 2\pi]$ with small radius $r > 0$. The first two coordinates f^1, f^2 are then travelled around twice: φ and $\varphi + \pi$ have images with the same argument. But under $\varphi \mapsto \varphi + \pi$ the third coordinate f^3 undergoes a sign change, $\sin 3\varphi = -\sin 3(\varphi + \pi)$, and so the third coordinate distinguishes the two images. Now $\sin 3\varphi$ is positive on three subintervals of $[0, 2\pi]$ and negative on the complementary three intervals. Thus the surface self-intersects in three curves which emanate from the branch point.

Remark. Our present discussion parallels the discussion of the Enneper surface at infinity, see II,2.4.

The second property we would like to discuss is that the Henneberg surface factors to a non-orientable surface. To see that, note the following symmetries. Clearly, $f(x, y+2\pi) = f(x,y)$. In fact, f has more symmetry:

$$f(-x, y+\pi) = f(x,y). \tag{31}$$

So if we consider f on the strip $(x,y) \in \mathbb{R} \times [0,\pi)$ then all other values will be repeated. But (31) means that the boundaries $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{\pi\}$ are glued together in opposite orientation. Thus the result of the gluing is a Möbius strip.

Under the symmetry (31), the Gauss map undergoes a sign change $\nu(x,y) = -\nu(-x, y+\pi)$. This means that the symmetry (31) is orientation reversing. Thus, on a general minimal surface, there need not exist a Gauss map with images in $\mathbb{S}^2$.

Using higher level terminology, we would say that Henneberg's surface is a minimal immersion of a Möbius strip. Here, the Möbius strip is the abstract manifold given as $\mathbb{R}^2$ modulo the equivalence relation

$$(x,y) \sim (u,v) \qquad :\Longleftrightarrow \qquad \exists k \in \mathbb{Z}: \quad u = (-1)^k x, \quad v = y + k\pi.$$

The Möbius strip is non-orientable, but its double cover, represented by $\mathbb{R} \times [0, 2\pi)$, is orientable. The parameterization f covers the immersed Möbius strip infinitely often.

Remarks. 1. The symmetry (31) shows that the set S maps to two different points, namely $f(0, k\pi) = (1,0,0)$ while $f(0, \frac{\pi}{2} + k\pi) = (-1,0,0)$ where $k \in \mathbb{Z}$.

2. Henneberg's surface is algebraic, see [N], p.141.

3. A geometric way to introduce Henneberg's surface is via *Björling's problem*: Consider an analytic curve $c(t)$ and any unit vector field $\nu(t) \perp c'(t)$. Is there a minimal surface which contains the curve c and has surface normal ν along this curve? Schwarz solved this problem in the affirmative in terms of an integral representation, see [N, p.136ff] or [DHKW, I.3.4]. Henneberg's surface contains *Neil's parabola* $c(t) = (t^2, t^3)$. Indeed,

$$x \mapsto f(x,0) = \big(\cosh 2x, 0, -\sinh x + \frac{1}{3}\sinh 3x\big) = \big(1 + 2\sinh^2 x, 0, \frac{4}{3}\sinh^3 x\big)$$

so that the plane $\{y = 0\}$ intersects Henneberg's surface in the translated parabola $2(f^1 - 1)^3 = 3(f^3)^2$.

References. [N], p. 140–142; [DHKW], p. 159–169

`35. Lecture, Wednesday 26.4.06` ______________________

4.3. **Branch points and generalized minimal surfaces.** The solution of the Plateau problem is a conformal harmonic map $f\colon D \to \mathbb{R}^n$. Suppose this map is an immersion, that is, it represents a surface in the sense of differential geometry. Then for $n = 3$, this surface is minimal by II(2).

Definition. Let $f\colon U \to \mathbb{R}^n$ be differentiable. Points $p \in U$ with $df_p = 0$ are called *branch points*.

Examples. 1. Henneberg's surface has branch points.
2. For any $k \geq 2$, the parameterization of the plane

$$z = (x,y) \mapsto (z^k, 0) = (\operatorname{Re} z^k, \operatorname{Im} z^k, 0),$$

has a branch point at 0. In this case, away from the branch point p, the parameterization is a k-fold covering of the image set. More precisely, there is a neighbourhood $U(p) \subset \mathbb{R}^n$ of the branch point such that for each point of $U(p) \setminus \{p\}$ there are either 0 or k preimages. Such branch points are called *false*, the others (which can be identified from the image alone) are called *true*.

A local analysis can be used to yield two important facts about branch points, which are familiar from Henneberg's example:
1. The normal extends smoothly to a branch point, that is, a branch point has a well-defined tangent plane.

2. Suppose the tangent plane at the branch point agrees with the xy-plane. Let f be conformal and consider the sign of f^3 as a function of D. Then unless f is a plane there is $k \in \mathbb{N}$ such that there are $2(k+2)$ curves in D whose images have $f^3 = 0$; they meet at equal angles in 0.

A conformal map $f\colon D \to \mathbb{R}^n$ has either rank 0 or 2. Hence after removing the branch points from the domain of f we are left with an immersion.

As asserted in Thm. 16 a harmonic map f is analytic. Thus its derivative f' is also analytic. If its zero set is not discrete then f' vanishes identically and then f is constant. Otherwise:

Theorem 30. *Let $f\colon D \to \mathbb{R}^n$ be a non-constant conformal harmonic map. The set of its branch points $\{p \in D : f_x(p) = f_y(p) = 0\}$ is isolated in D, and thus consists of at most countably many points.*

In view of this property, the following notation becomes meaningful:

Definition. Let $U \subset \mathbb{R}^2$ be a nonempty domain. A mapping $f \in C^1(U, \mathbb{R}^n)$ is called a *branched immersion* if f is an immersion except at a discrete set. Branched minimal immersions are also called *generalized minimal surfaces.*

Many theorems which hold for minimal surfaces also hold for generalized minimal surfaces. For example, Cor.II.17 generalizes to:

Theorem 31. *A generalized minimal surface cannot be compact.*

Proof. Since the surface M is compact, each coordinate function attains an interior maximum. But each coordinate function is harmonic. Thus by the maximum principle f must be constant, contradicting the discreteness of the set of points where f is not an immersion. $\square$

Let us mention two general properties of branched minimal immersions. The Gauss maps extends through the branch points, and there is a local expansion in a neighbourhood of a branch point which analyses the local behaviour.

The main reason to study branched immersions is that they come up naturally in constructions like the Plateau problem. They are not objects of primary interest.

4.4. **Branch points in Plateau solutions.** For area minimizers in 3-space, branch points are absent:

Theorem 32 (Osserman 1970)**.** *Let $f \in C^1(D, \mathbb{R}^3) \cap C^0(\overline{D}, \mathbb{R}^3)$ be a branched immersion whose boundary parameterizes a piecewise C^1 Jordan curve Γ homeomorphically. If f minimizes area over $\mathcal{C}$, then $f|_D$ is an immersion without branch points.*

Papers of Gulliver, Alt, and Gulliver-Osserman-Royden also contributed to the proof (cf. [DHKW, p.279/280]).

In particular, the Plateau solution for $\mathbb{R}^3$ does not have any branch points in D. We will not give the formal proof of Osserman's theorem. Nethertheless we want to use this property later and so we want to present at least an indication why Osserman's theorem is true. We will present the main idea for a somewhat special branch point. See [N, p.339-346] for details.

We first show that minima of area do not have kinks. Let $D^+ := \{(x, y) \in D\colon y > 0\}$ and $D^- := \{(x, y) \in D\colon y < 0\}$.

Lemma 33. *Suppose $f \in C^0(\overline{D}, \mathbb{R}^3)$ with restrictions to the half disks $f^\pm := f|_{D^\pm}$ which are C^1. Suppose moreover that at 0 the one-sided partial derivatives exist but differ, $\partial_y f^+ \neq \partial_y f^-$. Then f cannot minimize area among maps with the boundary values $f|_{\partial D}$.*

Idea of proof. In case the two mappings $f^\pm$ are linear we can modify f in the neighbourhood of the x-axis by locally replacing the edge with a rectangular face (shortcutting the image wedge); we get back to the edge outside this neighbourhood with two triangles. This idea will work if the rectangle is thin enough.

For the general case, the same idea can be applied to the linearization. The error is of lower order and thus the result for the linear case extends. $\square$

The next statement gives us a map which we will use to compose f with. We first introduce notation.

Let X be the interval on the x-axis $X := \{(x, 0) \in D : -\frac{1}{2} \le x \le \frac{1}{2}\}$. Let Y be the open interval on the y-axis, $Y := \{(0, y) : -1 < y < 1\}$, and $Y^\pm := Y \cap D^\pm$.

Lemma 34. *There exists a continuous map $\sigma\colon \overline{D}\backslash X \to \overline{D}\backslash Y$ with the following properties:*
(i) $\sigma|_{\partial D} = \mathrm{id}$,
(ii) σ maps $D \setminus \{X \cup Y\}$ diffeomorphically to $D \setminus Y$.
(iii) σ maps the sets $Y^\pm$ to $\pm i$. It has one-sided limits for each $(x, 0) \in X$, namely

$$\lim_{(s,t)\to(x,0)} \sigma(s,t) = \begin{cases} \big(0, \min\{1+2x, 1-2x\}\big) & \text{when all } t > 0, \\ \big(0, -\min\{1+2x, 1-2x\}\big) & \text{when all } t < 0. \end{cases} \tag{32}$$

(iv) Moreover, $(\sigma^1, \sigma^2)(-x, y) = (-\sigma^1, \sigma^2)(x, y)$, that is, σ respects reflection symmetry about the y-axis,

Proof. To begin with, we define σ only on the subset $(D \setminus X) \cap \{x \ge 0\}$ by a composition of two maps, $\sigma = \tau \circ \rho$.

First, let us consider mappings which maps rays through the origin linearly onto itsself: $re^{i\varphi} \mapsto \ell(\varphi)re^{i\varphi}$. We want to choose a function $\ell(\varphi)$ such that the points on the unit circle with $x > 0$ are mapped to a circle of radius 1 with midpoint 1; the points $\pm i$ will map to the origin. This means

$$1 \stackrel{!}{=} (\ell\cos\varphi - 1)^2 + (\ell\sin\varphi)^2 = \ell^2\cos^2\varphi - 2\ell\cos\varphi + 1 + \ell^2\sin^2\varphi^2 = 1 + \ell(\ell - 2\cos\varphi),$$

which yields $\ell(\varphi) = 2\cos\varphi$. Thus we define for $re^{i\varphi} \in D \cap \{x \geq 0\}$

$$\rho(re^{i\varphi}) = 2r\cos\varphi\, e^{i\varphi} \qquad \text{for } -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}, \quad 0 \leq r \leq 1.$$

The range of ρ is the closed disk $\overline{D_1}(1,0)$ of radius 1 with midpoint 1. Let us now remove X from the domain of ρ. This removes the interval $I := \{(x,0) : 0 \leq x \leq 1\}$ from its range $\overline{D_1}(1,0)$.

Second, the mapping τ is defined on $\overline{D_1}(1,0) \setminus I$ by setting

$$\tau(1 + re^{i\vartheta}) := re^{i\vartheta/2} \qquad \text{for } -\pi < \vartheta < \pi, \quad 0 < r \leq 1.$$

The map τ is onto $\overline{D} \cap \{x > 0\}$.

We now extend the maps ρ and τ by requiring the symmetry property (iv) for each. This defines a unique extension of $\sigma = \rho \circ \tau$ to $D \setminus X$ which again is subject to (iv). Note that the two maps agree on the sets $Y^\pm$ common to their domains, which they map to $\pm i$. This completes the definition of σ.

Let us now check the properties claimed for σ. By construction, σ is continuous on $D \setminus X$. The diffeomorphism property (ii) is immediate from the fact that ρ and τ are diffeomorphisms on the appropriate sets. The limits (32) can be explicitely computed. In order to show (i), consider how the right hemicircle gets mapped under ρ:

$$\begin{aligned}\rho(e^{i\varphi}) &= 2\cos\varphi e^{i\varphi} = 2\cos^2\varphi + 2i\cos\varphi\sin\varphi = 1 + \cos^2\varphi - \sin^2\varphi + 2i\cos\varphi\sin\varphi \\ &= 1 + \cos 2\varphi + i\sin 2\varphi = 1 + e^{2i\varphi}\end{aligned}$$

Thus as desired $\tau(\rho(e^{i\varphi})) = e^{i\varphi}$. □

36. Lecture, Tuesday 2.5.06

Consider f with a branch point at 0. Take coordinates such that the normal at 0 is the z-vector. It is a fact that the sign of the z- coordinate of f alternates on $2k$ wedges. Suppose in particular that the z-component vanishes along the y-axis of D. The main advantage of the parameterization $f \circ \sigma$ then is that across the arc X two regions with the same sign of the z-component are adjacent (sketch!). This produces a kink in the parameterization $f \circ \sigma$.

Let us state this more formally:

Proposition 35. *Suppose that $f \colon D \to \mathbb{R}^3$ has a branch point at 0, and that the y-axis maps to a self-intersection line, namely,*

$$f(0,y) = f(0,-y) \qquad \text{for all } -1 \leq y \leq 1. \tag{33}$$

Moreover assume the branch point is true, so that

$$\partial_x f(0,y) > 0 \qquad \text{for all } 0 < |y| < 1 \tag{34}$$

Then $f \circ \sigma$ has the following properties:
(i) It is continuous on $\overline{D}$.
(ii) $f|_{\partial D} = f \circ \sigma|_{\partial D}$.
(iii) $f \circ \sigma$ has a kink along the set X, that is for $p \in X \setminus \{0\}$ the derivative $\partial_y(f \circ \sigma)(p)$ has different one-sided limits.

For a general branch point, it must be shown that (33) and (34) hold after a suitable change of parameters.

Idea of proof. (i) Continuity of $f \circ \sigma$ needs only be checked along X. But the two one-sided limits (32) of σ are mapped under f to the same points, as prescribed by (33).

(ii) follows from Lemma 34(i).

To prove (iii) it must be shown that the one-sided derivatives of σ at X do not vanish. □

How does the parameterization $f \circ \sigma$ look like? Consider

$$f(D \setminus Y) = f\big(D \cap \{x > 0\}\big) \cup f\big(D \cap \{x < 0\}\big) =: M_1 \cup M_2,$$

as well as the self-intersection line

$$L := f(Y) = f\big(Y^+ \cup \{0\}\big) = f\big(Y^- \cup \{0\}\big).$$

In the original parameterization f there are two preimages of L: one glues together one side of M_1 to one side of M_2, while the other one glues the remaining two sides together. In $f \circ \sigma$, there are also two preimages of L; however, the first (which is $X \cap \{x > 0\}$) glues the boundaries of M_1 to oneanother, while the second (which is $X \cap \{x < 0\}$) glues the two sides of M_2 to oneanother. The two resulting disks $(f \circ \sigma)\big(D \cap \{x > 0\}\big)$ and $(f \circ \sigma)\big(D \cap \{x < 0\}\big)$ are then glued together by the image of the arc Y, mapping to the single point $f \circ \sigma(Y) = f(i)$. Note in particular that the branch point $f(0)$ has the two distinct preimages $(\pm\frac{1}{2}, 0)$ under $f \circ \sigma$.

Let us finally indicate how Osserman's theorem can be proved. Either a branch point is true. Then the above arguments can be adapted to obtain a parameterization $f \circ \sigma$ which has the same area and boundary values as f, but has an interior kink. Thus $f \circ \sigma$ cannot be area minimizing and f itself cannot be area minimizing either. Else a branch point is false. Then by analyticity $f(\partial D)$ also has multiple preimages, contradicting the fact that f maps the boundary homeomorphically.

Remark. For smooth boundary points, we can also define branch points at the boundary. It is known they do not show up for analytic Jordan curves, but up to today unclear if they arise for smooth curves.

4.5. **Uniqueness and embeddedness results.** The following issues for Plateau solutions deserve special interest: 1. Uniqueness, 2. Embeddedness, 3. Regularity at the boundary.

By Osserman's theorem, a Plateau solution $f\colon D \to \mathbb{R}^3$ is an immersion. Thus the uniqueness results of Part II implies that the Dirichlet problem for minimal graphs can be solved over convex domains:

Theorem 36 (Radó 1930)**.** *Let Γ be a Jordan curve which admits a 1-1 projection π to a plane E such that $\pi(\Gamma)$ bounds a convex domain $U \subset E$. Then the solution to the Plateau problem for Γ is contained in the cylinder $\overline{U} \times \mathbb{R}$ and graph over U. In particular, it is embedded and unique.*

Proof. By Thm. II.35 the Plateau solution M is contained in the convex hull of Γ; in particular it is contained in $\overline{U} \times \mathbb{R}$. For convex U, Thm. II.36(ii) says that there is at most one such minimal surface which is graph. Thus M must be this unique graph. $\square$

There is a direct and beautiful proof of this result, bypassing the use of Osserman's theorem, which depends on a local expansion of a surface at a branch point; it is contained in many books, see for instance [DHKW, pp.272/73].

`37. Lecture, Wednesday 3.5.06`

Example. We want to show that convexity of the boundary projection is essential for the Plateau solution to be graph. To see this we let $h \in \mathbb{R}$ and define a quadrilateral $\Gamma = \Gamma(h)$ by its four vertices

$$p_1 := (0,1,0), \quad p_2 := (1,0,0), \quad p_3 := (0,-1,0), \quad p_4 := (\frac{1}{2}, 0, h).$$

Note that p_1, p_2, p_3 lie in the xy-plane. For the projection π to the xy-plane we have

$$\pi\big(\Gamma(h)\big) = \Gamma(0).$$

Let $\overline{U}$ be the compact component of $\mathbb{R}^2$, bounded by $\Gamma(0)$. It can be shown that the Plateau solution $M = M(h)$ has a projection $\pi(M) \not\subset \overline{U}$ whenever $h \neq 0$ (see Nitsche §410). This would be easy to show if we knew that the solution has a tangent plane at the boundary point p_4: Then the solution will project to points $(x, 0)$ in the xy-plane with $x < \frac{1}{2}$. In the

problems we will use a catenoid piece to see that at least for some $h > 0$ the solution is not graph.

Let us now collect further results without proofs.

Embeddedness is known more generally:

Theorem (Meeks-Yau 1982)**.** *Suppose that $K \subset \mathbb{R}^3$ is compact and C^2 such that a piecewise C^1-Jordan curve Γ is contained in ∂K. Let $M \subset K$ be any area minimizer of disk-type which bounds Γ. Then M is embedded, provided K is convex or, more generally, K is mean convex.*

Here, *mean convex* means that the mean curvature of M is positive with respect to the interior normal.

Example. Any C^1-Jordan curve in the 2-sphere $\mathbb{S}^2$ bounds an embedded minimal disk.

It is worth knowning that embeddedness can always be achieved by allowing for possibly higher topological type:

Theorem (Hardt-Simon 1979)**.** *Each Jordan curve Γ of class C^2 bounds at least one embedded orientable minimal surface.*

Thus we can achieve embedded or disk-type solutions of the Plateau problem, but in general not both together!

Example. The clover knot bounds a disk with self-intersections or an embedded Möbius strip.

In general, the Plateau problem can have many solutions; there are examples which bound a continuum of solutions. Besides the situation of Radó's Theorem, there are cases when uniqueness of the solution to Plateau's problem is implied. Let us mention but the most famous one:

Theorem (Nitsche 1972)**.** *If $\Gamma \subset \mathbb{R}^3$ is a real analytic Jordan curve with total curvature $\int_\Gamma k(t)\,dt < 4\pi$ then Γ bounds a unique minimal surface of disk-type.*

Remember that if a space curve c has arc-length parameterization then its curvature is $k(t) = |c''(t)|$. For example, a circle has total curvature 2π, while its double cover has 4π.

A more recent extension of Nitsche's theorem is:

Theorem (Ekholm/White/Wienholtz 2002)**.** *Let $\Gamma \subset \mathbb{R}^3$ is a Jordan curve which is piecewise C^2 and has total cuvature $T(\Gamma) < 4\pi$. Then any branched minimal surface M with boundary Γ has an embedded interior and no interior branch points.*

For a piecewise C^2-curve, the total curvature is the sum $T(\Gamma) = \int_\Gamma k(t)\,dt + \sum \alpha_i$, where α_i are the exterior angles at the singular points of Γ.

Let us finally comment on boundary regularity. The Plateau solution is just continuous at the boundary. If the bounding Jordan curve is better, so is the Plateau solution:

Theorem (Hildebrandt 1969)**.** *Suppose $\Gamma \subset \mathbb{R}^3$ is a Jordan curve which is C^{k+1}, $k \geq 0$, then the Plateau solution f is differntiable up to the boundary, $f \in C^k(\overline{D}, \mathbb{R}^3)$.*

The actual statement of the theorem employs Hölder continuity, and shows the solution is as good as its boundary values: If Γ is $C^{k,\alpha}$ then $f \in C^{k,\alpha}(\overline{D}, \mathbb{R}^3)$. In any case, these statements mean that f can locally be extended accross ∂D in the given class. For the special case of straight boundary arcs we will establish this regularity in the next section.

5. Existence of complete periodic minimal surfaces

We will obtain periodic minimal surfaces by taking Plateau solutions for polygons which we extend by reflection across the boundary arcs.

5.1. **The reflection principle.** Many of the examples we discussed contain straight lines: For instance, the helicoid is ruled by them, and the Enneper surface contains two straight lines. Not only contain the surfaces lines, but these lines are arcs of symmetry: When the surface is rotated by 180 degrees about these lines, it becomes congruent to itself. This is evident from the construction of the helicoid or from the polar representation of Enneper's surface.

We want to formulate this as a principle, which we will use to extend Plateau solutions. Remember that Plateau solutions are smooth in the interior, but all we have established at the boundary is that they are continuous. This is precisely what the following statement assumes.

As before we denote the half-disks with $D^\pm := D \cap \{\pm y > 0\}$ and their common boundary on the x-axis with $I := \{(x, 0) : -1 < x < 1\}$.

Theorem 37. *Let $f \in C^0(D^+ \cup I, \mathbb{R}^3) \cap C^\infty(D^+, \mathbb{R}^3)$ be a conformally parameterized minimal surface on D^+, such that $f(I)$ is contained in a straight line L. Let $\sigma_L \colon \mathbb{R}^3 \to \mathbb{R}^3$ be reflection about the line L, that is $180°$ rotation about L. Then the extension*

$$\tilde{f}(x,y) := \begin{cases} f(x,y) & \text{for } (x,y) \in D^+ \cup I, \\ \sigma_L\big(f(x,-y)\big) & \text{for } (x,y) \in D^-, \end{cases} \tag{35}$$

is conformal and minimal on D, in particular it is smooth, $\tilde{f} \in C^\infty(D)$.

If we were to assume the boundary regularity granted by Thm. 4.5 the proof would become easy: Then at the boundary point we could write the surface as a C^2-graph, say with 0-boundary values. As shown in Problem 32 c) such a graph extends by reflection. We will not make use of Thm. 4.5, however, as it is not too hard to assert the necessary boundary regularity directly for the present case.

Proof. For simplicity, we assume L coincides with the z-axis, so that

$$\sigma_L(x^1, x^2, x^3) = (-x^1, -x^2, x^3).$$

Then the second line of (35) becomes

$$\tilde{f}^1(x,y) = -f^1(x,-y), \qquad \tilde{f}^2(x,y) = -f^2(x,-y), \qquad \tilde{f}^3(x,y) = f^3(x,-y) \tag{36}$$

for $(x,y) \in D^-$.

Now f^1 and f^2 are harmonic functions which vanish on I. Thus we can apply the Schwarz Reflection principle, Thm. 27, to give that the odd reflections $\tilde{f}^1$, $\tilde{f}^2$ are functions which are smooth on all of D.

We cannot argue the same way for the even reflection $\tilde{f}^3$ with its nonconstant values f^3 along I. The trick here is to consider its derivative $\partial_y \tilde{f}^3$ instead, which is odd in y and does allow for Schwarz Reflection. We first need to prove that $\partial_y f^3$ exists on I and vanishes. Clearly this will only be true for a conformal parameterization: On I then $\partial_x f$ is vertical, and so $\partial_y f$ is horizontal.

On I we have $f^1, f^2 = 0$ and so $\partial_x f^1 = \partial_x f^2 = 0$. Using this, we obtain for any sequence $z_n \in D^+$ which converges to $z \in I$

$$\partial_x f^3(z_n)\, \partial_y f^3(z_n) \overset{f \text{ conf.}}{=} -\partial_x f^1(z_n)\, \partial_y f^1(z_n) - \partial_x f^2(z_n)\, \partial_y f^2(z_n) \to 0; \tag{37}$$

indeed the right hand side tends to 0 as $\partial_y f^i(z_n)$ is the derivative of a harmonic function, hence bounded, while $\partial_x f^i(z_n) \to 0$ for $i = 1, 2$. The last formula means that after passing to a subsequence, if necessary, at least one of the two subsequences $\partial_x f^3(z_n)$ or $\partial_y f^3(z_n)$ will be convergent and tends to 0.

Let us now identify this as $\partial_y f^3(z_n)$. The other conformality relation gives

$$\begin{aligned} |\partial_x f^3(z_n)|^2 - |\partial_y f^3(z_n)|^2 &= -|\partial_x f^1(z_n)|^2 + |\partial_y f^1(z_n)|^2 - |\partial_x f^2(z_n)|^2 + |\partial_y f^2(z_n)|^2 \\ &\to |\partial_y f^1(z)|^2 + |\partial_y f^2(z)|^2 \geq 0. \end{aligned} \tag{38}$$

So combining (37) with (38) we obtain $\partial_y f^3(z_n) \to 0$, meaning

$$\partial_y f^3(z) = 0 \qquad \text{for } z \in I.$$

As $\tilde f^3(x,y)$ is even in y, its derivative $\partial_y \tilde f^3(x,y)$ is odd. Hence we can consider $\partial_y \tilde f^3$ as the result of odd reflection of the harmonic function $\partial_y f^3$ on D^+ which vanishes on I. By Schwarz reflection $\partial_y \tilde f^3$ is smooth on all of D.

`38. Lecture, Tuesday 9.5.06`

It remains to show that also $\tilde f^3$ itself is smooth. The function

$$g(x,y) := \tilde f^3(x,y) - \int_0^y \partial_y \tilde f^3(x,\eta)\,d\eta$$

is continuous on D and satisfies $\partial_y g(x,y) = 0$ on $D \setminus I$; thus $g(x,y) = \tilde f^3(x,0)$. But for $y \neq 0$ the function $x \mapsto g(x,y)$ is smooth and hence also $\tilde f^3(.,0) \in C^\infty(I)$. This gives, finally, as desired

$$\tilde f^3(x,y) = \tilde f^3(x,0) + \int_0^y \partial_y \tilde f^3(x,\eta)\,d\eta \quad \in C^\infty(D).$$

□

Similarly, a minimal surface can be extended across a plane if it is C^1 up to and including this plane and the normal along the plane is parallel to the plane:

Theorem 38. *Let $f \in C^0(D^+ \cup I, \mathbb{R}^3) \cap C^\infty(D^+, \mathbb{R}^3)$ be a conformally parameterized minimal surface on D^+, such that $f(I)$ is contained in a plane P and has normal $\nu(I)$ parallel to P. Let $\sigma_P\colon \mathbb{R}^3 \to \mathbb{R}^3$ be mirror reflection across P. Then the extension*

$$\tilde f(x,y) := \begin{cases} f(x,y) & \text{for } (x,y) \in D^+ \cup I, \\ \sigma_P\big(f(x,-y)\big) & \text{for } (x,y) \in D^-, \end{cases} \tag{39}$$

is conformal and minimal on D, in particular it is smooth, $\tilde f \in C^\infty(D)$.

For the proof see [DHKW, Thm.4.8,2]. We refer to the curve $f(I)$ in this case as an *arc of planar reflection*. Such arcs can be characterized by the fact that they are curvature lines and that they are geodesics of the surface (sse Lemma 10 below). We will need this version when we

5.2. **Plateau solutions with polygonal boundary: Extension and symmetries.** Let us consider a polygon and pick a vertex of angle α. In the next theorem we successively use reflections σ to extend the surface across its bounding edges. We expect the resulting surface will close smoothly provided $\alpha = \pi/k$ with $k \in \mathbb{N}$. It is, however, not sufficient to assume $\alpha = 2\pi/k$: The boundary conditions on the two bounding edges of the wedge are different and so repeat only after 2α. Therefore we need indeed $2\alpha = 2\pi/k$.

Remark. For any rational angle $\alpha = \pi \cdot p/q$ with p,q reduced we must reflect q times for the surface to close. However, we necessarily create a branch point: the tangent plane will be covered p times!

We can now assert that Plateau solutions to polygons with good angles admit extensions which are (smooth) generalized minimal surfaces, perhaps with branch points at the boundary:

Theorem 39. *Suppose $\Gamma \subset \mathbb{R}^3$ is a polygon without self-intersections with (interior) angles of the form π/k for integer $k \geq 2$. Let $f\colon D \to \mathbb{R}^3$ be the Plateau solution.*
(i) For each $p \in \partial D$ which maps to an edge of Γ, but not to a vertex there is a smooth local extension of f across ∂D which is harmonic and conformal.
(ii) Suppose $p \in \partial D$ maps to a vertex of Γ. Then there is conformal map

$$\zeta\colon W_k := \{z \in \mathbb{C} : 0 < \arg(z) < \frac{\pi}{k}\} \to D$$

which extends continuously to ∂W_k and maps $0 \in \partial W_k$ to $p \in \partial D$ with the following property: At 0 the map $F := f \circ \zeta$ has a local extension as a smooth conformal harmonic map.

For a general Jordan curve it is an interesting to decide whether a minimal surface admits a smooth local extension.

Proof. (i) Recall the map $\eta\colon \overline{H} \to \overline{D} \setminus \{1\}$ defined in Lemma 19, is a conformal diffeomorphism. In the following we will often use the fact that composition with conformal maps preserves conformality and harmonicity (see Lemma 12). Under η, the interval $I = \{(x,0) : -1 < x < 1\} \subset \partial H$ is mapped to some interval $\eta(I)$ in $\mathbb{S}^1$.

The inverse image $f^{-1}(e) \subset \partial D$ is also some interval in $\mathbb{S}^1$. By composing f with a Möbius transformation $\omega \in \mathrm{Aut}(D)$, if necessary, withouth loss of generality we can assume

$$f^{-1}(e) = \eta(I); \tag{40}$$

Indeed, by Prop. 18 there exists a Möbius transformation mapping the two endpoints of one interval to the endpoints of the other. For any choice of third points within the intervals, we obtain (40).

Let us now consider the composition

$$F := f \circ \eta \quad \in C^0(\overline{H}, \mathbb{R}^3) \cap C^2(H, \mathbb{R}^3),$$

which is conformal and harmonic in H and maps I to e.

Now we can apply the reflection principle Thm. 37 to the restriction of F to $I \cup D^+$. The result is an extension $\tilde F\colon D \to \mathbb{R}^3$ across I. As D^- is still in the domain of η, we can invoke

η^{-1} to construct

$$F \circ \eta^{-1} \colon D \cup I \cup \eta(D^-) \to \mathbb{R}^3. \tag{41}$$

This map is smooth, harmonic and conformal and extends f, as desired.

(*ii*) Call $e_1, e_2 \subset \Gamma$ the edges incident to p. Let e now denote the union $e_1 \cup \{p\} \cap e_2$. In addition to requiring (40), we may assume that $f^{-1}(p) = \eta(0,0)$; altogether this will impose exactly a 3-point condition on the Möbius transformation $\omega \in \operatorname{Aut}(D)$.

Surprisingly, standard reflection suffices to reflect around a vertex, as shown by the following argument. We compose $f \circ \eta$ as above with the k-th power: That is, we set $\zeta(z) := \eta(z^k)$. Then $F = f \circ \zeta$ conformally parameterizes the Plateau solution with the wedge W_k. The map F extends continuously to ∂W_k. By construction, one bounding arc of $W_k \cap D$ will map to e_1, the other to e_2.

We can apply then reflection principle $k-1$ times to extend this map smoothly by odd reflection about the edges e_1 or e_2 to become defined on all of the half-disk D^+ with a continuous extension to its boundary on the real line I. By a final extension of the reflection principle, we obtain a smooth extension to the entire disk D. □

The continuation of the Plateau solution across ∂D is smooth but may have branch points:

Example. We consider a hexagon $\Gamma \subset \mathbb{R}^3$ contained in the xy-plane with vertices $(0,0)$, $(0,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$, $(1,0)$. As a polygon in 3-space, Γ has six right angles. The Plateau solution $f \colon D \to \mathbb{R}^3$ for Γ is unique and parameterizes conformally the compact component of the xy-plane bounded by the hexagon. Thus the surface has a 270°-angle at the origin. When we extend by reflection using the previous theorem the resulting surface contain a false branch point at the origin; it covers the tangent plane four times. Note that if we were considering $\Gamma \subset \mathbb{R}^2$ then the orientation of $\mathbb{R}^2$ means that all angles are signed (meaning that we can distinguish a left and right turn by a right angle); in $\mathbb{R}^3$ however, this is impossible.

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The smooth local extension of Thm. 39 allows us to define that $p \in D$ is a *boundary branch point* if the extended conformal map satisfies $df_p = 0$. Let us state a theorem, which will allow us to rule out boundary branch points for polygonal boundaries. We say that a plane $E \subset \mathbb{R}^3$ is a *barrier* for the set $S \subset \mathbb{R}^3$ at $p \in S$, if $p \in E$ and S does not meet one of the two connected components of $\mathbb{R}^3 \setminus E$.

Lemma 40. *Suppose $f \in C^2(D^+ \cup I, \mathbb{R}^3)$ is a conformal harmonic map. Suppose there is a barrier E for $f(D^+ \cup I)$ at $f(0)$ and assume that $f(D^+ \cup I)$ is not contained in the plane E. Then 0 cannot be a branch point of f.*

Proof. By a motion, we can assume E is the xy-plane, $f(0) = 0 \in \mathbb{R}^3$ and $f^3(x,y) \le 0$ for $(x,y) \in D^+ \cup I$. The assumption $f(D^+ \cup I) \not\subset E$ implies $f^3 \not\equiv 0$. From the strong maximum principle for harmonic functions, Thm. II.33, we conclude $f^3(x,y) < 0$ for all $(x,y) \in D^+$. Note, moreover, that by our assumption the normal derivative $\frac{\partial}{\partial \nu} f^3 = \partial_y f^3$ exists.

Thus all assumptions of the Hopf Boundary Point Lemma II.32 are satisfied. We conclude that $\partial_y f^3(0) < 0$. In particular, $df(0) \neq 0$, meaning that 0 is not a branch point. □

We can apply the lemma to edges and vertices of a polygon alike:

Theorem 41. *Let $\Gamma \subset \mathbb{R}^3$ be an embedded polygon with all angles of form π/k for $k \in \mathbb{N} \ge 2$ subject to the following property:*
(i) For each p which is not a vertex of Γ, there is barrier E for Γ at p.
(ii) At each vertex $v \in \Gamma$, consider the union S of k copies of Γ, obtained by succesive 180°-rotation about the incident edges. We require that there is a barrier E for S at v.
Then the Plateau solution for Γ does not have a boundary branch point.

Proof. This is an immediate consequence of the Lemma. □

Let us finally make a remark on symmetries:

Proposition 42. *Let $\Gamma \subset \mathbb{R}^n$ be a Jordan curve and*

$$\operatorname{symm}(\Gamma) := \{\textit{motions of } \mathbb{R}^n \textit{ which preserve } \Gamma \textit{ as a set}\}.$$

If the Plateau solution M for Γ is unique in the class $\mathcal{C}$ then it is invariant under $\operatorname{symm}(\Gamma)$.

Proof. Any $\varphi \in \operatorname{symm}(\Gamma)$ preserves Γ. Thus $\varphi(M)$ is also bounded by Γ; the assumed uniqueness implies $\varphi(M) = M$. □

A typical application of this proposition is for graphs, that is, when Radó's theorem grants uniqueness.

5.3. **The D-surface.** The D or *diamond surface* was found by Schwarz using Weierstrass data. Let Γ be a polygonal Jordan curve in $\mathbb{R}^3$ containing six of the twelve edges of a standard cube $C := [0,1]^3 \subset \mathbb{R}^3$. To be precise, let us define Γ by demanding that it contains the vertices $(0,0,0)$, $(1,0,0)$, $(1,0,1)$, $(1,1,1)$, $(0,1,1)$, $(0,1,0)$ in cyclic order.

Γ does not have a 1-1 projection for any coordinate parallel direction. But nevertheless:

Lemma 43. *The Plateau solution M for Γ is a unique embedded surface, which is contained in the unit cube C. It is graph with respect to the space diagonal direction $v := (-1,-1,1)$.*

Proof. By Osserman's Theorem 32, the Plateau solution has no interior branch points and hence is a minimal surface.

Consider the orthogonal projection π_v to the plane $v^\perp$. We claim $\pi_v(\Gamma)$ is a regular hexagon. To see this note that the $120°$ rotation R_3 about v preserves Γ. Thus $\pi(\Gamma)$ is invariant under R_3. Moreover, Γ is invariant under the motion $R_{\overline{6}}$, which reflects in the plane $v^\perp$ followed by $60°$ rotation about v. Since the reflection has no effect on the projection, this means that $\pi(\Gamma)$ is invariant under $60°$-rotation R_6 about v. Hence $\pi(\Gamma)$ must be a regular hexagon; in particular $\pi(\Gamma)$ is convex.

By Radó's Theorem 36 the Plateau solution for the hexagon is a unique graph. By the Convex Hull Theorem II.35 the Plateau solution is in the unit cube (in fact, it is contained in the convex hull of Γ, which is what?). $\square$

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We now apply the reflection principle to extend the D-surface as a triply periodic minimal surface.

Theorem 44. *The continuation by reflection of the Plateau solution M for Γ yields an embedded minimal surface D. The surface together with its normal is invariant under addition of points in the lattice*

$$\Lambda_{or} := \{m(2,2,0)+n(2,0,2)+\ell(0,2,2)\colon m,n,\ell\in\mathbb{Z}\}.$$

Its quotient D/Λ_{or} is a compact surface which has genus 3. The surface M is also invariant under addition of points in the lattice

$$\Lambda_{unor} := 2\mathbb{Z}^3 \supset \Lambda_{or}.$$

Translations in Λ_{or} preserve points of M together with their normal, while translations in Λ_{unor} may change the sign of the normal.

We make a few remarks about the quotient surface, although the details go beyond the material covered in the present class. Translations in Λ_{or} induce an equivalence relation on $\mathbb{R}^3$: We call two points are called equivalent, iff they differ by a translation in Λ_{or}. The set of equivalence classes forms a 3-torus $T^3 := \mathbb{R}^3/\Lambda_{\mathrm{or}}$, in particular it is compact. Since T^3 is a Riemannian manifold of constant curvature 0, minimality is defined just as in $\mathbb{R}^3$. The quotient manifold D/Λ_{or} now is a compact surface contained in the compact manifold T^3.

Proof. Let us first rule out boundary branch points of M by appealing to Thm. 41. To do that, consider two successive edges e_1, e_2 of Γ, which meet at a vertex v. Then e_1 and e_2 span a plane E, which serves as a barrier along the edges e_1 and e_2. It also serves as a barrier for the two endpoints of e_1 and e_2 different from v.

Let us now continue M by reflection to a smooth immersed surface $\tilde M$. In the plane $\mathbb{R}^2$, two $180°$-rotations about points p and q generate a translation by $2(p-q)$. Similarly, two $180°$-rotations in $\mathbb{R}^3$ about parallel axes generate a translation by twice the orthogonal distance between the axes. Since opposite edges of the cube form rotation axes, this implies that $\tilde M$ is invariant under Λ_{or}.

We need to show D is embedded, that is, the analytic continuation by reflection leads to no two copies of M intersecting eachother. Let us consider the rectangular box

$$B = \{(x,y,z)\in\mathbb{R}^3 : 0\le x<4,\ 0\le y,z<2\}.$$

Then B forms a *fundamental domain* for the lattice Λ_{or}, meaning that the translated copies $B_t := t+B = \{x+t : x\in B\}$ have the properties
(i) if $s\ne t\in\Lambda_{\mathrm{or}}$ then $B_t\cap B_s=\emptyset$ and
(ii) $\bigcup_{t\in\Lambda_{\mathrm{or}}} B_t = \mathbb{R}^3$.
But the closure of B contains eight (of 16) cubes which contain a copy of M, and these are consistent with analytic continuation of M: For this it is sufficient to assert that the 8 respective space diagonals are consistent. Finally, the continuation by translation of B is also consistent with the rotation symmetries. Thus the copies of M in each cube coincide and D is embedded.

To check the genus, let us compute the Euler characteristics. Since eight hexagons tesselate P/Λ_{or}, we have

$$\chi(P/\Lambda_{\mathrm{or}}) = E-K+F = \frac{8\cdot 6}{4}-\frac{8\cdot 6}{2}+8 = 12-24+8=-4 \tag{42}$$

and so from $\chi(P/\Lambda_{\mathrm{or}}) = 2-2g(P/\Lambda_{\mathrm{or}})$ we conclude $g=3$. $\square$

The diamond surface D appears in various natural interface systems. Note that $\mathbb{R}^3\setminus D$ has two components. Any $180°$-rotation exchanges side of the D-surface, and hence will flip the two components.

Remark. As the fundamental hexagon contains additional straight lines, it is possible to generate the surface from a smaller quadrilateral. This quadrilateral consists of two half-edges of Γ, whose one endpoint is connected with the midpoint of the cube.

5.4. **The P-surface.** We now come to the primitive or P-surface, also discovered by Schwarz using Weierstrass data.

Consider the four points in $\mathbb{R}^3$,

$$p_1 = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}, \quad p_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \quad p_3 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}, \quad p_4 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

These points are the vertices of a quadrilateral Γ. Our choice of coordinates for this quadrilateral is such that p_2 and p_4 are midpoints of two cubes of a cube tesselation, while p_1 and p_3 are edge midpoints. Obviously Γ has right angles at p_1 and p_3 (the plane containing a hinge at one of these points is axis-parallel). The angles at p_2 and p_4 are $60°$.

Lemma 45. Γ *bounds a unique embedded minimal surface M; it is symmetric with respect to reflection in the yz-plane.*

Proof. The quadrilateral has an injective projection to a convex quadrilateral in the xy-plane. Thus the Plateau solution is unique and embedded by Radó's theorem. $\square$

Theorem 46. *The continuation by reflection of the Plateau solution M yields an embedded minimal surface P. The surface together with normal is invariant under addition of points in the lattice*

$$\Lambda_{or} := 2\mathbb{Z}^3$$

Its quotient $M = P/\Lambda_{or}$ has genus 3. The surface M itself is also invariant under addition of points in the lattice

$$\Lambda_{unor} := 2\mathbb{Z}^3 \supset \Lambda_{or}.\{m(1,1,1) + n(1,1,-1) + \ell(1,-1,-1) \colon m, n, \ell \in \mathbb{Z}\}.$$

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Proof. We give an outline. The piece M has a reflection symmetry. Consider a symmetric half of M, call it M^+, with boundary in the yz-plane. We may assume $M^+ \subset [0,1]^3$. By construction, M^+ is contained in a "little" cube of edgelength $\frac{1}{2}$, eight copies of which make up for $[0,1]^3$. Let us now reflect M^+ by $180°$-rotation about the two straight edges. This will lead to six copies within $[0,1]^3$, each contained in a little cube of edgelength $1/2$. Consequently, the six copies of M^+ are disjoint.

The result is a minimal surface in $[0,1]^3$ which we call the hexagon M'. Since our original fundamental piece M had a reflection symmetry, we can also relflect the hexagon in any of the six (out of eight) faces which intersect M'. Just as the reflection extends M^+ to M, it will extend M' to a reflected copy of M' (using unique analytic continuation).

Now note that the hexagon M' has the same symmetries as the D-hexagon; thus it is uniquely described by a space diagonal. Thus, similarly to the D-surface we obtain a fundamental domain of the P surface, containing eight copies of the hexagon M', related by reflection in faces of their containing cubes; this time however, relfection leaves no cubes empty.

Since eight copies of the hexagon form a fundamental domain, the calculation (42) is literally valid for M/Λ_{or} and shows it also has genus 3. $\square$

5.5. **Other periodic surfaces generated from polygons.** There are other periodic minimal surfaces, which can be generated by the Plateau solution of a quadrilateral which is continued by $180°$ reflection. These are the Schwarz CLP- and H-surface, as well as Neovius' surface. Schoenflies proved 1891 that there are exactly 6 quadrilaterals whose Plateau solutions generate complete surfaces without branch points. Fischer and Koch studied polygonal boundaries which have more than 4 edges or which are not connected

Diploma theses: 1. Discuss the six Schoenflies surfaces systematically.
2. Study the work of Fischer and Koch: Which of their surfaces can be proven to be embedded?

`41. Lecture, Wednesday 17.5.06`

5.6. **Symmetries, lattices, and crystallographic groups.** Let us review facts about symmetries which are relevant to minimal surfaces.

An *isometry* or *motion* of $\mathbb{R}^n$ is a map φ such that

$$p, q \in \mathbb{R}^n \quad \Rightarrow \quad |\varphi(p) - \varphi(q)| = |p - q|.$$

The set of isometries forms a group $\mathsf{Isom}(n)$ under composition.

Examples of isometries:
1. Translations.
2. Special orthogonal maps $\mathsf{SO}_n := \{A \in \mathrm{End}(\mathbb{R}^n) : \det A = 1\}$ which for $n = 3$ can be identified with rotations (why?).
3. Orthogonal maps $\mathsf{O}_n := \{A \in \mathrm{End}(\mathbb{R}^n) : |\det A| = 1\}$. Maps in $\mathsf{O}_n \setminus \mathsf{SO}_n$ include, for instance, reflections in hyperplanes.

It is a theorem that these examples generate all isometries:
Each motion is the composition of a translation with an orthogonal map.

Let $X \subset \mathbb{R}^n$ be any set. The *symmetry group* of X is the subgroup of those motions in $\mathsf{Isom}(n)$ which preserve X as a set.

Examples of symmetry groups $\subset \mathsf{O}(3)$ for subsets of $\mathbb{R}^3$:
- D_k for a regular k-gon P_k (with vertices on $\mathbb{S}^1$), where $k \geq 2$;
- C_k for $X = P_k \cup (0,0,1)$ for $k \geq 2$; moreover let C_1 be generated by $\pm\,\mathrm{id}$ and reflection in a plane;

• T for a regular tetrahedron,
• O for a regular octahedron or cube (groups coincide by duality),
• I for a regular icosahedron or dodecahedron (dto.).

For G any of the above groups, we also consider the orienatation preserving subgroup $G^+ := G \cap \mathsf{SO}(3)$. Furthermore, we consider a third kind of subgroup $G^* \ni \{C_k^*, D_k^*, T^*\}$, which is generated by the respective group G^+ together with the inversion $\{-\operatorname{id}\}$.

The groups we have mentioned make up for all finite subgroups of $\mathsf{O}(3)$ or *point groups*:

A finite subgroup of $\mathsf{O}(3)$ *is conjugate to one of the following groups:* $D_k \supset D_k^* \supset D_k^+$ *for* $k \geq 2$, $C_k \supset C_k^* \supset C_k^+$ *for* $k \geq 1$, $T \supset T^* \supset T^+$, $O \supset O^+$, $I \supset I^+$. *None of the groups listed is conjugate to another, and no pair is isomorphic as abstract groups.*

Here, a group $G = \{A_1, \ldots, A_k\}$ is conjugate to G', if there exists $X \in \mathsf{O}(n)$ such that $G' = \{X^{-1}A_1X, \ldots, X^{-1}A_kX\}$.

See Chapter 1 of Knörrer, Geometrie (Vieweg 1996) for a detailed account and proof of the classification of point groups.

Remark. It is interesting to investigate the algebra of the point groups. For instance, the group D_k^+ is isomorphic to the cyclic group generated by one element x under the relation $x^k = 1$. The proper point of view then is to view the above symmetry groups as *presentations [Darstellungen]* of abstract groups in the orthogonal group $\mathsf{O}(n)$.

When we deal with symmetry groups of periodic objects, the set of translations will be infinite: A *lattice* Λ of rank $k \leq n$ is a subgroup of $(\mathbb{R}^n, +)$ which is generated by k linearly independent vectors $v_1, \ldots, v_k$, that is,

$$\Lambda = \Big\{ \sum_{i=1}^{k} \lambda_i v_i : \lambda_i \in \mathbb{Z} \Big\}.$$

Examples of lattices with rank 3 *in* $\mathbb{R}^3$:
1. The *primitive cubic lattice* $\mathbb{Z}^3$,
2. the *face centred cubic lattice* (FCC), $\{(k, l, m) \in \mathbb{Z}^3 : k + l + m \in 2\mathbb{Z}\}$,
3. the *body-centred cubic lattice* (BCC), $2\mathbb{Z}^3 \cup \{2\mathbb{Z}^3 + (1, 1, 1)\}$.

The primitive lattice is generated by the edges, the FCC-lattice by face diagonals, and the BCC-lattice by space diagonals of a cube. These *cubic lattices* are distinguished from all other lattices in that they admit symmetries exchanging three linearly independent directions pairwise. We call a surface in $\mathbb{R}^3$ *singly, doubly,* or *triply periodic* if it is invariant under a lattice of rank 1,2, or 3, respectively.

Examples. The helicoid is singly periodic. Scherk's doubly periodic surface is invariant under the planar square lattice and hence doubly periodic. The P and D-surface are triply periodic.

We call a group G of motions *discrete* if for each point $p \in \mathbb{R}^n$ there is $d(p) > 0$ such that if $A \in G$ satisfies $Ap \neq p$ then $|Ap - p| > d(p)$. This property means that distinct images of p under the group (the so-called *orbit* of p) is a *discrete set*, that is, it does not have any accumulation points. A *crystallographic group* is a subgroup of $\mathsf{Isom}(n)$ which contains a lattice of maximal rank n as the subgroup of translations. The following famous fact was established around 1900:

The number of non-isomorphic crystallographic groups of $\mathbb{R}^n$ *is finite for each n; in particular, there exist 17 non-isomorphic planar groups* ($n = 2$) *and* 219 *space groups* ($n = 3$).

This classification can also be viewed as a classification up to certain conjugations and homothety. In any case, the classification leaves free angle parameters of the lattice for the "less symmetric" groups.

The *International Tables of Crystallography* list all these groups; in fact, they list 230 space groups as they admit only the more special conjugation in $\mathsf{SO}(3)$. Crystallographic groups contain point groups as subgroups. Namely, the subgroup which fixes some given point p is a point group. Thus it must be one of the groups we listed above; but only certain of these groups can actually occur: In case $n = 3$, groups from the infinite series can only occur with $k = 1, 2, 3, 4, 6$, and moreover they can be of type T or O, but not I.

References. Wolf, Spaces of constant curvature, Sect. 3.1 and 3.2, contains all results in precise form. For the classification of the planar groups a more popular presentation is Armstrong, Groups and symmetries (Springer 1988). Buser (L'enseignement mathématique 31, p.137-145, 1983) gives a proof that the number of crystallographic groups is finite in any dimension.

Each triply periodic minimal surface M has a pair of crystallographic groups associated: the group which preserves M and its subgroup which preserves M together with its orientation. A 180°-rotation about a straight line in the surface exchanges the normal and hence does only belong to the first group, not to the second.

Example. For P, the crystallographic groups in international notation are $Im\overline{3}m$ and $Pm\overline{3}m$, for D, they are $Pn\overline{3}m$ and $Fd\overline{3}m$.

An open problem is to classify the triply periodic minimal surfaces with a given symmetry group.

`42. Lecture, Tuesday 23.5.06`

Let us conclude this section with a theorem, which sounds surprising on the first sight: A triply periodic surface is a maximum of area when compared to its parallel surfaces! To make this precise recall that a periodic minimal surface has infinite area. Thus it is reasonable to consider the area of a fundamental domain, $A(M/\Lambda)$.

Theorem 47. *Let $M = M_0$ be a complete (connected) triply periodic minimal surface in $\mathbb{R}^3$ with lattice Λ (orientation preserving). Let ν be the Gauss map and consider for sufficiently small $\varepsilon > 0$ the family of parallel surfaces $M_t = \{p + t\nu(p) : p \in M\}$ where $0 < |t| < \varepsilon$. Then*

$$A(M_t/\Lambda) < A(M/\Lambda) \quad \text{for all } t \neq 0.$$

In particular, this means that the surface M_t/Λ is not a local minimum of area in the torus $\mathbb{R}^3/\Lambda$.

Proof. In Corollary II.20 we stated that the parallel surface in distance t to a minimal surface has area

$$A(M_t/\Lambda) = A(M/\Lambda) + \frac{t^2}{2}\int_{M/\Lambda} K\, dS + O(t^3).$$

Since $K \leq 0$, we conclude $A(M_t) \leq A(M)$. In fact, unless M is planar, even $A(M_t) < A(M_0)$. But M is a triply periodic connected minimal surface and hence cannot be planar. □

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Part 4. **The Weierstrass representation**

`43. Lecture, Wednesday 24.5.06`

The formulas we will establish allow to represent each minimal surface by a pair of holomorphic functions. These formulas yield conformal harmonic maps, meaning that once again we work in the class of generalized minimal surfaces.

Many examples were found by systematic guesses of suitable holomorphic functions, in the 19th century, in the 20th, and still up to today. There is also a global version of the representation formulas, where the holomorphic functions are defined on Riemann surfaces. Most naturally, it applies to surfaces of finite total curvature, in which case the Riemann surface is compact. In this form, the Weierstrass formulas also present a powerful tool for analyzing such surfaces.

1. The representation formula

1.1. **Holomorphic null curves.** Let us recall that a map $f \in C^2(U,\mathbb{R}^n)$ is called a a *generalized* or *branched minimal surface* if it a conformally parameterized harmonic mapping, that is

$$|f_x|^2 = |f_y|^2, \quad \langle f_x, f_y\rangle = 0 \qquad \text{and} \qquad \Delta f = 0 \qquad \text{for all} \quad z \in U.$$

Example. Henneberg's surface $f\colon \mathbb{C} \to \mathbb{R}^3$ is a generalized minimal surface, but not a minimal surface

For $U \subset \mathbb{C}$, let us consider smooth mappings $\Phi\colon U \to \mathbb{C}^n$. Being the complex analogue of a curve $c\colon I \to \mathbb{R}^n$, such maps are also called *curves in* $\mathbb{C}^n$. The mapping or curve is called *holomorphic*, if the Cauchy Riemann equations hold componentwise, that is for $\partial_{\bar z}\Phi := \frac{1}{2}(\partial_x\Phi + i\,\partial_y\Phi)$ we have $\partial_{\bar z}\Phi = 0$, see III(26).

Definition. A curve $\Phi = (\Phi_1,\dots,\Phi_n)\colon U \to \mathbb{C}^n$ is called a *null curve* if

$$\Phi_1^2 + \ldots + \Phi_n^2 = 0. \tag{1}$$

Here Φ_k^2 denotes the standard complex product,

$$\Phi_k^2 = (\operatorname{Re}\Phi_k + i\operatorname{Im}\Phi_k)^2 = \operatorname{Re}^2\Phi_k - \operatorname{Im}^2\Phi_k + 2i\operatorname{Re}\Phi_k\operatorname{Im}\Phi_k.$$

Example. $(z_1,z_2) = (1,i) \in \mathbb{C}^2$ has $z_1^2 + z_2^2 = 1^2 + i^2 = 0$.

Another notation for (1) arises from the standard scalar product $\langle .,.\rangle$ on $\mathbb{R}^n$, extended in complex linear way to a bilinear form $\langle\langle .,.\rangle\rangle$ on $\mathbb{C}^n$,

$$\langle\langle a+ib, c+id\rangle\rangle := \langle a,c\rangle - \langle b,d\rangle + i\big(\langle b,c\rangle + \langle a,d\rangle\big) \qquad \text{for} \quad a,b,c,d \in \mathbb{R}^n.$$

Then indeed

$$\langle\langle\Phi,\Phi\rangle\rangle = \langle\operatorname{Re}\Phi,\operatorname{Re}\Phi\rangle - \langle\operatorname{Im}\Phi,\operatorname{Im}\Phi\rangle + 2i\langle\operatorname{Re}\Phi,\operatorname{Im}\Phi\rangle = \sum_{k=1}^n \Phi_k^2.$$

Note, however, that the extended scalar product does not define a norm –the null curve condition is not equivalent to $\Phi \equiv 0$. A norm will only arise from the skew-symmetric Hermitian scalar product on $\mathbb{C}^n$.

We can now characterize generalized minimal surfaces f in terms of $\partial_z f := \frac{1}{2}(\partial_x f - i\partial_y f)$:

Proposition 1. *Consider a C^2-mapping $f\colon U \to \mathbb{R}^n$ and set $\Phi := \partial_z f$. Then*
(i) Φ is holomorphic iff f is harmonic and
(ii) Φ is a null curve iff f is conformal.

Proof. (*i*) We need to show $\partial_{\bar z}\Phi = 0 \Leftrightarrow \Delta f = 0$. But

$$\partial_{\bar z}\Phi = \partial_{\bar z}\partial_z f = \frac{1}{4}(\partial_x + i\,\partial_y)(\partial_x - i\partial_y)f = \frac{1}{4}\big(\partial_{xx} + \partial_{yy}\big)f = \frac{1}{4}\Delta f. \tag{2}$$

(*ii*) Using the extended scalar product, we can write

$$4\langle\langle\Phi,\Phi\rangle\rangle = \langle\langle\partial_x f - i\partial_y f,\, \partial_x f - i\partial_y f\rangle\rangle = |\partial_x f|^2 - |\partial_y f|^2 - 2i\langle\partial_x f, \partial_y f\rangle.$$

(Compare with the function ψ of III(25)). □

The proposition indicates the following approach to finding minimal surfaces:
1. Specify holomorphic null curves Φ in terms of holomorphic functions (easy).
2. Integrate holomorphic null curves Φ to obtain minimal surfaces f.

1.2. **Cauchy's theorem.** To fulfill the second task mentioned before, we recall facts from complex analysis.

First, let us express holomorphicity in real terms. Suppose $f\colon U^2 \to \mathbb{R}^2$ is partially differentiable. Then

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \partial_x f(z)$$

and

$$\lim_{h\to 0}\frac{f(z+ih)-f(z)}{ih} = \frac{1}{i}\partial_y f(z) = -i\partial_y f(z).$$

But if f is holomorphic, these limits agree and are denoted by

$$f'(z) := \partial_x f(z) = -i\partial_y f(z). \tag{3}$$

Clearly, as $i\binom{x}{y} = \binom{-y}{x}$ this represents the Cauchy-Riemann equations: $\binom{\partial_x \operatorname{Re} f}{\partial_x \operatorname{Im} f} = \binom{\partial_y \operatorname{Im} f}{-\partial_y \operatorname{Re} f}$

Remark. Upon addition we find

$$f'(z) = \frac{1}{2}\big(\partial_x f(z) - i\partial_y f(z)\big).$$

In general, for $f\colon U \to \mathbb{R}^2$ not necessarily holomorphic, the shorthand notation $\partial_z f$ for the right side is used. Then we can consider $\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar z}$, a different choice of basis for the tangent space; this basis is favourable when studying holomorphicity.

Let us recall from complex analysis the notion of a a *path integral*. Suppose $\gamma\colon [a,b] \to U \subset \mathbb{C}$ is differentiable and $\varphi\colon U \to \mathbb{C}$. Then, assuming integrability,

$$\int_\gamma \varphi(w)\,dw := \int_a^b \varphi\big(\gamma(t)\big)\gamma'(t)\,dt,$$

where the right hand side is in terms of the standard complex product.

Theorem 2. *Suppose $f\colon U \to \mathbb{C}$ is holomorphic. Then for any path $\gamma\colon [0,1] \to U$ with $\gamma(0) = z_0$, $\gamma(1) = z$*

$$\int_\gamma f'(w)\,dw = f(z) - f(z_0);$$

in particular, the path integral depends on the endpoints of γ alone.

Proof. We prove whe result by direct computation, using real notation:

$$\begin{aligned}
f(z) - f(w) &\overset{\text{fund.thm.calc.}}{=} \int_0^1 \frac{d}{dt} f\big(\gamma(t)\big)\,dt = \int_0^1 \begin{pmatrix} \frac{d}{dt}\operatorname{Re} f\big(\gamma(t)\big) \\ \frac{d}{dt}\operatorname{Im} f\big(\gamma(t)\big)\end{pmatrix} dt \\
&\overset{\text{chain rule}}{=} \int_0^1 \begin{pmatrix} \partial_x \operatorname{Re} f\big(\gamma(t)\big)\operatorname{Re}\gamma'(t) + \partial_y \operatorname{Re} f\big(\gamma(t)\big)\operatorname{Im}\gamma'(t) \\ \partial_x \operatorname{Im} f\big(\gamma(t)\big)\operatorname{Re}\gamma'(t) + \partial_y \operatorname{Im} f\big(\gamma(t)\big)\operatorname{Im}\gamma'(t)\end{pmatrix} dt \\
&\overset{\text{Cauchy-Riemann}}{=} \int_0^1 \begin{pmatrix} \partial_x \operatorname{Re} f\big(\gamma(t)\big)\operatorname{Re}\gamma'(t) - \partial_x \operatorname{Im} f\big(\gamma(t)\big)\operatorname{Im}\gamma'(t) \\ \partial_x \operatorname{Im} f\big(\gamma(t)\big)\operatorname{Re}\gamma'(t) + \partial_x \operatorname{Re} f\big(\gamma(t)\big)\operatorname{Im}\gamma'(t)\end{pmatrix} dt
\end{aligned}$$

Now recall the formula

$$zw = (\operatorname{Re} z + i\operatorname{Im} z)(\operatorname{Re} w + i\operatorname{Im} w) = (\operatorname{Re} z\operatorname{Re} w - \operatorname{Im} z\operatorname{Im} w) + i(\operatorname{Re} z\operatorname{Im} w + \operatorname{Im} z\operatorname{Re} w)$$

to see that

$$\begin{aligned}
f(z) - f(z_0) &= \int_0^1 \Big(\partial_x \operatorname{Re} f\big(\gamma(t)\big) + i\,\partial_x \operatorname{Im} f\big(\gamma(t)\big)\Big)\big(\operatorname{Re}\gamma'(t) + i\operatorname{Im}\gamma'(t)\big)\,dt \\
&= \int_0^1 \partial_x f\big(\gamma(t)\big)\gamma'(t)\,dt \overset{(3)}{=} \int_0^1 f'\big(\gamma(t)\big)\gamma'(t)\,dt = \int_\gamma f'(w)\,dw. \qquad \square
\end{aligned}$$

Clearly, the more interesting direction is the converse: If we are given a holomorphic function φ and consider $\int_\gamma \varphi(w)\,dw$, does it define an antiderivative f of φ?

Example. The function $\varphi(z) = \frac{1}{z}$ is holomorphic over $\mathbb{C}\backslash\{0\}$, but Cauchy's integral $\int \frac{1}{z}\,dz$ is not well-defined. However, there are local antiderivatives of φ: As long as the argument is well-defined over $U \subset \mathbb{C}\setminus\{0\}$, the function $\log z = \log|z| + i\arg z = \int_1^z \frac{1}{z}\,dz$ is welldefined,

Thus we need the appropriate condition for U:

Definition. Let $U \subset \mathbb{R}^n$ be a domain, that is open and connected. Then U is *simply connected*, if for each closed continuous curve $h_0\colon [0,1] \to U$ with $c(0) = c(1) =: p$ extends to a homotopy $h = h_s(t)\colon [0,1]\times[0,1] \to U$, such that h_s is closed, $h_s(0) = h_s(1) = p$ for all s, and it contracts to p, that is, $h_1(t) = p$ for all t.

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Thus in a simply connected domain there is no obstacle to shrinking any given closed curve h_0 to a point. A particular example of simply connected domains are star-shaped domains: if U is star-shaped with respect to 0, then the homotopy can be given explicitely as $h_s(t) = sh_0(t)$.

Example. An annulus or $\mathbb{C}\setminus\{0\}$ are not simply connected. On the other hand the slit domain $\mathbb{C}\setminus\{z = (x,0) : x \geq 0\}$ is star-shaped and simply connected; the set $\{z \in \mathbb{C} : 0 < r < |z| < R, z \neq (x,0) \text{ for } x > 0\}$ representing an annulus without a slit is also simply connected but not star-shaped.

We can now state Cauchy's theorem in general form:

Theorem 3. *Suppose $\varphi\colon U \to \mathbb{C}$ is holomorphic and U is simply connected. Then*

$$f(z) := \int_\gamma \varphi(w)\,dw := \int_0^1 \varphi(\gamma(t))\gamma'(t)\,dt, \qquad \textit{for } \gamma \textit{ a path with } \gamma(0) = z_0,\ \gamma(1) = z,$$

is well-defined and holomorphic; moreover f is an antiderivative of φ, that is, $f' = \varphi$.

1.3. **The local Weierstrass representation.** Applying Cauchy's theorem componentwise we obtain:

Theorem 4. *Let U be simply connected and Φ a holomorphic null curve. Then*

$$f\colon U \to \mathbb{R}^n, \qquad f(z) := \operatorname{Re}\int_{z_0}^z \Phi(w)\,dw, \tag{4}$$

defines a conformal harmonic map.

The converse is also true, even without assuming that U is simply connected: Any conformal harmonic map $f\colon U \to \mathbb{R}^n$ with $\Phi = \partial_z f$ can be represented by (4) up to adding a constant.

Proof. Cauchy's Thm. 3 gives that $F(z) := \int_{z_0}^{z} \Phi(w)\,dw$ is holomorphic in each component. Thus the real part $f = \operatorname{Re} F$ is harmonic in each component. By Thm. 3 we also have $\partial_z F = \Phi$. Since Φ is assumed to be null, it follows from Prop. 1 that F is conformal, in particular $f = \operatorname{Re} F$ is conformal. $\square$

We now turn to the task to choose holomorphic null curves in order to determine new minimal surfaces.

A triple $a, b, c \in \mathbb{N}$ is called Pythagorean, if $a^2 + b^2 = c^2$. For any pair of natural numbers $m < n$, the triple $a = n^2 - m^2$, $b = 2mn$, $c = n^2 + m^2$ clearly is Pythagorean; in fact all Pythagorean triples arise that way (why?).

For dimension $n = 3$, we now want to represent Φ with $\Phi_1^2 + \Phi_2^2 + \Phi_3^2 = 0$ similarly. This gives the local version of the Weierstrass representation formula.

Recall that a function $g\colon U \to \overline{\mathbb{C}}$ is *meromorphic*, if for each $z \in U$ either $g(z) \in \mathbb{C}$ holds or else $\frac{1}{g(z)}$ has a zero of order $k \in \mathbb{N}$. That is, at each $z_0 \in U$ the function g has a Laurent expansion

$$g(z - z_0) = a_{-k}(z - z_0)^{-k} + \ldots + a_0 + a_1(z - z_0) + \ldots$$

with $k \in \mathbb{N}$. If the leading coefficient a_{-k} is nonzero, we say g has a *pole* of order k at z_0.

Theorem 5 (Weierstrass). *Suppose $h\colon U \to \mathbb{C}$ is holomorphic and $g\colon U \to \overline{\mathbb{C}}$ is meromorphic such that at each zero or pole of order k of g the function h has a zero of order at least k. Then*

$$\Phi = (\Phi_1, \Phi_2, \Phi_3) = h\left(\frac{1}{2}\Big(\frac{1}{g} - g\Big), \frac{i}{2}\Big(\frac{1}{g} + g\Big), 1\right) \tag{5}$$

is a holomorphic null curve and consequently for fixed $z_0 \in U$

$$f(z) = \operatorname{Re} \int_{z_0}^{z} \Phi(w)\,dw$$

is a branched minimal surface. It has no no branch points if at each zero of order k of h the function g has a pole of order k. Moreover, any holomorphic null curve $\Phi\colon U \to \mathbb{C}^3$ can be represented this way.

We call g, h the *Weierstrass data* of the minimal surface f.

Proof. Let us check the null curve condition:

$$\Phi_1^2 + \Phi_2^2 + \Phi_3^2 = h^2\left(\frac{1}{4}\Big(\frac{1}{g^2} + g^2 - 2\Big) - \frac{1}{4}\Big(\frac{1}{g^2} + g^2 + 2\Big) + 1\right) = h^2(-1 + 1) = 0$$

The condition on the zeros of h implies that Φ is holomorphic.

Let us now check that any null curve Φ admits a representation (5). We rewrite the null condition as

$$0 = \Phi_1^2 + \Phi_2^2 + \Phi_3^2 = (\Phi_1 + i\Phi_2)(\Phi_1 - i\Phi_2) + \Phi_3^2. \tag{6}$$

We set

$$g := -\frac{\Phi_1 + i\Phi_2}{\Phi_3} \qquad h := \Phi_3. \tag{7}$$

Since $\Phi_1 + i\Phi_2 = -gh$ then (6) becomes equivalent to $\Phi_1 - i\Phi_2 = \frac{1}{g}h$. Equivalently,

$$2\Phi_1 = -gh + \frac{1}{g}h, \qquad 2i\Phi_2 = -gh - \frac{1}{g}h.$$

Finally $\Phi_1(z) = \Phi_2(z) = \Phi_3(z) = 0$ if and only if $h(z) = 0$. Thus for $h(z) \neq 0$ we have $f'(z) = \Phi(z) \neq 0$, and so the holomorphic map f must have rank 2, meaning it is immersion. $\square$

This represents all branched minimal surfaces in terms of a pair of functions g, h as

$$f(z) = \operatorname{Re} \int_{z_0}^{z} \left(\frac{1}{2}\Big(\frac{1}{g} - g\Big), \frac{i}{2}\Big(\frac{1}{g} + g\Big), 1\right) h\,dw.$$

Remarks. 1. The representation is local, that is, on simply connected domains. We would need to introduce a universal covering if we were to drop this assumption.
2. Many books state the following version of the Weierstrass representation

$$\Phi = \big(1 - \tilde{g}^2, i(1 + \tilde{g}^2), 2\tilde{g}\big)\tilde{h}. \tag{8}$$

This time $h := \Phi_1 - \Phi_2$ and $g := \frac{\Phi_3}{\Phi_1 - i\Phi_2}$. Our choice (5), however, is more geometric, as the next Proposition will show.
3. If $\tilde{g}'(z) \neq 0$ holds, we may compose $\tilde{g}, \tilde{h}$ with $\tilde{g}^{-1}$ to obtain the representation

$$\Phi = \big(1 - z^2, i(1 + z^2), 2z\big)R(z)$$

depending on solely one holomorphic function R. Note that this removes the conformal invariance from the Weierstrass representation: While Φ has three components, or three degrees of freedom, the null (or conformality) condition removes one, and the choice of conformal parameterization removes another. Thus ultimately a surface is described by just one holomorphic function.

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Examples. 1. For Enneper's surface (Sect. II 2.3.) the Weierstrass data are

$$U = \mathbb{C}, \qquad g(z) = z, \qquad h(z) = 2z,$$

so that

$$\Phi(z) = \Big(1 - z^2, i(1 + z^2), 2z\Big)$$

which gives upon integration from 0 to z:

$$f(z) = \operatorname{Re}\Big(z - \frac{1}{3}z^3, i(z + \frac{1}{3}z^3), z^2\Big) = \Big(x - \frac{1}{3}x^3 + xy^2, -y + \frac{1}{3}y^3 - x^2y, x^2 - y^2\Big),$$

using that $\frac{1}{3}(x+iy)^3 = \frac{1}{3}x^3 + ix^2y - xy^2 - \frac{i}{3}y^3$.

2. The catenoid (Sect. II 2.1) has Weierstrass data

$$U = \mathbb{C} \setminus \{(x,0) : x \ge 0\}, \qquad g(z) = z, \qquad h(z) = \frac{1}{z},$$

so that

$$\Phi(z) = \Big(\frac{1}{2}(\frac{1}{z^2} - 1), \frac{i}{2}(\frac{1}{z^2} + 1), \frac{1}{z}\Big).$$

Upon integration we find

$$f(z) = \operatorname{Re}\Big(\frac{1}{2}(-\frac{1}{z} - z),\ \frac{i}{2}(\frac{1}{z} + z),\ \log z\Big),$$

that is, in polar coordinates $z = \rho e^{i\varphi}$,

$$\begin{aligned} f(\rho e^{i\varphi}) &= \operatorname{Re}\Big(\frac{1}{2}(-\frac{1}{\rho}e^{-i\varphi} - \rho e^{i\varphi}), \frac{i}{2}(-\frac{1}{\rho}e^{-i\varphi} + \rho e^{i\varphi}), \log(\rho e^{i\varphi})\Big) \\ &= \Big(\frac{-1}{2}(\frac{1}{\rho} + \rho)\cos\varphi, \frac{-1}{2}(\frac{1}{\rho} + \rho)\sin\varphi, \log\rho\Big) \end{aligned}$$

(recall that $\log z = \log|z| + i\arg z$). Now setting $r = e^\rho$ we obtain

$$f(\log r e^{i\varphi}) = \big(-\cosh r\cos\varphi, -\cosh r\sin\varphi, r\big),$$

which we recognize as the parameterization of the catenoid (up to a 180° rotation about the z-axis). While the Weierstrass data and Φ are defined on $\mathbb{C}^*$, the integral $\int\Phi$ does not continuously extend to the positive real axis. Nevertheless, inspection shows that the parameterization $f = \operatorname{Re}\int\Phi$ does extend.

Remark. If we were to declare Φ on the universal covering of $\mathbb{C}^*$ then $\int\Phi$ and f were well-defined.

Remark. Any minimal surface can locally be represented with Weierstrass data. We will not prove this fact here; the proof involves the Poincaré lemma or an equivalent tool. See, e.g., [HK]. The crucial step of this problem is to construct for a harmonic function h on a simply connected domain a so-called conjugate function h^*, such that $h + ih^*$ becomes holomorphic.

1.4. **Gauss map, first fundamental form, Gauss curvature.** Let us introduce the *stereographic projection*

$$(9) \qquad \mathrm{st}\colon \overline{\mathbb{C}} \to \mathbb{S}^2, \qquad \mathrm{st}(z) := \begin{cases} \frac{1}{|z|^2+1}\big(2\operatorname{Re} z, 2\operatorname{Im} z, |z|^2 - 1\big) & \text{for } z \in \mathbb{C}, \\ (0,0,1) & \text{for } z = \infty. \end{cases}$$

To derive (9), let $N := (0,0,1)$ be the north pole. A projection by straight lines means $\mathrm{st}(z) = \lambda z + (1-\lambda)N$ with $\lambda \ne 0$. It maps to $\mathbb{S}^2$ iff

$$1 \overset{!}{=} |\operatorname{st}(z)|^2 = \lambda^2|z|^2 + (1-\lambda)^2 = 1 - 2\lambda + \lambda^2\big(|z|^2 + 1\big).$$

Thus $0 = -2 + \lambda(|z|^2 + 1) \Longrightarrow \lambda = \frac{2}{|z|^2+1}$, so that indeed (9) follows. It can be checked that st is a diffeomorphism, and st is conformal (see problems).

We can now explain the geometric meaning of g, h:

Proposition 6. *Let $f\colon U \to \mathbb{R}^3$ be a minimal surface with Weierstrass data g, h. Then:*
(i) g is stereographic projection of the Gauss map ν of f, that is,

$$\nu = \mathrm{st}(g) = \frac{1}{|g|^2 + 1}\Big(2\operatorname{Re} g, 2\operatorname{Im} g, |g|^2 - 1\Big).$$

Moreover, ν is conformal.
(ii) The function $\operatorname{Re} h$ is the derivative of the height function f_3, that is $f_3' = \operatorname{Re} h$.

Proof. (i) It remains to show that $st(g)$ is perpendicular to the tangent space. From

$$\partial_x f - i\partial_y f = \partial_z f = \Phi = \operatorname{Re}\Phi + i\operatorname{Im}\Phi$$

we conclude that

$$(10) \qquad \partial_x f = \operatorname{Re}\Phi \qquad \text{and} \qquad \partial_y f = \operatorname{Im}\Phi,$$

meaning that real and imaginary parts of Φ span the tangent space. Noting that $\mathrm{st}(g)$ is a real vector, we compute

$$\begin{aligned} \langle\operatorname{Re}\Phi, \mathrm{st}(g)\rangle + i\langle\operatorname{Im}\Phi, \mathrm{st}(g)\rangle &= \big\langle\big\langle\Phi, \mathrm{st}(g)\big\rangle\big\rangle \\ &= \bigg\langle\bigg\langle h\Big(\frac{1}{2}\Big(\frac{1}{g} - g\Big), \frac{i}{2}\Big(\frac{1}{g} + g\Big), 1\Big),\ \frac{1}{|g|^2+1}\Big(2\operatorname{Re} g, 2\operatorname{Im} g, |g|^2 - 1\Big)\bigg\rangle\bigg\rangle \\ &= \frac{h}{|g|^2+1}\bigg((\operatorname{Re} g)\Big(\frac{1}{g} - g\Big) + i(\operatorname{Im} g)\Big(\frac{1}{g} + g\Big) + |g|^2 - 1\bigg) \\ &= \frac{h}{|g|^2+1}\Big(\frac{g}{g} - g\overline{g} + |g|^2 - 1\Big) = 0. \end{aligned}$$

Thus $\partial_x f$ and $\partial_y f$ are perpendicular to $\mathrm{st}(g)$.

Conformality follows from the fact that g is conformal and st is conformal.

(ii) is immediate from $\partial_z f = \operatorname{Re}\Phi$. ☐

Examples. The catenoid and Enneper surface have $g(z) = z$, so that $\nu(z) = \mathrm{st}(z)$. Thus they are parameterized by (the stereographic projection of) the Gauss map. In particular, their Gauss image is $\mathbb{S}^2$ up to sets of measure zero, and so the total curvature is $-A(\mathbb{S}^2) = -4\pi$.

We can now express the differential geometry of f in terms of the Weierstrass data:

Proposition 7. (i) *The first fundamental form of f is*

$$ds^2(X,Y) := \Big(\frac{1}{2}\big(|g| + \frac{1}{|g|}\big)|h|\Big)^2 \langle X, Y\rangle \tag{11}$$

(Since the letter g is reserved for Weierstrass data, we use ds^2 for the first fundamental form instead.)
(ii) *The Gauss curvature of f is*

$$K = \frac{-16}{(|g| + \frac{1}{|g|})^4}\left|\frac{g'}{g}\right|^2 \frac{1}{|h|^2}. \tag{12}$$

Proof. (i) Since f is conformal, we have

$$|\partial_x f|^2 = |\partial_y f|^2 \overset{(10)}{=} |\operatorname{Re}\Phi|^2 = |\operatorname{Im}\Phi|^2,$$

and so $|\Phi| = \sqrt{2}|\operatorname{Re}\Phi|$. Our claim is now verified by calculation:

$$\begin{aligned} 2|\partial_x f|^2 &= |\Phi|^2 = \Phi_1\overline{\Phi}_1 + \Phi_2\overline{\Phi}_2 + \Phi_3\overline{\Phi}_3 \\ &= |h|^2\Big(\frac{1}{4}\Big(\frac{1}{g} - g\Big)\Big(\frac{1}{\overline{g}} - \overline{g}\Big) + \frac{1}{4}\Big(\frac{1}{g} + g\Big)\Big(\frac{1}{\overline{g}} + \overline{g}\Big) + 1\Big) \\ &= |h|^2\frac{1}{2}\Big(\frac{1}{|g|^2} + |g|^2 + 2\Big) = \frac{1}{2}|h|^2\Big(|g| + \frac{1}{|g|}\Big)^2 \end{aligned}$$

(ii) Recall from II(6) that the Gauss curvature can be written as

$$K = \det S = \det\big(-(df)^{-1}d\nu\big) = \frac{\det d\nu}{\det df}.$$

Now f and the Gauss map ν are conformal. Thus the determinants are given by squares of vectors: $\det df = |\partial_x f|^2 = |\partial_y f|^2$, and similarly so for $\det \partial\nu$ with a minus sign. We conclude

$$K = -\frac{|\partial_x \nu|^2}{|\partial_x f|^2}.$$

Now we use the fact that stereographic projection has a conformal factor $4(1+|g|^2)^{-2}$; this must be multiplied with $|g'|^2$. Together with (11) this gives the desired expression

$$K = -|g'|^2\frac{4}{(1+|g|^2)^2}\,\frac{4}{(|g| + \frac{1}{|g|})^2|h|^2} = -\frac{|g'|^2}{|h|^2|g|^2}\frac{4}{(|g| + \frac{1}{|g|})^2}\,\frac{4}{(|g| + \frac{1}{|g|})^2}. \qquad \square$$

`46. Lecture, Tuesday 6.6.06`

1.5. Second fundamental form and symmetry properties.

Proposition 8. *The second fundamental form is*

$$b(X,X) = -\operatorname{Re}\Big(\frac{g'}{g}hX^2\Big). \tag{13}$$

Consequently, X is an asymptote direction iff $\frac{g'}{g}hX^2 \in i\mathbb{R}$ and X is a principal curvature direction iff $\frac{g'}{g}hX^2 \in \mathbb{R}$.

Proof. By definition, see I.(16), the second fundamental form is

$$b(X,X) = \langle d^2 f(X,X), \nu\rangle = \langle \operatorname{Re}\Phi' X^2, \nu\rangle$$

Let us differentiate using the product rule:

$$\Phi' = h\,\frac{\Phi}{h} = h\Big(\frac{1}{2}\Big(\frac{-g'}{g^2} - g'\Big), \frac{i}{2}\Big(\frac{-g'}{g^2} + g'\Big), 0\Big) + h'\frac{\Phi}{h}.$$

Thus, using that $\nu = \mathrm{st}(g) \perp \Phi$,

$$\begin{aligned} b(X,X) &= \big\langle \operatorname{Re}\Phi' X^2, \mathrm{st}(g)\big\rangle \\ &= \operatorname{Re}\Big(h\frac{g'}{g}X^2\Big\langle\Big\langle\Big(\frac{1}{2}\Big(\frac{-1}{g} - g\Big), \frac{i}{2}\Big(\frac{-1}{g} + g\Big), 0\Big), \frac{1}{|g|^2+1}\Big(2\operatorname{Re} g, 2\operatorname{Im} g, |g|^2 - 1\Big)\Big\rangle\Big\rangle\Big) \\ &= \operatorname{Re}\Big(h\frac{g'}{g}X^2\frac{1}{|g|^2+1}\Big(\frac{-g}{g} - \overline{g}g\Big)\Big) = -\operatorname{Re}\Big(h\frac{g'}{g}X^2\Big). \end{aligned}$$

$\square$

In the examples of surfaces constructed with Weierstrass data, it will be crucial to know the symmetry properties of special curves. We need a few facts from the differential geometry of surfaces.

Definition. Let $f\colon U^n \to \mathbb{R}^{n+1}$ be a hypersurface. A *geodesic* is a curve $\gamma\colon I \to U$ whose image curve $c := f\circ\gamma$ has constant speed such that the curvature vector is normal, that is,

$$c''(t) \in N_{c(t)}f := \{T_{c(t)}f\}^\perp. \tag{14}$$

Examples. 1. Great circles of $\mathbb{S}^2$ are geodesics.
2. Straight lines in $\mathbb{R}^2 \subset \mathbb{R}^3$ are geodesics.
3. More generally, the meridian of any surface of revolution is a geodesic. The only latitude circles which are geodesics are those whose radius is critical for the radius.

The condition (14) says a geodesic does not curve measured within the surface; all its curvature serves to let it stay inside the surface. Therefore, it is not surprising that there is also an intrinsic characterization of geodesics, namely as curves which minimize length on small intervals. It is a fact which we do not want to establish here that geodesics are the solution of an ordinary equation with smooth coefficients, and so are uniquely determined by initial values $c(t_0)$ and $c'(t_0)$. Moreover, the coefficients of the equation depend only on the first fundamental form of f (and its first and second derivatives). Recall that maps which preserve the first fundamental form are called isometries. It follows that geodesics are invariant under isometries.

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In the following examples, we will use these facts to realize that rather special curves are geodesics:

Lemma 9. *Let γ be a non-constant curve which parameterizes a subset of a line L in $U \subset \mathbb{C}$. Suppose that $g \circ \gamma$ and $h \circ \gamma$ are each contained in a line through the origin. Then, up to parameterization, γ is a geodesic for f.*

Proof. Let σ be the reflection across L. Since σ is linear, $d\sigma = \sigma$, and so $\langle X, X\rangle = \langle d\sigma(X), d\sigma(X)\rangle$.

Let now ρ_1, ρ_2 be reflection about the straight line of the images of g, h, respectively. Then g and $\rho_1 \circ g \circ \sigma$ are holomorphic and agree along L. But they are analytic and so must coincide, meaning that g respects the symmetry, $g = \rho_1 \circ g \circ \sigma$. Similarly for h.

Now $|\rho_i(z)| = |z|$ for reflection about a line. We conclude

$$
\begin{aligned}
ds_z^2(X,X) &= \left(\frac{1}{2}\Big(|g| + \frac{1}{|g|}\Big)|h|\right)^2 \langle X, X\rangle = \left(\frac{1}{2}\Big(|\rho_1(g)| + \frac{1}{|\rho_1(g)|}\Big)|\rho_2(h)|\right)^2 \langle X, X\rangle \\
&= \left(\frac{1}{2}\Big(|g\circ\sigma| + \frac{1}{|g\circ\sigma|}\Big)|h\circ\sigma|\right)^2 \langle d\sigma(X), d\sigma(X)\rangle = ds^2_{\sigma(z)}(d\sigma(X), d\sigma(X)).
\end{aligned}
$$

This means that σ is an isometry for f. Hence it maps geodesics onto geodesics.

Consider now a geodesic δ whose initial conditions agree at some point with γ, that is, the initial conditions are $\gamma(t_0)$ and $\gamma'(t_0) \neq 0$. On the one hand, this condition remains invariant under σ. On the other hand, L is the only curve which is invariant under σ. Thus by the uniqueness of geodesics we have $\delta = \delta \circ \sigma$ and so δ must agree with γ up to parameterization. This shows γ is geodesic. □

We have the following characterization of symmetry curves, that is, curves which, in the context of minimal surfaces, allow for applying the Schwarz reflection principles Thms. 37 and th:mirrorreflminimal.

Lemma 10. *(i) A nonconstant curve c parameterizes a straight line of $\mathbb{R}^{n+1}$ if and only if c is a geodesic of the surface f and asymptotic line.*
(ii) Assume that $c'' \neq 0$. Then c is an arc of planar reflection, that is, c and $\nu \circ c$ are contained in a hyperplane of $\mathbb{R}^{n+1}$, if and only if c is a geodesic of the surface f and curvature line.

Suppose f is minimal. Then in the first case the surface has a 180° rotational symmetry about c. In the second case, a version of the Schwarz reflection applies to show that the surface has a mirror symmetry in the hyperplane containing c.

Proof. (i) Clearly, if c is straight then $c'' = 0$ and so it is geodesic. Moreover, for a straight line, the derivative of ν is always orthogonal to c', that is, $g(Sc', c') = b(c', c') = 0$, and so it is asymptotic.

Let us now prove the converse. On the one hand, c is geodesic and so $c'' \parallel \nu$. On the other hand, c is an asymptote line, meaning $\frac{d}{dt}\nu \perp c'$. Using this in

$$\langle \nu, c'\rangle = 0 \quad \Rightarrow \quad \langle \nu, c''\rangle + \langle \nu', c'\rangle = 0$$

we obtain that the second term vanishes, so that $\nu \perp c''$. We conclude that $c'' = 0$ for each t meaning that c is a straight line.

(ii) The proof is similar and left as an exercise. □

1.6. **Schwarz reflection for holomorphic functions.** We discussed Schwarz reflection for harmonic functions in III 3.5, and for minimal surfaces in III 5.1. For holomorphic functions the principle reads:

Theorem 11 (Schwarz reflection principle)**.** *Let $U^+ \subset \mathbb{C}$ be an open domain of the halfspace $\{\operatorname{Im} z > 0\}$ whose boundary contains an open subset T of the hyperplane $\{\operatorname{Im} z = 0\}$. Suppose $f \in C^\infty(U^+, \mathbb{C}) \cap C^0(U^+ \cup T, \mathbb{C})$ is holomorphic on U^+ with boundary values $\operatorname{Im} f|_T = 0$. Denote by $U^- := \{z \in \mathbb{C} : \overline{z} \in U^+\}$ the reflection of U^+ across T. Then the extension of f by odd reflection,*

$$\tilde f(z) := \begin{cases} f(z) & \text{if } \operatorname{Im} z \geq 0, \\ \overline{f(\overline{z})} & \text{if } \operatorname{Im} z < 0, \end{cases} \tag{15}$$

is in $C^\infty(U^+ \cup T \cup U^-)$ and holomorphic.

The same holds if T instead is contained in an arbitrary line L_1 through the origin, and if $u|_T$ is contained in some line L_2 through the origin. Indeed, this follows from the theorem by composition with suitable rotations.

Geometically, (15) means that we precompose f with the reflection $z \mapsto \overline{z}$ about T in the domain and postcompose with another reflection $f \mapsto \overline{f}$ about $f(T)$ in the range. If T or $f(T)$ are subset of $\mathbb{S}^1$ we can replace the mirror reflection in domain or range by the inversion in $\mathbb{S}^1$ given by $z \mapsto \frac{1}{\overline{z}}$.

Proof. It is straightforward to check that on the reflected image U^- of U^+ the extension (15) preserves the Cauchy-Riemann equations. Indeed, for $\operatorname{Im} z < 0$ we have

$$\partial_x \tilde{u}(z) = \partial_x(u(\overline{z})) \overset{\text{chain rule}}{=} (\partial_x u)(\overline{z}), \qquad \partial_y \tilde{v}(z) = -\partial_x(v(\overline{z})) = (\partial_x v)(\overline{z});$$

so that the Cauchy-Riemann equation on U^+ implies the equation on U^-. Similarly for the other equation.

Clearly, the extension $\tilde{f}$ is continuous along T. To show it is smooth along T, consider the imaginary part $\operatorname{Im} f$. It is harmonic, and the extension $\operatorname{Im} \tilde{f}$ is harmonic and hence smooth by the Schwarz reflection principle Thm. 27 for harmonic functions. By the Cauchy-Riemann equations this means the real part is also smooth, hence $\tilde{f}$ is smooth. □

Remark. A harmonic function to reflect vanishes on the boundary and is extended by odd reflection. A minimal surface to reflect in the z-axis has first two components which vanish on the boundary (odd reflection), and one which does not vanish (even reflection). The holomorphic function with zero imaginary part has one component which vanishes (namely the imaginary part) and is extended by odd reflection, and one other (namely the real part) which does not vanish and is extended by even reflection.

2. Surfaces constructed from Weierstrass data

We construct three examples of minimal surfaces. For the third example we also need to introduce some theory of elliptic functions. This section is based on [Ka].

2.1. **Higher order Enneper surfaces.** Let $k \geq 1$; the case $k = 1$ corresponds to the classical Enneper surface. The Weierstrass data are

$$U = \mathbb{C} \setminus \{0\}, \qquad g(z) = z^k, \qquad h(z) = z^k.$$

The order of the zero agrees for g and h, so the requirement of Thm. 5 is met. We find

$$\Phi(z) = \Big(\frac{1}{2}(1 - z^{2k}), \frac{i}{2}(1 + z^{2k}), z^k\Big)$$

which gives upon integration from 0 to z:

$$f(z) = \operatorname{Re} \frac{1}{2}\Big(z - \frac{1}{2k+1} z^{2k+1}, i(z + \frac{1}{2k+1} z^{2k+1}), \frac{2}{k+1} z^{k+1}\Big).$$

Let L be any straight line through the origin. Then $g(L) = h(L) = L$ are also straight and hence L is geodesic for f by Lemma 9.

The first fundamental form is

$$ds = \frac{1}{2}\Big(|z^k| + \frac{1}{|z^k|}\Big)|z^k| = \frac{1}{2}\big(|z^{2k}| + 1\big).$$

Thus each curve from 0 to ∞ has infinite length and so the Enneper surfaces are complete.

Note that the metric depends only on $|z|$, that is, for each k the Enneper surfaces have an intrinsic isometry, namely the rotation $z \mapsto e^{i\varphi} z$ for all $\varphi \in \mathbb{R}$.

The second fundamental form is

$$b_z(X, X) = -\operatorname{Re}\Big(\frac{g'(z)}{g(z)} h(z) X^2\Big) = -\operatorname{Re}\Big(\frac{k}{z} z^k X^2\Big).$$

Let us consider the radial curves $\gamma(t) = tz$ where X is $\gamma'(t) = z$. Then

$$\frac{g'(\gamma(t))}{g(\gamma(t))} h\big(\gamma(t)\big)\big(\gamma'(t)\big)^2 = \frac{k(tz)^{k-1}}{(tz)^k}(tz)^k z^2 = k t^{k-1} z^{k+1}.$$

Thus we find

$$\Big(\frac{g'h}{g} \circ \gamma\Big)(\gamma')^2 \in \mathbb{R} \qquad \Longleftrightarrow \qquad (k+1)\arg z = 0 \bmod \pi,$$
$$\Big(\frac{g'h}{g} \circ \gamma\Big)(\gamma')^2 \in i\mathbb{R} \qquad \Longleftrightarrow \qquad (k+1)\arg z = \frac{\pi}{2} \bmod \pi.$$

Hence we can apply Lemma 10: For the first case, the radial lines $t \mapsto tz$ are straight, while for the second they are contained in a mirror plane of the surface.

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Let us finally consider a circle $\gamma(\varphi) := Re^{i\varphi}$ for R large. Then $f \circ \gamma$ winds $2k+1$ times about the vertical axis. In particular, none of the Enneper surfaces can be embedded.

Let us also consider deformations of the Enneper surfaces. If p is a polynomial of degree $k-1$ and $\varepsilon \in \mathbb{R}$ is an arbitrary parameter, then we set

$$g(z) = h(z) := z^k + \varepsilon p(z).$$

Again there are no branch points and the behaviour at ∞ remains. Clearly in general the surface $f = f_\varepsilon$ will no longer have the symmetries of the Enneper surface.

2.2. **Simply periodic Scherk surfaces.** The Weierstrass data are

$$U = D, \qquad g(z) = z, \qquad h(z) = \frac{z}{z^4 + 1} = \frac{1}{z}\frac{1}{z^2 + \frac{1}{z^2}},$$

where the second representation of h works for $z \neq 0$. As $h(0) = 0$, the order of the zero of h is indeed the order of g, as required in Thm. 5. Note that h is singular at the four points $p_{1,2} := e^{\pm i\pi/4}$ and $p_{3,4} := e^{\pm i3\pi/4}$, while it extends to ∂D elsewhere.

Moreover, the metric is

$$ds = \frac{1}{2}\Big(|z| + \frac{1}{|z|}\Big)\frac{|z|}{|z^4 + 1|} = \frac{|z|^2 + 1}{2|z^4 + 1|} \quad \text{and} \quad = \frac{1}{2}\Big(|z| + \frac{1}{|z|}\Big)\frac{1}{|z^2 + \frac{1}{z^2}|}\frac{1}{|z|},$$

the four singular points p_1 to p_4 occur at infinite distance from 0. Let us show this for instance at the point $p_1 = \frac{1+i}{\sqrt{2}}$. Consider $\gamma(t) = \frac{1+i}{\sqrt{2}}t$ on $t \in [0,1)$. To compute $L(\gamma)$ note first

$$|\gamma^4(t)+1| = 1 - t^4 = (1-t^2)(1+t^2) = (1-t)(1+t)(1+t^2).$$

Consequently

$$L(\gamma) = \int_0^1 ds\big(\gamma(t)\big)|\gamma'(t)|\,dt = \int_0^1 \frac{1}{2}\frac{1}{(1-t)(1+t)}\,dt \to \infty. \tag{16}$$

Let us now discuss symmetry properties. We verify that Lemma 9 applies to certain curves in D:

1. $\gamma(t) = t \quad \Rightarrow \quad g(\gamma(t)) = t \in \mathbb{R}, \quad h(\gamma(t)) = \dfrac{t}{t^4+1} \in \mathbb{R}$
2. $\gamma(t) = it \quad \Rightarrow \quad g(\gamma(it)) = it \in i\mathbb{R}, \quad h(\gamma(t)) = \dfrac{it}{t^4+1} \in i\mathbb{R}$
3. $\gamma(t) = (1+i)t \quad \Rightarrow \quad g(\gamma(t)) = (1+i)t \in (1+i)\mathbb{R}, \quad h(\gamma(t)) = \dfrac{(1+i)t}{-4t^4+1} \in (1+i)\mathbb{R}$
4. $\gamma(t) = (1-i)t \quad \Rightarrow \quad g(\gamma(t)) = (1-i)t \in (1-i)\mathbb{R}, \quad h(\gamma(t)) = \dfrac{(1-i)t}{-4t^4+1} \in (1-i)\mathbb{R}$

Since each curve is contained in D, so is their image under g. Thus the Gauss image $\nu = \mathrm{st}(g)$ lies in the upper hemisphere.

For these curves, the second fundamental form test gives

$$\Big(\frac{g'}{g}\circ\gamma\Big)h(\gamma')^2 = \frac{1}{\gamma^4+1}(\gamma')^2 \in \begin{cases} \mathbb{R} & \text{in cases 1., 2.} \qquad \text{, i.e., curv. lines,} \\ i\mathbb{R} & \text{in cases 3., 4.} \qquad \text{, i.e., asympt. lines;} \end{cases}$$

We apply Prop. 8 and Lemma 10(ii) to give that the curves $\mathbb{R}\cap D$ and $i\mathbb{R}\cap D$ (cases 1. and 2.) map to symmetry planes of the minimal surface. Which planes are these? Let us assume $f(0) = 0$. Then, for 1. $\mathrm{st}(\mathbb{R}) = xz$-plane, and since the mirror plane contains $f(0) = 0$ it must be the xz-plane itself. Simlilarly for 2.: Here $\mathrm{st}(i\mathbb{R}) =$ showing that the mirror plane is the yz-plane.

By Lemma 10(i), the diagonals of cases 3. and 4. map to straight lines, which moreover contain $f(0) = 0$. These curves must run at the diagonal directions in the xy-plane, as $\mathrm{st}(g) = \nu$ is on the great circles in the planes spanned by $(1,\pm1,0)$ and $(0,0,1)$. By (16) they run to infinity of $\mathbb{R}^3$

We also wish to consider curves

$$5. \qquad \gamma(t) = e^{it} \quad \Rightarrow \quad g(\gamma(t)) = e^{it} \in \mathbb{S}^1,$$

avoiding p_1, p_2, p_3, p_4. Their normal $\mathrm{st}(g\circ\gamma)) = \nu\circ\gamma$ is horizontal. i.e., contained in the xy-plane.

Although Lemma 9 does not apply directly, we can use similar arguments to show that γ, that is $\mathbb{S}^1$, is geodesic. We claim that for the first fundamental form

$$ds^2(X,X) = \Big(|z| + \frac{1}{|z|}\Big)^2 \frac{1}{\big|z^2 + \frac{1}{z^2}\big|^2}\Big|\frac{X}{z}\Big|^2,$$

an inversion in the circle, $z \mapsto \sigma(z) = \frac{1}{\bar z}$, is again an isometry. Indeed, the first two factors are preserved when we replace z by $\sigma(z)$. Moreover, $d\frac{1}{z} = -\frac{1}{z^2}$ and $z \to \bar z$ is length preserving. Hence $|d\sigma(X)| = \frac{1}{|z|^2}|X|$, and so

$$\frac{1}{|\sigma(z)|}|d\sigma(X)| = |\bar z|\frac{1}{|z|^2}|X| = \frac{1}{|z|}|X|.$$

This establishes again the desired isometry property, $ds_z^2(X,X) = ds^2_{\sigma(z)}\big(d\sigma(X), d\sigma(X)\big)$.

Finally,

$$\frac{(\gamma')^2}{\gamma^4+1} = \Big(\frac{\gamma'}{\gamma}\Big)^2\frac{1}{\gamma^2+\gamma^{-2}} = \Big(\frac{ie^{it}}{e^{it}}\Big)^2\frac{1}{e^{2it}+e^{-2it}} = -\frac{1}{2\,\mathrm{Re}(e^{2it})} \in \mathbb{R}$$

so that γ is a geodesic curvature line.

Thus each of the four arcs of $\mathbb{S}^1 \setminus \{p_1,p_2,p_3,p_4\}$ is contained in a mirror plane. So the four curves must therefore be contained in horizontal planes. Due to the reflection symmetries in the xz- and yz-planes opposite pairs of these curves are in fact contained in the same planes!

This gives two horizontal symmetry planes, which can be shown to be different. As two successive reflections create a translation, we have that $\overline{f}(D)$ together with its mirror image forms a translational fundamental domain F. In particular, the resulting surface is singly periodic. The vertical period d is precisely two times the vertical distance between the two different symmetry planes. On the interior of F, the Gauss map is injective, and on F it is surjective to $\mathbb{S}^2$. Hence the area 4π of the Gauss image is the total curvature of F, see II 1.3.

Let γ_{ij} be the open arc in $\mathbb{S}^1$ from p_i to p_j. Then f extends analytically across the two opposite arcs γ_{12} and γ_{43} to $f\colon \mathbb{C}\setminus\{\overline{\gamma}_{24}\cup\overline{\gamma}_{41}\} \to \mathbb{R}^3$. This extension parameterizes a non-simply connected domain of the surface. This extension will, however be discontinuous along the pair of arcs $\gamma_{24}\cup\gamma_{41}$; in fact, the one-sided limits of f differ by $(0,0,d)$.

There are versions of the Scherk surface of higher symmetry: Let us set for $k \geq 2$ (with $k = 2$ corresponding to the classical example)

$$U = D, \qquad g(z) = z^{k-1}, \qquad h(z) = \frac{z^{k-1}}{z^{2k}+1} = \frac{1}{z}\frac{1}{z^k + \frac{1}{z^k}},$$

then the $2k$-th roots of -1 are the singular points.

Remarks. 1. It can also be shown that the surfaces are embedded and have ends which are asymptotically planar.
2. For any angle $0 < \alpha < \pi$, there are versions of the Scherk surface for which the angle between the ends is not $\pi/2$ but α.
3. It is an open problem if the family of Scherk surfaces as in 2. are the only embedded singly periodic surface, for which the quotient has genus 1 and the ends are planar.

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2.3. **Elliptic functions.** To construct the Costa surface in the next section, we need special functions:

Definition. A meromorphic function $f: \mathbb{C} \to \overline{\mathbb{C}}$ which is invariant under some lattice $\Lambda = \{\lambda_1 v_1 + \lambda_2 v_2\}$ of rank 2, that is,

$$f(z+w) = f(z) \qquad \text{for all } w \in \Lambda,$$

is called an *elliptic function*.

Note that the only elliptic functions which are holomorphic are the constants. Indeed, such functions are bounded and defined on all of $\mathbb{C}$. Hence they must be constant by Liouville's theorem.

Other interesting facts can be derived from the residue theorem. To state them, let us denote a fundamental domain by $Q := \{sv_1 + tv_2 : 0 \leq s, t < 1\}$.

Proposition 12. *Let f be elliptic with poles at $b_1, \dots, b_k \in Q$. Then*

$$0 = \sum_{j=1}^{k} \operatorname{res}_{b_j} f.$$

Proof. If ∂Q contains no pole then

$$2\pi i \sum_{j=1}^{k} \operatorname{res}_{b_j} f = \int_{\partial Q} f(w)\, dw = \Big(\int_{[0,v_1]} + \int_{[v_1,v_1+v_2]} - \int_{[v_2,v_1+v_2]} - \int_{[0,v_2]} \Big) f(w)\, dw,$$

using interval notation for straight paths in $\mathbb{C}$. But due to periodicity of f, the sum of the first and third, as well as the sum of the second and fourth integral must vanish. If ∂Q contains a pole, we find a parallel translate of Q whose interior will contain all poles; the above proof then applies. $\square$

Corollary 13. (i) *There is no elliptic function with only one pole of first order.*
(ii) *Over its fundamental domain Q, an elliptic function takes each value the same number of times; here we count the number with multiplicity (at branch points).*

Proof. To prove (ii), apply the proposition to the elliptic function f'/f. We first show that f has as many poles as zeros. Suppose f has a zero of order $n = n_j$ at $b = b_j$, where $j = 1, \dots, Z$ and $n_j \in \mathbb{N}$, then

$$\frac{f'}{f} = \frac{na_n(z-b)^{n-1} + \dots}{a_n(z-b)^n + \dots} = n\frac{1}{z-b} + \dots \qquad \Rightarrow \quad \operatorname{res}_b \left(\frac{f'}{f}\right) = n.$$

Similarly, if f has a pole of order $m = m_j$ at $a = a_j$, for $j = 1, \dots, P$ then

$$\frac{f'}{f} = \frac{-ma_m(z-a)^{-m-1} + \dots}{a_m(z-a)^{-m} + \dots} = -m(z-a)^{-1} + \dots \qquad \Rightarrow \quad \operatorname{res}_a \left(\frac{f'}{f}\right) = -m.$$

Thus when counted with multiplicity,

$$\sum_{j=1}^{M} \operatorname{res} f = \sum_{j=1}^{Z} n_j - \sum_{j=1}^{P} m_j \tag{17}$$

which vanishes by the proposition.

Now apply the result to f replaced with $f - c$, where $c \in \mathbb{C}$ is arbitrary to conclude that the number of points where $f = c$ minus the number of poles of f agrees with (17). $\square$

2.4. **Existence of elliptic functions.** Are there any non-constant elliptic functions at all? The simplest elliptic functions which are admissible in view of (i) would have two poles and zeros, either distinct or coincident. We will construct such functions by the Riemann mapping problem and Schwarz reflection. We present the construction of elliptic functions suggested by Karcher. It is a simple planar analogue of our construction of the triply periodic minimal surfaces. This constructs the elliptic functions together with symmetries. These symmetries in turn will be essential for applying them to construct minimal surfaces. We will see this on the example of the Costa surface. Our approach is also explained in *Ausgewählte Kapitel aus der Funktionentheorie* by Fischer-Lieb, Sect VI,8.

Elliptic functions are a traditional subject in complex analysis, see, e.g. [FL, Kap.VII,7]. The classical approach to introduce these functions is via certain series.

To simplify the exposition, we consider exclusively the particular case of a primitive lattice

$$\Lambda := 4\mathbb{Z}^2, \qquad Q := [-2,2) \times [-2,2) \subset \mathbb{C} \tag{18}$$

Our first elliptic function has two distinct poles and zeros of first order.

Lemma 14. *There exists a function $q: \mathbb{C} \to \overline{\mathbb{C}}$ which is elliptic w.r.t. $\Lambda = 4\mathbb{Z}^2$. It has*

$$\textit{first order zeros at } 0 \textit{ and } 2+2i, \qquad \textit{first order poles at } 2 \textit{ and } 2i.$$

These are all poles and zeros up to the action of Λ. Special values of q are:

$$q(\mathbb{R}) = \overline{\mathbb{R}}, \qquad q(i\mathbb{R}) = i\overline{\mathbb{R}},$$
$$q\big((1 \pm i)\mathbb{R}\big) = (1 \pm i)\mathbb{R}, \qquad q(2i + \mathbb{R}) = \overline{\mathbb{R}}, \qquad q(2 + i\mathbb{R}) = i\overline{\mathbb{R}}$$

Note that $\mathbb{C}$ compactifies with one point ∞, while $\mathbb{R}$ usually compactifies with two points $\pm\infty$. However, when we consider $\mathbb{R}$ as a subset of $\mathbb{C}$, we will compactify $\mathbb{R}$ with just one point ∞ as well. Thus we use, e.g., the notation $\overline{\mathbb{R}}$ for $\mathbb{R} \cup \infty \subset \overline{\mathbb{C}}$.

Proof. Consider a triangle and an eighth of the unit disk,

$$\Delta := (0,1) \times (0,1) \cap \{x < y\}, \qquad C := \{z = re^{i\varphi} : 0 < r < 1,\ \ 0 < \varphi < \pi/8\}.$$

By the Riemann mapping theorem III.29 there is biholomorphic map $q\colon \Delta \to C$ subject to the three point condition $q(0) = 0$, $q(1) = 1$, $q(1+i) = \frac{1+i}{\sqrt{2}}$. The map q is holomorphic and extends continuously to the boundary.

Note that the continuous boundary extension $q\colon \overline{\Delta} \to \overline{C}$ maps $[0,1]$ to itself and $[0,1+i]$ to $[0, \frac{1}{2}(1+i)]$. Thus we can use the Schwarz reflection Thm. 11 to extend u across these straight arcs in domain and range. The resulting reflection is of the form $u(z) = \sigma(u(\rho(z)))$, where ρ, σ are the appropriate reflections of domain and range, respectively. More precisely, we proceed in three steps.

1. Four copies of Δ map $(-1,1)\times(0,1)$ to the upper disk $D\cap\{x>0\}$; the map q is smooth on the interior of this set. This extension maps the bounding interval $(-1,1)$ into itself. Let us apply one further Schwarz reflection, this time of the standard type $q(z) = \overline{q(\overline{z})}$. As guaranteed by Thm. 11 the resulting map is smooth at 0. This way we obtain a surjective map $q\colon (-1,1)^2 \to D$; it is biholomorphic, as the reflected copies are disjoint.

As in the proof of Riemann mapping theorem we conclude that q has no branch point at 0. Thus 0 is a zero of first order. Moreover, q has no further zeros in $[-1,1]^2$, nor poles.

2. We now extend q to the rectangle $R := (-2,2) \times (-1,1)$ by Schwarz reflecting the images in the bounding circle arcs. Note that these arcs are of the form e^{it} where either $|t| < \frac{\pi}{4}$ or $|t - \pi| < \frac{\pi}{4}$. The horizontal boundary arcs of ∂R are mapped to the complement of these arcs on $\mathbb{S}^1$. The image of the rectangle is $\mathbb{C}$.

The points $2, 2i \in \partial R$ are mirror image of 0, and hence under the inversion $z \mapsto \frac{1}{z}$ map to ∞. For this reason these are poles of first order.

3. In a third step we extend q to all of Q. This can be done with respect to a tesselation of Q with altogether 32 copies of the original triangle (or by reflecting larger pieces at once). A second zero at $2+2i$ is created.

Further extension leads to a periodic function. The special values and the location of poles and zeros follow directly from the construction. □

There are further properties of q which our construction yields, but which will not need in the sequel: The values $q(\pm i + \mathbb{R}) \subset \mathbb{S}^1$, $q(\pm 1 + i\mathbb{R}) \subset \mathbb{S}^1$ are immediate. Moreover, q maps $(-1,1)\times(-1,1)$ biholomorphically to D. that these points represent poles of first order, as stated in (*ii*).

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Our second elliptic function has a double pole and a double zero:

Lemma 15. *There exists an function $p\colon \mathbb{C} \to \overline{\mathbb{C}}$ which is elliptic w.r.t. $\Lambda = 4\mathbb{Z}^2$. It has*

a zero of second order at 0, *a pole of second order at* $2+2i$;

these are all poles and zeros up to the action of Λ. Besides 0, the only branch points are 2 and $2i$ modulo Λ. Special values of p are:

$$p(2) = 1, \quad p(2i) = -1, \qquad p(\mathbb{R}) = [0,1], \quad p(i\mathbb{R}) = [-1,0],$$
$$p\big((1+i)\mathbb{R}\big) = i[0,\infty) \cup \infty, \qquad p\big((1-i)\mathbb{R}\big) = i(-\infty,0] \cup \infty,$$
$$p(2i + \mathbb{R}) = (-\infty,-1] \cup \infty, \qquad p(2 + i\mathbb{R}) = [1,\infty) \cup \infty$$

Note that at 0 the special images are consistent with 0 being a branch point of order 2.

Proof. Let C be a "triangle" bounded by $[0,i]$, $[0,\rho]$ for $\rho > 0$ and a circular arc through i and ρ. The point $\rho > 0$ is chosen such that the circular arc meets the imaginary axis at 45° and the real axis at 90° (the circle has its midpoint on the negative real axis). The Riemann mapping theorem gives a unique map p, mapping the triangle Δ (defined above) biholomorphically onto C, such that $0 \mapsto 0$, $1+i \mapsto i$, and 1 maps to the third vertex.

Note that at the vertex 0, the map q doubles the 45° angle of Δ to the 90° angle of C. Thus the Taylor series starts with z^2, that is, 0 is a zero of second order.

We can extend p stepwise by mapping reflected images to reflected images. Along circular arcs in the domain, we use inversion.
1. Reflect Δ along $[1, 1+i]$. The original "triangle" C can be complemented by its image inverted in the bounding circular arc. The result is the disk $D_4 := D \cap \{x > 0, y > 0\}$. Indeed, the image of $[0,i]$ is a circle arc which meets both the real and imaginary axis at a right angle. Thus the doubled triangle Δ maps to the quarter disk D_4.
2. We can now use the extension constructed in 1. to reflect further. At this stage, we only need planar reflection and (in the image:) inversion in the unit circle.

All properties claimed are obvious from the construction. For instance, the reflected images 2, $2i$ of the origin also become branch points, and no further branch points are created. $\square$

It is easy to create elliptic functions with zeros and poles of higher order, for instance by differentiation of p or q. We collect the following facts about the derivative p' which will become useful to study the Gauss map of the Costa surface below,

Lemma 16. *The function $p'\colon \mathbb{C} \to \overline{\mathbb{C}}$ is elliptic w.r.t.* (18) *and satisfies the differential equation*

$$p'q = cp \qquad \text{for some } c \in \mathbb{R}_+. \tag{19}$$

It has the following special values:

$$p'(0) = p'(2) = p'(2i) = 0 \quad (1.\ order), \qquad p'(2+2i) = \infty \quad (3.\ order),$$
$$p'(\mathbb{R}) \subset \mathbb{R}, \qquad p'(i\mathbb{R}) \subset i\mathbb{R}, \qquad p'\big((1 \pm i)\mathbb{R}\big) \subset (1 \pm i)\overline{\mathbb{R}},$$
$$p'(2i + \mathbb{R}) \subset \overline{\mathbb{R}}, \qquad p'(2 + i\mathbb{R}) \subset i\overline{\mathbb{R}}$$

Moreover, there are exactly four branch points modulo Λ, namely one in each interval $(0, 2)$, $(-2, 0)$, $(0, 2i)$, $(-2i, 0)$.

Proof. To establish (19), we claim that the zeros and poles on the two sides coincide.

We start with p' itself. It has simple zeros at the branch points of p, namely only at 0, 2, $2i$ modulo Λ. Since p has double pole at $2 + 2i$, the function p' has a triple pole there; it has no more poles. Moreover, q has simple poles at 2, $2i$, making these points removable for $p'q$. The simple zeros of q at 0, $2 + 2i$ creating a double zero of $p'q$ at 0, and a double pole of $p'q$.

Thus $p'q$ has the same zeros and poles as p. To establish (19), consider the quotient $\frac{p'q}{p}$, which is elliptic as well. Zeros and poles of numerator and denominator coincide Thus the fraction is bounded and its singularities are removable. By Liouville's theorem, the quotient is constant $c \in \mathbb{C}$ The values of numerator and denominator on $(0, 2)$ are positive real. Indeed, to see this for p', note that the Riemann mapping theorem gives that p is injective on $(0, 1)$ hence monotone; there is no branch point on $(0, 1)$ hence the derivative is positive. We conclude that our constant c is a positive real.

To determine the special values $p' \circ \gamma$ along some arc γ, note that by the chain rule

$$p' \circ \gamma = \frac{1}{\gamma'} \frac{d}{dt}(p \circ \gamma)$$

Using the values of $p \circ \gamma$ of the Lemma 15, we can assert the following. On $\mathbb{R}$ we have $\gamma' < 0$ and $p \in \mathbb{R}$, so that p' is also real. On $i\mathbb{R}$ the tangent is i while $p > 0$, thus p' is

imaginary. Finally, on $(1 \pm i)\mathbb{R}$ the tangent is $1 \pm i$ while p is imaginary; thus p' is a real multiple of

$$\frac{1}{1 \pm i} i = \frac{1 \mp i}{1 - i^2} i = \frac{1}{2}(i \pm 1).$$

Since $p'(0) = p'(2) = 0$ and p' is real on $(0, 2)$, there must be a critical point of p' on $(0, 2)$. Thus p'' has at least one zero, or p' at least one branch point on $(0, 2)$. The same argument applies to the other three intervals on the coordinate axes. Since p'' has exactly one pole of order 4 we can apply Corollary 13(ii) to conclude that there are no more branch points. $\square$

There is one more relation of elliptic functions which we need:

$$\frac{q^2}{p} = \frac{2}{1 - p^2} \tag{20}$$

As in the proof of (19), the two side of the equation are proportional as they have the same zeros and poles. The exact value of the proportionality constant can be read off by picking a particular point (which one?) in the domain. We leave the verification as an exercise.

Remarks. 1. Elliptic functions exist for arbitrary lattices Λ. The symmetry method to construct them is, however, limited to the rectangular lattices with $v_1 \perp v_2$.
2. We have introduced two elliptic functions p, q. In the following sense, two functions are sufficient: *Each elliptic function is a rational function of p, q.* See [FL].
2. The Weierstraß $\wp$-function, defined in many books, differs from our p-function by translation and scaling.

`51. Lecture, Wednesday 21.6.06` ________________________________

2.5. **The Costa surface.** The catenoid is an embedded minimal surface of finite total curvature. It is the only surface with these properties we have discussed so far, and it had been the only known such surface up to 1982. In that year the Brasilian mathematician Costa discovered Weierstrass data of a further finite total curvature surface. Only later, Hofmann and Meeks showed that Costa's surface is indeed embedded.

It is not only for historical reasons that we present the surface here. Also it is a fine example for the main problems faced when appying the Weierstrass construction:
• Solution of the period problem,
• embeddedness proof.

The Weierstrass data for the Costa surface involve a real parameter $r > 0$ to be determined later:

$$U = (-2, 2) \times (-2, 2), \qquad g = rp' = cr\frac{p}{q}, \quad h = q,$$

where $c > 0$ is determined by Lemma 16.

Proposition 17. *Let* $P := 2+2i$, $C_- := 2$, $C_+ := 2i$, *and define a singular set*

$$S := \{P, C_-, C_+\} + 4\mathbb{Z}^2.$$

Then for any $r > 0$, *the Costa surface* $f_r\colon U \to \mathbb{R}^3$ *extends continuously to* $\overline{U}\backslash S$, *and neighbourhoods of* P *map to a planar end while neighbourhoods of* $C_\pm$ *each map to a catenoid end.*

Proof. Note that the only zero of g on $\overline{U} \setminus S$ is at $z = 0$. There, h has a zero as well, so by Thm. 5 the Weierstrass data define a minimal surface over U. The asymptotics of the ends will be asserted below. □

Proposition 18. *For any* $r > 0$, *consider the Costa surface* $f_r\colon U \to \mathbb{R}^3$ *translated so that* $f_r(0) = 0$. *It has the following symmetry properties:*
(i) *The lines* $\mathbb{R}$ *and* $i\mathbb{R}$ *map to lines of planar reflection contained in the* $xz-$ *and* yz*-planes, respectively.*
(ii) *The lines* $(1 \pm i)\mathbb{R}$ *map to straight arcs of* $180°$*-rotation on the* $x \mp y$*-lines through the origin.*
(iii) *The two boundary arcs* $(-2+2i, 2i)$ *and* $(2i, 2+2i)$ *map to mirror symmetry planes of the surface, each of them parallel to* xz. *Likewise, the curves* $(2-2i, 2)$ *and* $(2, 2+2i)$ *map to mirror planes of the surface, each parallel to* yz.

Proof. (i) We consider the coordinate axes:

$$g(\mathbb{R}) = p'(\mathbb{R}) \subset \mathbb{R}, \quad h(\mathbb{R}) = q(\mathbb{R}) \subset \mathbb{R}, \qquad g(i\mathbb{R}) = p'(i\mathbb{R}) \subset i\mathbb{R}, \quad h(i\mathbb{R}) = q(i\mathbb{R}) \subset i\mathbb{R}.$$

By Lemma 9, the curves map to geodesics of the surface.

Without calculation, this suffices to assert that along the curve γ parameterizing $(-2, 2)$ with tangent $\gamma' = 1 \in \mathbb{R}$,

$$\frac{g'}{g}\gamma' = \frac{(g \circ \gamma)'}{g} \in \mathbb{R}, \quad h \in \mathbb{R} \quad \Rightarrow \quad \frac{g'}{g}\gamma' h\gamma' \in \mathbb{R}.$$

Similarly, along the curve γ parameterizing $(-2i, 2i)$ with tangent $\gamma' = i$,

$$\frac{g'}{g}\gamma' = \frac{(g \circ \gamma)'}{g} \in \mathbb{R}, \quad h(i\mathbb{R}) \in i\mathbb{R} \quad \Rightarrow \quad \frac{g'}{g}\gamma' h\gamma' \in \mathbb{R};$$

By Lemma 10 combined with Proposition 8(ii), this shows that f_r maps these curves to mirror planes of the surface. Since $g(\mathbb{R}) \subset \mathbb{R}$, we have $\mathrm{st}(g) = \nu$ is in the xz-plane, and so the mirror plane is the xz-plane. Similarly on $i\mathbb{R}$.

(ii) We reason as in (i) for the images of the diagonals. Since

$$g\big((1 \pm i)\mathbb{R}\big) = p'\big((1 \pm i)\mathbb{R}\big) \subset (1 \pm i)\mathbb{R}, \quad h\big((1+i)\mathbb{R}\big) = q\big((1 \pm i)\mathbb{R}\big) \subset (1+i)\mathbb{R}$$

the diagonals map to geodesics.

Let γ parameterize the diagonals $(1 \pm i)\mathbb{R}$ with tangent $\gamma' \in (1 \pm i)\mathbb{R}$. Then

$$\frac{g'}{g}\gamma' = \frac{(g \circ \gamma)'}{g} \in \mathbb{R} \quad \Rightarrow \quad \frac{g'}{g}\gamma' h\gamma' \in (1 \pm i)^2\mathbb{R} = i\mathbb{R}.$$

Lemma 10 combined with Proposition 8(i) asserts the curves are straight lines.

Moreover, g maps to $(1 \pm i)\mathbb{R}$, so that $\nu = \mathrm{st}(g)$ is in the plane spanned by z and $x \pm y$. The straight image lines must run in the direction of the orthogonal complement of these planes, which gives our claim.

(iii) Let γ be a curve parameterizing a connected arc in $(2i + \mathbb{R}) \setminus S$. Such an arc is a geodesic curvature line, hence a curve of planar reflection:

$$g(\gamma) = p'(\gamma) \in \mathbb{R}, \quad h(\gamma) = q(\gamma) \in \mathbb{R}, \qquad \frac{g'}{g}\gamma' = \frac{(g \circ \gamma)'}{g} \in \mathbb{R} \Rightarrow \frac{g'}{g}\gamma' h\gamma' \in \mathbb{R},$$

On the other hand, let now γ parameterize a connected arc of $(2 + i\mathbb{R}) \setminus S$. Any arc γ defines a mirror plane as well:

$$g(\gamma) = p'(\gamma) \in i\mathbb{R}, \quad h(\gamma) = q(\gamma) \in i\mathbb{R}, \qquad \frac{g'}{g}\gamma' = \frac{(g \circ \gamma)'}{g} \in \mathbb{R} \Rightarrow \frac{g'}{g}\gamma' h\gamma' \in i^2\mathbb{R} = \mathbb{R}.$$

The values of g give the claim on the containing planes. □

We call the eight components of $\partial U \setminus S$ *boundary arcs*. Each boundary arc has length 2. We call a pair of arcs *opposite* if they differ by a translation in $4\mathbb{Z}^2$.

Part (iii) of Proposition 18 asserts that the image of the boundary arcs creates a vertical symmetry plane of the surface which allows for Schwarz reflection to continue the surface across the boundary arc. This continuation agrees with the continuation of the Weierstrass data f to any simply connected domain which contains the arc in its interior.

Lemma 19. (i) *The image of two opposite boundary arcs differs (pointwise) by a vector perpendicular to the containing symmetry planes.*
(ii) *If the image of one pair of opposite arcs is contained in the same plane then the immersion* f_r *agrees on all four pairs and extends to a doubly periodic map*

$$f_r\colon \mathbb{C} \setminus S \to \mathbb{R}^3 \quad \text{with} \quad f(z+w) = f(z) \text{ for all } w \in 4\mathbb{Z}^2.$$

Proof. We deal with the case that opposite boundary arcs $I_1, I_2 \subset \partial U$ have their image contained in a pair of symmetry planes parallel to the yz-plane. The case of arcs with xz-symmetry plane is analogous.

(i) By Prop. 18(i) there is a third reflection plane parallel to the yz-plane, namely $f(i\mathbb{R})$. Reflection in the yz-plane leaves the last two coordinates invariant, that is, $f^j(z) = f^j(z+4)$ for $z \in I_1$ and $j = 2, 3$.

(*ii*) If f agrees on the pair of opposite arcs I_1, I_2, parallel to the yz-plane then this means precisely $f^1(z) = f^1(z+4)$ for $z \in I_1$. Together with (*i*) this gives $f(z) = f(z+4)$ for $z \in I_1$.

Let us now use the rotations about the straight lines in the $x \pm y$-direction stated in Prop. 18(*ii*). Starting with one pair of opposite arcs, the two rotations map them to the remaining three pairs of opposite arcs. Thus the distances between the symmetry planes containing a pair of opposite arcs must agree. ☐

`52. Lecture, Tuesday 27.6.06`

In order to obtain a nice closed surface we wish to choose r such that the symmetry planes coicide.

The two yz-mirror planes containing the curves $f_r\big((-2,-2+2i)\big)$ and $f_r\big((2,2+2i)\big)$ have a distance

$$d(r) := f_r^1(z) - f_r^1(z+4) = \operatorname{Re}\int_\Gamma \frac{1}{2}\Big(\frac{1}{g} - g\Big) h\,dw,$$

where Γ represents a path in U from z to $z+4$. Since $d(r)$ measures the orthogonal distance of the two planes. It is independent of the choice of $z \in (-2,-2+2i)$ and Γ. For convenience, we let Γ parameterize $(-2+i, 2+i)$. We call $d(r)$ a *period* of the surface. The *period problem* amounts to determining r such that $d(r) = 0$. For a general minimal surface, it will be necessary to determine several parameters which lead to the vanishing of several periods.

Lemma 20 (Solution of period problem)**.** *There exists an $r > 0$, such that $d(r) = 0$.*

Proof. With $c > 0$ as in (19) we have

$$\frac{1}{g}h = \frac{q}{crp}q = \frac{1}{cr}\frac{q^2}{p} \quad \text{and} \quad gh = \frac{crp}{q}q = crp.$$

Thus

$$d(r) = \operatorname{Re}\int_\Gamma \frac{1}{2}\Big(\frac{1}{g} - g\Big) h\,dw = \frac{1}{r}\frac{1}{2c}\operatorname{Re}\int_\Gamma \frac{q^2}{p}\,dw - r\frac{c}{2}\operatorname{Re}\int_\Gamma p\,dw.$$

We want to pick r such that the sum of the integrals vanishes. We will show that the real parts of both integrals are positive,

$$\text{(21)} \qquad \operatorname{Re}\int_\Gamma \frac{q^2}{p}\,dw \overset{(20)}{=} 2\operatorname{Re}\int_\Gamma \frac{1}{1-p^2}\,dw > 0 \quad \text{and} \quad \operatorname{Re}\int_\Gamma p\,dw > 0,$$

so that the intermediate value theorem yields such an r. Since we only know p well along the symmetry lines, in the following we must reduce the verification of (21) to the values of p on symmetry lines.

Let us first consider the second integral of (21). We consider a path $\tilde\Gamma$ consisting of the three arcs $(-2+i,-2)$, $(-2,2)$, and $(2,2+i)$. Let R be the rectangle bounded by the closed curve $\Gamma \cup \tilde\Gamma$. Note that the only pole of p is at $2+2i$ so that p has no pole on $\overline{R}$; thus Cauchy's Theorem 3 applies to R:

$$\int_\Gamma p\,dw \overset{\text{Cauchy's thm.}}{=} \int_{\tilde\Gamma} p\,dw = \Big(\int_{(-2+i,2)} + \int_{(-2,2)} - \int_{(2,2+i)}\Big) p\,dw \overset{p \text{ periodic}}{=} \int_{(-2,2)} p\,dw > 0,$$

where we used $p\big((-2,2)\big) \subset [0,1]$ in the last step.

We use the same kind of argument for the first integral of (21). Here we let $\tilde\Gamma$ be defined as the concatination of the four intervals

$$(-2+i,-2+2i), \quad (-2+2i,0), \quad (0,2+2i), \quad (2+2i,2+i).$$

Together with Γ, it bounds a subset $\Omega \subset U$, consisting of three open sets (triangles). We claim that 2 is the only zero of $1-p^2$. Indeed, by Lemma 15, $p(2) = 1$ is a branch point, and moreover p has only one zero of second order. Thus by Corollary 13, the function p takes each value twice, thus only 2 maps to 1. Similarly, only $2i$ maps to -1. This proves the claim, and we can apply Cauchy's theorem to Ω:

$$\begin{aligned}\int_\Gamma \frac{1}{1-p^2}\,dw &\overset{\text{Cauchy's thm.}}{=} \int_{\tilde\Gamma} \frac{1}{1-p^2}\,dw \\ &= \Big(\int_{(-2+i,-2+2i)} + \int_{(-2+2i,0)} + \int_{(0,2+2i)} - \int_{(2+i,2+2i)}\Big)\frac{1}{1-p^2}\,dw \\ &\overset{p \text{ periodic}}{=} \int_{(-2+2i,0)} \frac{1}{1-p^2}\,dw + \int_{(0,2+2i)} \frac{1}{1-p^2}\,dw\end{aligned}$$

We have $p \in \pm i\mathbb{R}$ along the two diagonals implying $p^2 \in \mathbb{R}_-$ and $\frac{1}{1-p^2} > 0$. Thus for $\gamma(t) = -2+2i+(1-i)t$ we find

$$\operatorname{Re}\int_{(-2+2i,0)} \frac{1}{1-p^2}\,dw = \operatorname{Re}\int_0^2 \frac{1}{1-p^2(\gamma(t))}(1-i)\,dt = \int_0^2 \frac{1}{1-p^2(\gamma(t))}\,dt > 0.$$

Similarly for the other integral. We conclude $\operatorname{Re}\int_\Gamma \frac{1}{1-p^2}\,dw > 0$ and (21) is established. ☐

`55. Lecture, Wednesday 28.6.06`

Theorem 21. *The Costa surface f_r with r chosen as in Lemma 20, is embedded and has finite total curvature 12π.*

Proof. The triangle Δ with vertices $0, 2, 2+2i$ parameterizes an eighth of the Costa surface. We may assume $f(0) = 0$. It is sufficient to prove:

- $f(\Delta)$ is embedded and contained in an octon of $\mathbb{R}^3$.

• The eight different copies of the triangle tesselating Q are each mapped to a different octon of $\mathbb{R}^3$.

The second property follows directly from the first: The symmetry properties allow us to conlude that the eight copies of Δ which tesselate U each map under f to a different octon of $\mathbb{R}^3$.

To show the first property, consider the Gauss image $\nu = \mathrm{st} \circ g$ of Δ. We claim that Δ is mapped under g biholomorphically to

$$M = \{z = x + iy : x > 0, y < x\} \setminus g([0, 2]).$$

There is a unique branch point of rp' on $(0, 2)$, see Lemma 16. Thus $g = rp'$ maps the arc $[0, 2]$ onto some interval $[0, x_0]$, where x_0 is the image of the branch point.

There are no branch points on Δ. Using the argument given in the proof of the Riemann mapping Theorem 29 Thus each value in M is attained the same number of times. This number is one in a neighbourhood of $\partial\Delta$, and hence one everywhere.

The point $(1, 0, 0) = \mathrm{st}(1)$ is the midpoint of the hemisphere $\{z \in \mathbb{C}, 0 < \langle \mathrm{st}(z), \mathrm{st}(1)\rangle\} = \{z, \operatorname{Re} z > 0\}$. All points of M have positive real part, and so for $z \in \Delta$ the Gauss map $g = rp'$ makes a positive scalar product with $\mathrm{st}(1)$. Thus the normal has a nonzero x-component, that is, the tangent plane never contains the x-direction. Hence the projection of f to the yz-plane, $\pi_x \circ f : \Delta \to \mathbb{R}^2$, is an immersion.

We would like to assert that $\pi_x \circ f$ is injective, so that $f : \Delta \to \mathbb{R}^3$ is graph with respect to the x-direction. To see this, let us analyze $f(\partial\Delta)$. It is sufficient to From the special values stated in Lemma 15(iii) we can assert the following.

a) The curve $(0, 2 + 2i)$ is mapped under f to a straight ray in the $x - y$ direction of $\mathbb{R}^3$. Thus under π_x it maps to the negative y-axis.

b) The image of the arc $(0, 2)$ is contained in the xz-plane, the curve runs from the saddle to one end. Since $h = q$ is real and positive on $(0, 2)$ we have that $f^3 \circ \gamma = \operatorname{Re} \int h > 0$. Thus the curve projects to the positive z-axis.

c) The curve $(2, 2 + 2i)$ is mapped under f to a planar symmetry curve in the yz plane; it runs from catenoid end to planar end. We claim that the Gauss curvature K is strictly negative along this arc. Indeed, on $(2, 2 + 2i)$ neither g nor h have zeros or poles, and g' is nonzero as g has no branch point on the arc. Thus K is nonzero and so the curvature line $f(2, 2 + 2i)$ has nonzero curvature; moreover the limiting positions of the normal are opposite, $g(2i) = 0$ and $g(2 + 2i) = \infty$. Consequently, the z-component along this curve is strictly monotone, while the y-component is strictly monotone on As π_x leaves the curve invariant, and the mean curvature of f vanishes we must have that $\pi_x \circ f(2, 2 + 2i)$ is concave w.r.t. the side of the immersion $f(\Delta)$.

Since $\pi_x \circ f$ is an immerion, the image of Δ locally sits to one side of the arcs (a) and (b). Moreover, (c) also has the image to one side. Since $\pi_x \circ f$ is an immersion, (c) does not intersect (a) or (b). Hence $\pi_x \circ f(\Delta)$ bounds a domain in a quadrant of $\mathbb{R}^2$, and $f(\Delta)$ is a graph over this domain, hence embedded.

We need to show that $f(\Delta)$ is contained in an octon. Since the projection is in a quadrant, $f^2(p) < 0$, $f^3(p) > 0$ for all $p \in \Delta$. Moreover, all three boundary arcs $f(\partial\Delta)$ satisfy $x \geq 0$, and this holds as well at a neighbourhood of the singular points 2 and $2i$ which map to the ends; this will follow from our analysis of ends below. By the maximum principle, we also have $f^1(p) > 0$ for all $p \in \Delta$. This establishes the embeddedness in an octon.

`54. Lecture, Tuesday 4.7.06`

We have seen $g(\Delta) = M$. The Gauss image $\nu(\Delta) = (st \circ g)(\Delta)$ is bounded by two great circle arcs which make a 135°-angle. Thus it has area $A(\mathrm{st}(M)) = \frac{3}{8}4\pi$. Since g extends to the fundamental domain U by reflection in great circles, we have, as claimed,

$$-\int_U K\, dS \overset{I1.3}{=} A_U(\nu) = 8A_\Delta(\nu) = 8A(\mathrm{st}(M)) = 12\pi. \qquad \square$$

In reality, Weierstrass data are not given but must be derived. If sufficient symmetries can be assumed, this task can be performed systematically. Let us show how this can be done on the example of the Costa surface:

1. We determine the topology: It is a torus with three punctures.
2. We determine the geometry of U: The fundamental domain has two perpendicular diagonals of reflection making the fundamental domain a rhombus. Moreover, there are two planar symmetry curves (through the saddle), mapping to mirror planes, and so the fundamental domain is also a rectangle. In conclusion the domain must be a square. The boundary of the square also maps to planar symmetry curves. The locations of the ends now become obvious: The planar end is on the intersection of the boundary curves, the catenoids on the intersection of ∂U with the coordinate axes.
3. The Gauss map g is $st^{-1}(0, 0, 1) = 0$ at the two catenoid ends and at the saddle origin. It is $st^{-1}(0, 0, -1) = \infty$ on the planar middle end. Assuming the zeros are simple (as supported by the end asymptotics) and the planar end is the unique pole, the pole must be of third order. This determines the Gauss map as the elliptic function p' up to a complex factor. If we assume that the x-axis maps to the xz-plane, that is g is real there, then this leaves only a real constant r. Altogether we conclude $g = rp'$ for some $r \in \mathbb{R}$.
4. We determine h: The only finite zero or pole of g is 0; thus $h(0) = 0$. The only other location of zeros and poles for h is at the three singular points. At the planar end the height differential is $h = 0$. At all other points, the tangent plane is not horizontal, and so $h \neq 0$. Moreover, at the catenoidal ends $h = \infty$ (these assertions are yet to be verified).

This determines the height differential as p, up to a complex factor. Since the x-axis is a planar symmetry curve on which g is real, so is h. This leaves a real factor, which we can set to 1 by scaling the entire surface. Thus $h = q$.
5. It must then be checked that the Weierstrass data U, g, h indeed lead to a surface with the desired properties of symmetry, period closure, and embeddedness. The above statements and proofs verify this for the Costa surface.

Remarks. 1. The Costa surface generalizes to higher dihedral symmetry with higher genus. To get an idea of the geometry, note that two (vertically stacked) octons of $\mathbb{R}^3$ contain a generating piece of the standard Costa surface; the boundary halfplanes each contain two different curves of planar reflection. This wedge of angle $\pi/2$ is replaced by a wedge of angle π/k for $k \geq 3$. Our construction generalizes, including the construction of the necessary meromorphic functions (which are no longer elliptic). See [HK].

2. The Costa surface is embedded and has three ends. All such minimal surfaces have been analyzed by Hoffman, Meeks, and Karcher. They form a continuous one-parameter family. For these generalized surfaces the middle end may also be catenoidal, and the two outer ends no longer have the same asymptotic growth. See [HK]. Up to today, no other (nontrivial) minimal surface family has been completely analyzed.

3. Global Weierstrass representation

To present the global version of the Weierstrass representation we need the notion of a Riemann surface.

3.1. **Riemann surfaces.** In short, a Riemann surface is a manifold of real dimension 2 with a complex structure.

Definition. Let R be a metric space. A *chart* (U, φ) of X is a topological map $\varphi\colon U \to R$ where $U \subset \mathbb{C}$ is open. Two charts (U, φ), (V, ψ) are *biholomorphically compatible* if the coordinate change

$$\psi^{-1} \circ \varphi$$

is a biholomorphic mapping between the subsets of $\mathbb{C}$ where the mapping is defined.

Compatibility is an equivalence relation. A family of compatible charts

$$\mathcal{A} = \{(U_\alpha, \varphi_\alpha), \alpha \in A\}$$

which cover, $\bigcup_{\alpha\in A} U_\alpha = R$, is called an *atlas* on R. Here A is an arbitrary index set. Two atlasses $\mathcal{A}, \mathcal{A}'$ are *equivalent* if their charts are pairwise compatible. We call an equivalence class of atlasses a *holomorphic structure* on R.

Definition. A *Riemann surface* is a connected metric space with a holomorphic structure.

Examples. 1. Each open connected subset $\Omega \subset \mathbb{C}$ is a Riemann surface. An atlas is given by $\mathrm{id}\colon \Omega \to \Omega$. Another (maximal) atlas is

$$\mathcal{S} = \{(U, \varphi) : U \text{ open} \subset \mathbb{C},\ \varphi\colon U \to \Omega \text{ biholomorphic onto its image}\}.$$

2. A closed subset of $\mathbb{C}$ (with nonempty boundary) is not a Riemann surface: A boundary point does not have a neighbourhood mapping homeomorphically to an open subset of $\mathbb{C}$.

3. The *Riemannian sphere* is the set $R = \overline{\mathbb{C}} = \mathbb{C} \cup \infty$ with the two charts $(\mathbb{C}, \mathrm{id})$ and $(\mathbb{C}, \frac{1}{z})$. The second chart is under the convention $\frac{1}{0} = \infty$. Clearly, the two charts cover R. The surface R is topologically $\mathbb{S}^2$ and thus is an example of a compact Riemann surface.

4. Let Λ be a lattice of rank 1 or 2. Then the quotient $T := \mathbb{C}/\Lambda$ is a compact Riemann surface, namely a cylinder or torus, respectively. Here the quotient denotes the set of equivalence classes under the relation

$$z \sim z + w \quad \text{for } z \in \mathbb{C} \quad \text{and } w \in \Lambda.$$

We denote the equivalence class of z by $z + \Lambda$ and call the map from points to classes

$$\pi\colon \mathbb{C} \to T, \qquad \pi(z) := z + \Gamma$$

a projection. A set $U \subset T$ is open iff $\pi^{-1}(U)$ is open. By this very definition π is continuous. Let us now define charts: Provided π maps a set $V \subset \mathbb{C}$ injectively to T we consider $(V, \pi = \pi_V)$ a chart. Then clearly π_V is a homeomorphism, and the charts cover. Let us check two charts are compatible: For each z, the coordinate change $\pi_V^{-1} \circ \pi_U(z)$ is a translation by some $w(z) \in \Lambda$. By continuity, $w(z)$ is constant, hence w is constant on connected components, hence a holomorphic translation, hence biholomorphic.

`54. Lecture, Wednesday 5.7.06` ______

5. Historically, Riemann surfaces were introduced in order to create domains on which non-biholomorphic functions, become welldefined. Let us consider the simplest example of the square root. Writing in polar coordinates we should have $\sqrt{z} = \sqrt{re^{i\varphi}} = \sqrt{r}e^{i\varphi/2}$. But the half-angle cannot be definded continuously on $\mathbb{C}$, and indeed $\sqrt{r}e^{i(\pi+\varphi/2)}$ is another root of z. So we are led to consider the numbers $R := \{z = re^{it}, r > 0, 0 \leq t < 4\pi\}$ with the obvious identification at the bounding ray. Then $\sqrt{\ }\colon R \to \mathbb{C} \setminus \{0\}$ becomes a welldefined bijection; we can in fact include 0 to R the function $\sqrt{\ }$ remains welldefined. The point 0 is, however, a *branch point* of $R \cup \{0\}$: its tangent space has a total angle of 4π. Thus $R \cup \{0\}$ is a generalized Riemann surface, called a *branched Riemann surface*. It is a branched cover of $\mathbb{C}$.

We need functions on and between Riemann surfaces.

Definition. (i) A function $f\colon R \to \mathbb{C}$ is *holomorphic* or *meromorphic*, if for each chart (U,φ) the composition $f\circ\varphi\colon U \to \mathbb{C}$ is holomorphic or meromorphic, respectively.
(ii) A function $f\colon R_1 \to R_2$ between two Riemann surfaces R_1, R_2 is holomorphic if for each chart (U,φ) of R_1 and (V,ψ) of R_2 the composition $\psi^{-1}\circ f\circ\varphi$ is holomorphic.

Note that holomorphicity is well-defined as coordinate changes preserve it. In case (i), any specific chart (U,φ) gives rise to a series representation of f, namely the power or Laurant series of $f\circ\varphi$; similarly for a pair of charts in case (ii).

Remark. Meromorphic functions f on a Riemann surface R can as well be defined by saying that f is a map from the Riemann surface R to the sphere $\overline{\mathbb{C}}$ which is holomorphic and not the constant ∞.

Local properties of holomorphic functions of the complex plane extend to holomorphic functions on Riemann surfaces. For instance, the strong maximum principle generalizes to Riemann surfaces and leads to an interesting global property:

Proposition 22. (i) Strong maximum principle*: If a holomorphic function $f\colon R \to \mathbb{C}$ has a modulus $|f|$ which attains a maximum then it must be constant.*
(ii) *A holomorphic function f on a compact Riemann surface R must be constant.*

Proof. (i) Suppose $|f|$ takes maximum at $p \in R$. By the covering property of an atlas, there is a chart (U,φ), such that $p \in \varphi(U)$. By the classical maximum principle, the holomorphic map $f\circ\varphi$ is constant on U. If (V,ψ) is any other chart with $\varphi(U)\cap\psi(V) \neq \emptyset$ then by unique analytic continuation $f\circ\varphi$ is also constant on V. Consequently the set of points there f is constant is an open subset of R, and anyway it is closed. By connectedness of R the function f is a constant.

(ii) Since R is compact, f must take a maximum. The result follows from the first part. $\square$

An interesting problem is to determine a Riemann surface up to biholomorphic equivlence.

Examples. 1. The punctured complex plane $\mathbb{C}^*$ is not conformally equivalent to any annulus $A_{r,R} = \{r < |z| < R\}$ with $0 < r < R$. Indeed assume on the contrary we had a biholomorphic map $f\colon \mathbb{C}^* \to A$. The singularity at 0 of this map must be removable, as the values of any neighbourhood of 0 are bounded (the singularity cannot be a pole or essential). Thus f has a holomorphic extension as a map from $\mathbb{C}$ to A. But Liouville's theorem contradicts this fact. It is more difficult to see that $A_{1,R}$ and $A_{1,R'}$ are not equivalent when $R \neq R'$.
2. The exponential map $e^z = e^{\operatorname{Re} z}e^{i\operatorname{Im} z}$ is surjective from $\mathbb{C}$ to $\mathbb{C}^*$. Moreover it is periodic, $e^z = e^{z+2\pi i}$, and hence welldefined on the quotient of $\mathbb{C}$ under the lattice $\Lambda = 2\pi i\mathbb{Z}$ of

rank 1. The quotient $\mathbb{C}/\Lambda$ is a cylinder, and so exp shows the cylinder is conformally equivalent to $\mathbb{C}^*$.

More generally, the uniformization theorem states (see [FK]):

Theorem. *A simply connected Riemann surface R is necessarily biholomorphic to one of the following.*
(Hyperbolic case) *If R is non-compact, it can be conformally equivalent to the open disk D; this case can be characterized by the existence of non-constant positive harmonic functions $h > 0$.*
(Parabolic case) *A non-compact R can as well be biholomorphic to all of $\mathbb{C}$. This case is characterized by the property that any harmonic function with one sign is constant.*
(Elliptic case) *If R is compact it is conformally the Riemann sphere $\overline{\mathbb{C}}$; this means there is a meromorphic map $\varphi\colon R \to \overline{\mathbb{C}}$ which is a homeomorphism. This case is characterized by the property that any harmonic function necessarily is constant.*

In the first case, if R is a bounded subdomain of $\mathbb{C}$ then suitable affine functions $h(z) = c+L(z)$ with L linear (and non-zero) can be taken for h. The term hyperbolic is justified by the fact the hyperbolic plane $\mathbb{H}^2$ is conformally equivalent to the disk D with the standard metric: the Poincaré model gives the equivalence. Note that the version of the Riemann mapping theorem presented in this class proves that each bounded domain is biholomorphic to D.

3.2. **On the global Weierstrass representation.** An oriented minimal surface M carries a holomorphic structure: Indeed, the first fundamental form includes the notion of an angle, and with respect to a choice of normal we can define the oriented 90°-rotation.

Recall from III 2.4 that a conformal map φ (defined on a connected domain) is always holomorphic ($\det d\varphi > 0$) or antiholomorphic ($\det d\varphi < 0$). Invoking a reflection if necessary we can assume that φ is holomorphic. Similarly, by a reflection of the parameter domain U, if necessary, we can assume that a conformal parameterization of a minimal surface is a holomorphic map from U to M where M is endowed with the above holomorphic structure,

A beautiful problem is to determine the conformal type of a minimal surface, that is, to find a Riemann surface R and a $\varphi\colon M \to R$ which is conformal. For finite total curvature surfaces, we will state a theorem which classifies the conformal type and hence makes it a lot simpler to determine Weierstrass data. Recall that a surface M has *finite total curvature* if

$$\int_M |K|\,dS < \infty.$$

For a minimal surface this is equivalent to $-\int K < \infty$. From our examples, the catenoid and Enneper's surface have finite total curvature $\int K = -4\pi$, while the helicoid and any other non-planar periodic surface have infinite total curvature.

Theorem (Osserman)**.** *Let M be a comlete minimal surface in $\mathbb{R}^3$ which has finite total curvature. Then M is conformally equivalent to a compact Riemann surface R, from which finitely many points $p_1, \ldots, p_k$ are removed.*

Remark. Osserman's theorem holds more generally for any surface with Gauss curvature $K \leq 0$; it need not be immersed into $\mathbb{R}^3$.

Example. A doubly connected minimal surface, such as the catenoid is topologically an annulus. By the theorem it must be conformally equivalent to $\mathbb{S}^2$ minus two points. We may assume these two points to be 0 and ∞ and so the theorem says the catenoid is conformally equivalent to $\mathbb{C} \setminus \{0\}$. By the example above, however, it cannot be equivalent to any annulus $r < |z| < R$ with $0 < r < R$.

The Weierstrass representation theorem then has a global analogue which we mention without proof:

Theorem. *Let M be a minimal surface, and R be its Riemann surface according to Osserman's theorem. Then there is a meromorphic function g on R, and a holomorphic h, such that $f = \operatorname{Re} \int \Phi$, with Φ as in (5). Conversely, for given Weierstrass data on some Riemann surface R there is a minimal surface $f\colon \tilde{R} \to \mathbb{R}^3$ defined on the universal covering $\tilde{R}$ of R. The mapping f is actually defined on R provided the* period $P_\gamma(\Phi) := \operatorname{Re} \int_\gamma \Phi = 0$ *vanishes for all closed curves γ.*

All what is necessary to prove the first part is to show that the integration is well defined on the Riemann surface. This follows just as the well-definedness of a surface integral from chain rule and substitution formula. We have not proven even the local version of the second part. So let us only point to one fact: If a component of Φ_j has a residue which is not purely imaginary, then $\int \Phi_j$ is not well-defined on a punctured set, but must be defined on its universal covering. Let us finally point out that if $P_\gamma(\Phi)$ vanishes for a curve γ it will vanish for all curves homotopic to γ.

We follow [O, §9] and sketch the main steps of the proof (see also [L, III §6]). A number of crucial topological and analytical arguments will only be cited, however.

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Suppose $U \subset \mathbb{C}$ is a domain and $f\colon U \to \mathbb{R}^3$ induces a first fundamental form g on U. In the following it will not matter that g is induced by f, but only that for each point of U it is a positive definite bilinear form. We then call g a *(Riemannian) metric* on U. A curve $\gamma\colon (a,b) \to U$ is called *divergent* if $\lim_{t\to b} \gamma(t) \in \partial U$ or $= \infty$. The metric g is called

complete if no divergent curve has finite length

$$L(\gamma, g) := \int_a^b \sqrt{g(\gamma', \gamma')}\, dt.$$

Lemma 23. *Let $U \subset \mathbb{C}$ be a domain and suppose $f\colon U \to \mathbb{R}^3$ induces a metric conformally equivalent to the euclidean metric, that is*

$$g_{ij} = \lambda^2 \delta_{ij} \qquad \text{for } \lambda\colon U \to (0,\infty).$$

Moreover, suppose U is complete with respect to g. If there exists a harmonic function $h\colon U \to \mathbb{R}$ such that

$$h(z) \geq \log \lambda(z) \qquad \text{for all } z \in U \tag{22}$$

then U can only be conformally equivalent to $\mathbb{C}$ or to $\mathbb{C}$ minus a point.

Proof. Let us consider a further metric $\tilde{g} := e^{2h}\langle .,. \rangle$. Then any divergent curve γ has length

$$L(\gamma, \tilde{g}) = \int e^h |\gamma'|\, dz \overset{(22)}{\geq} \int_\gamma \lambda |\gamma'|\, dz = L(\gamma, g) \overset{g \text{ complete}}{=} \infty,$$

meaning that $\tilde{g}$ is complete as well.

Let us now invoke a fact which we do not prove: Given a harmonic function h on a simply connected domain U there is a holomorphic function $f\colon U \to \mathbb{C}$, such that $\operatorname{Re} f = h$. If U is not simply connected then similarly f may be defined on the universal covering $\hat{U}$.

Now consider the complex curve integral

$$\psi\colon U \to \mathbb{C}, \qquad \psi(z) := \int_{z_0}^z e^{f(\zeta)}\, d\zeta.$$

Then

$$|\psi'(z)| = |e^{f(z)}| = |e^{\operatorname{Re} f(z)}| = e^h. \tag{23}$$

We can interpret (23) to say that

$$L(\gamma, \tilde{g}) = L(\psi \circ \gamma, \langle .,. \rangle) \qquad \text{for any curve } \gamma \text{ in } \hat{U}. \tag{24}$$

Indeed, by differentiation,

$$\frac{d}{dt} L\big(\gamma(t), e^h \langle .,. \rangle\big) = \big|\sqrt{\tilde{g}(\gamma', \gamma')}\big| = e^h |\gamma'|$$

while

$$\frac{d}{dt} L\big(\psi \circ \gamma(t)), \langle .,. \rangle\big) = \sqrt{\langle \psi'(\gamma)\gamma', \psi'(\gamma)\gamma' \rangle} = |\psi'||\gamma'|.$$

By (23) the latter two expressions agree. Equivalently we can say that ψ is a (local) isometry of $(\hat{U}, \tilde{g})$ to $(\mathbb{C}, \langle .,. \rangle)$.

Since $\hat{U}$ is simply connected, the identity (24) means that ψ extends to a distance preserving map (global isometry) from $(\hat{U},\tilde{g})$ to $(\mathbb{C},\langle .,.\rangle)$. By completeness of $(\hat{U},\tilde{g})$ the map must be onto $\mathbb{C}$.

Thus the mapping ψ gives us a conformal equivalence of the covering $\hat{U}$ with the plane $\mathbb{C}$. But then U is conformally equivalent to a quotient of $\mathbb{C}$ under some group of deck transformation. Either this group is the identity, in which case $U=\mathbb{C}$, or it is a lattice of rank 1, in which case U is $\mathbb{C}$ minus a point (cf. Ex. 2 above). It cannot be a lattice of rank 2: the quotient is compact and can therefore not be homeomorphic to an open domain of $\mathbb{C}$. □

While the previous lemma establishes a subharmonic function, we need in fact a harmonic function for Lemma 24 to apply. Such a function can be constructed using λ.

Lemma 24. *Let A be the annulus $\{z : 0<r<|z|<R\le\infty\}$ endowed with some Riemannian metric $g=\lambda^2\langle .,.\rangle$. Moreover, suppose there is a harmonic function $h\colon A\to\mathbb{R}$ satisfying* (22)*. If each curve γ which is divergent to the outer boundary $\{|z|=R\}$ has infinite length then $R=\infty$.*

Proof. Suppose in addition to the assumptions that each path divergent to the inner boundary also had infinite length. Then Lemma 23 would contradict $R<\infty$ and the statement was proved.

If $R<\infty$ we will modify the metric as to assign infinite length to the paths limiting in the inner boundary, while preserving all other properties.

Our claim is invariant by scaling of the annulus. Scaling specifically by $\frac{1}{\sqrt{rR}}$ we can assume that

$$A=\{z:\frac{1}{\rho}<|z|<\rho\}.$$

We claim that the metric

$$\tilde{g}:=\lambda^2(z)\lambda^2\Big(\frac{1}{z}\Big)\langle .,.\rangle$$

on A satisfies the assumptions of the lemma.

Indeed,

$$\log\left(\lambda(z)\lambda\Big(\frac{1}{z}\Big)\right)=\log\lambda(z)+\log\lambda\Big(\frac{1}{z}\Big)\le h(z)+h\Big(\frac{1}{z}\Big)=:\tilde{h}(z).$$

Now $\tilde{h}$ is harmonic. But $h(\frac{1}{z})$ is also harmonic as the composition of the harmonic map $h(z)$ with the conformal mapping $z\mapsto\frac{1}{z}$ (see Lemma III,12). Clearly, $\tilde{h}>0$, and now paths limiting in either component of ∂A attain infinite length. □

In view of the last two lemmas the claim of Osserman's theorem is evident, provided we had a harmonic function on the minimal surface. The next lemma gives us a little less, namely a subharmonic function.

Lemma 25. *Suppose a minimal surface $f\colon U\to\mathbb{R}^3$ is given by Weierstrass data, that is the first fundamental form is $\lambda^2\langle .,.\rangle$ with conformal factor $\lambda:=\frac{1}{2}\big(|g|+\frac{1}{|g|}\big)|h|>0$. Then the Gauss curvature $K=\frac{-16}{(|g|+\frac{1}{|g|})^4}\left|\frac{g'}{g}\right|^2\frac{1}{|h|^2}$ can also be expressed as*

$$K=-\frac{\Delta\log\lambda}{\lambda^2} \tag{25}$$

on U. In particular, $\log\lambda$ is a subharmonic function on U, that is $\Delta\log\lambda=-\lambda^2K\ge 0$.

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Proof. Rewriting the desired equation as $K\lambda^2=-\Delta\log\lambda$, we need to show that

$$4\frac{1}{\big(|g|+\frac{1}{|g|}\big)^2}\left|\frac{g'}{g}\right|^2=\Delta\log\left(\frac{1}{2}\Big(|g|+\frac{1}{|g|}\Big)|h|\right)>0$$

or equivalently

$$4\frac{|\partial_z g|^2}{(|g|^2+1)^2}=\Delta\log\Big(|g|+\frac{1}{|g|}\Big)+\Delta\log|h|. \tag{26}$$

Let us check that $\Delta\log|h|=0$. We rewrite $\Delta=\partial_{\bar z}\partial_z$ and recall that for h holomorphic the Cauchy-Riemann equations imply

$$\partial_z\overline{h}=(\partial_x-i\partial_y)(\operatorname{Re}h-i\operatorname{Im}h)=\partial_x\operatorname{Re}h-\partial_y\operatorname{Im}h-i(\partial_x\operatorname{Im}h+\partial_y\operatorname{Re}h)=0, \tag{27}$$

$$\partial_{\bar z}h=(\partial_x+i\partial_y)(\operatorname{Re}h+i\operatorname{Im}h)=\partial_x\operatorname{Re}h-\partial_y\operatorname{Im}h+i(\partial_x\operatorname{Im}h+\partial_y\operatorname{Re}h)=0. \tag{28}$$

Thus

$$\partial_z\log|h|=\frac{\partial_z|h|}{|h|}=\frac{\partial_z\sqrt{h\overline{h}}}{|h|}=-\frac{\partial_z(h\overline{h})}{2|h|^2}=-\frac{\overline{h}\,\partial_z h+h\,\partial_z\overline{h}}{2|h|^2}\overset{(27)}{=}-\frac{\overline{h}\,\partial_z h}{2|h|^2}$$

Consequently,

$$\Delta\log|h|=-\partial_{\bar z}\frac{\overline{h}\,\partial_z h}{2|h|^2}=-\frac{\partial_{\bar z}(\overline{h}\,\partial_z h)2|h|^2-\overline{h}\,\partial_z h\,\partial_{\bar z}(2|h|^2)}{4|h|^4} \tag{29}$$

But the numerator of (29) gives

$$\begin{aligned}\partial_{\bar z}(\overline{h}\,\partial_z h)&2|h|^2-\overline{h}\,\partial_z h\partial_{\bar z}(2|h|^2)\\ &=\partial_{\bar z}\overline{h}\,\partial_z h2|h|^2+\overline{h}\,\underbrace{\partial_{\bar z}\partial_z h}_{=\Delta h=0}2|h|^2-2\overline{h}\,\partial_z h\big(\overline{h}\,\underbrace{\partial_{\bar z}h}_{=0\text{ by }(28)}+h\,\partial_{\bar z}\overline{h}\big)\\ &=\partial_{\bar z}\overline{h}\,\partial_z h(2|h|^2-2\overline{h}h)=0\end{aligned}$$

This asserts $\Delta\log|h|=0$. We skip the similar, but more lengthy, calculation which verifies (26). □

Remarks. 1. In particular, the calculation given in the proof verifies $\Delta \log |z| = 0$ on $\mathbb{C}^*$. The harmonic function $\log |z|$ is called *fundamental solution* of Δ in dimension $n = 2$. It can also be determined by solving the equation $\Delta u = 0$ among functions which are rotationally symmetric, $u = u(|z|)$; in this case the Laplace equation simplifies to an ordinary differential equation.

2. Beyond minimal surfaces, the relation (25) is valid for arbitrary (two-dimensional) surfaces in conformal parameters. Given the fact that conformal parameters can always be introduced, it indicates that Gauss curvature depends on the first fundamental form alone (and not on the second fundamental form it was defined with). This is an important insight, which is the content of Gauss's *theorema egregium*: K belongs to the intrinsic geometry of surfaces.

Lemma 26. *Let U be a domain which carries a positive harmonic function and consider a metric $g = \lambda^2\langle .,. \rangle$ on U which is complete. Suppose that*

$$\Delta \log \lambda \geq 0 \quad \text{and} \quad \int_U |\Delta \log \lambda| \, dxdy < \infty. \tag{30}$$

Then there exists a harmonic function $h\colon U \to \mathbb{R}$ satisfying (22).

Note that the geometric meaning of (30) is that the metric has $K \leq 0$ and finite total curvature $-\int K dS = -\int \frac{K}{\lambda^2}(\lambda^2 dxdy) < \infty$.

Proof. We main step of this proof is a result we cite from Ahlfors and Sario, Riemann surfaces, IV.6. If U carries a positive harmonic function, then for $z \neq \zeta$ there is a Green's function $G(z, \zeta) > 0$ which is harmonic. Moreover, the integral

$$u(z) = \frac{1}{2\pi} \int_U G(z, \zeta) \Delta \log \lambda(z) \, dxdy$$

exists by (30) and, since the integrand is positive, also $u \geq 0$. By Poisson's formula, $\Delta u = -\Delta \log \lambda$. Thus $h(z) := u(z) + \log \lambda(z) \geq \log \lambda(z)$ is harmonic. $\square$

Proof of Osserman's theorem. We assume that M has finite topology, that is, M is homeomorphic to a compact Riemann surface minus finitely many points. (This property can in fact be infered from finite total curvature by invoking a theorem of Huber).

Consider a closed subset $M_0 \subset M$ which is compact, such that $M \setminus M_0$ is the disjoint union $\bigcup_{j=1}^k U_j$ of open topological annuli. We can assume that each annulus is bounded by an analytic Jordan curve Γ_j.

Each U_j can conformally be mapped to some standard annulus $A_j := \{1 < |z| < R_j \leq \infty\}$, where Γ_j maps to $|z| = 1$. We now apply Lemma 26 to A_j. Note that A_j has a positive harmonic function, namely $\log |z|$. By the Lemma there exists a harmonic function

$h\colon A_j \to \mathbb{R}$ such that $h(z) \geq \log \lambda(z)$. Thus we can appeal to Lemma 24 to obtain $R = \infty$. In particular, each A_j compactifies with one point p_j to a disk $\tilde{A}_j$.

We now let $\tilde{M}$ be the compact Riemann surface given by M_0 with the union of the $\tilde{A}_j$ "glued back" along Γ_j (it must be shown this defines a Riemann surface). $\square$

Essential singularities like ($e^{1/z}$ in 0) can occur in Weierstrass data. For instance, the helicoid can be parameterized with an essential singularity. However, for finite total curvature:

Proposition 27. *If a minimal surface f of a finite total curvature is given by Weierstrass data, then the Gauss map g extends to a meromorphic function on the compact Riemann surface R of Osserman's theorem.*

Proof. On $R \setminus \{p_1, \dots, p_k\}$, the function g is certainly meromorphic (that is, valued in the Riemann surface $\mathbb{S}^2$). But if any point p_j would be an essential singularity of g then by Picard's theorem g would assume each value infinitely often, with at most two exceptions. But then the spherical area of $A_R(\nu) = A_R(\mathrm{st} \circ g)$ would be infinite, contrary to the assumption of finite total curvature. Thus g has a pole at p_j and is meromorphic on R. $\square$

References. These results and further reading are contained in [O, §9] and [L].

3.3. **Ends.** We call $f\colon D \setminus \{0\}$ an *end*. More generally, if $f\colon R \setminus \{p_1, \dots, p_k\}$ is minimal then we call the restriction of f to an annular neighbourhood of p_j in $R \setminus \{p_1, \dots, p_k\}$ an end. As before, an end is *complete* if any curve divergent to 0 has length $L(f \circ \gamma) = \infty$.

The plane, the catenoid, the helicoid, and the Enneper surface have different types of ends. We have seen that k-Enneper ends are very flexible (lower order polynomials can be added to the Weierstrass data). However, when we assume embeddedness and finite total curvature, ends are rather rigid:

Theorem 28. *Suppose $f\colon D \setminus \{0\} \to \mathbb{R}^3$ is a complete embedded minimal end of finite total curvature and with Gauss map at 0 equal to $(0, 0, \pm 1)$. Then a neighbourhood of 0 can be reparameterized in graph form $\tilde{f}(x, y) = \big(x, y, u(x, y)\big)$ with*

$$u(x, y) = a \log \rho + b + \frac{1}{\rho^2}\big\langle V, (x, y)\big\rangle + O\Big(\frac{1}{\rho^2}\Big), \qquad \text{where } \rho(x, y) := \sqrt{x^2 + y^2}, \tag{31}$$

for some $a, b \in \mathbb{R}$, $V \in \mathbb{R}^2$.

The proof will also give the following: The first two components Φ_1, Φ_2 have poles of order two at 0 with no residues, while Φ_3 either has a simple pole (iff $\alpha \neq 0$) or it is regular (iff $\alpha = 0$).

Examples. For a plane, $a = 0$, while for the end of a catenoid, $a \neq 0$. We say an end is asymptotically planar or catenoidal if $a = 0$ or $a \neq 0$, respectively, in (31).

Proof. In general, g can be branched at 0, that is $g(z) = cz^k + O(z^{k+1})$ for $k \geq 2$. For simplicity, let us deal only with the unbranched case $k = 1$. Decreasing D if necessary, by the inverse function theorem we can assume $g(z) = z$. Then the Weierstrass representation gives

$$x_1 = \operatorname{Re}\int \frac{1}{2}\Big(\frac{1}{z} - z\Big)h\,dz \qquad x_2 = \operatorname{Re}\int \frac{i}{2}\Big(\frac{1}{z} + z\Big)h\,dz \qquad x_3 = \operatorname{Re}\int h\,dz. \tag{32}$$

As $\operatorname{Re}(iz) = \operatorname{Re}(ix - y)) = -\operatorname{Im} z$ we have

$$\begin{aligned} 2(x_1 - ix_2) &= \operatorname{Re}\int \frac{1}{z}h\,dz - i\operatorname{Re} i\int \frac{1}{z}h\,dz - \operatorname{Re}\int zh\,dz - i\operatorname{Re} i\int zh\,dz \\ &= \int \frac{1}{z}h\,dz - \overline{\int zh\,dz} \end{aligned}$$

The Laurant expansion of h at 0 lets us write

$$h = c_{-k}\frac{1}{z^k} + \ldots + c_{-1}\frac{1}{z} + c_0 + zw_1(z) \tag{33}$$

with w holomorphic. We have $c_0 = 0$ since otherwise the first integral of $x_1 - ix_2$ is not well-defined (a logarithm). Similarly the second integral implies $c_{-2} = 0$.

We claim h must have a pole at 0. Indeed, the metric (11)

$$\frac{1}{2}\Big(|g| + \frac{1}{|g|}\Big)|h| = \frac{1}{2}\Big(|z| + \frac{1}{|z|}\Big)|h|$$

must have a pole at 0 in order to be complete, and so given that $c_0 = 0$ we infer h has a pole.

We claim that h has a simple pole at 0. If not, let $c_{-k}\frac{1}{z^k}$ with $k \geq 3$ be the leading term in (33). Consider the (large) image of a small circle $|z| = \varepsilon$. Then

$$\int \frac{1}{z}h\,dz = \int c_{-k}z^{-k-1} + \ldots\,dz = \frac{-1}{k}c_{-k}z^{-k} + \ldots$$

and hence $x_1 - ix_2$ must have self-intersections. Since the normal at 0 is vertical and the end is embedded the image circle of $|z| = \varepsilon$ must project 1-1 to the xy-plane. We conclude that $c_{-k} = 0$ and so as claimed

$$h(z) = c_{-1}\frac{1}{z} + zw_1(z), \qquad c_{-1} \neq 0, \quad w_1 \text{ holomorphic.}$$

Moreover, for $x_3 = \int h\,dz$ to be well-defined we must have c_{-1} real (recall $\int \frac{1}{z} = 2\pi i$ over a circle). From (32) we conclude that the Weierstrass integrands Φ_1, Φ_2 have no residues at 0, but second order poles.

We summarize our results:

$$x_1 - ix_2 = -c_{-1}\frac{1}{z} + c + zw_2(z) + \overline{z}\overline{w_3(z)}, \tag{34}$$

$$x_3 = \text{const} + c_{-1}\log|z| + O(|z|^2), \tag{35}$$

where w_2, w_3 are holomorphic, and c is a constant. Since the end is complete, $x_1 - ix_2$ must be unbounded, and so $c_{-1} \neq 0$.

We will finally show that $x_3(x_1, x_2)$ has the desired form. First, observing $z - \overline{z} = 2\operatorname{Re} z$ we find

$$\begin{aligned} \rho^2 = x_1^2 + x_2^2 &= (x_1 + ix_2)(x_1 - ix_2) = \overline{(x_1 - ix_2)}(x_1 - ix_2) \\ &= |c_{-1}|^2\frac{1}{|z|^2} - 2\operatorname{Re}\Big(c \cdot \overline{c_{-1}\frac{1}{z}}\Big) + c + O(|z|); \end{aligned}$$

in particular $|z| = O(\frac{1}{\rho})$. Second, we multiply (34) with $\overline{c}$ and take the real part:

$$\operatorname{Re}\Big(\overline{c}\,c_{-1}\frac{1}{z}\Big) = -\operatorname{Re}\big(\overline{c}(x_1 - ix_2)\big) + |c|^2 + O(|z|)$$

We combine the last two equations

$$\begin{aligned} |c_{-1}|^2\frac{1}{|z|^2} &= \rho^2 + 2\operatorname{Re}\big(\overline{c}(x_1 - ix_2)\big) + \text{const} + O\Big(\frac{1}{\rho}\Big) \\ &= \rho^2\Big(1 + \frac{1}{\rho^2}\Big)\Big(2\operatorname{Re}\big(\overline{c}(x_1 - ix_2)\big) + O(\rho^2)\Big) \end{aligned}$$

The logarithm of both sides in conjunction with $\log(1 + x) = x + O(x^2)$ gives

$$\log|z| = \log|c_{-1}| - \log\rho - \frac{\operatorname{Re}(x_1 - ix_2) \cdot c}{\rho^2} + O\Big(\frac{1}{\rho^2}\Big)$$

Substitution in (35) yields

$$x_3(z) = \text{const} - c_{-1}\log\rho - \frac{1}{\rho^2}c_{-1}\operatorname{Re}(x_1 - ix_2) \cdot c + O\Big(\frac{1}{\rho^2}\Big)$$

which is the desired representation (31). □

Corollary 29. (*i*) *An end of an embedded finite total curvature minimal surface is asymptotic to either a plane ($h = b$) or a catenoid ($h = a\log\rho + b$).*
(*ii*) *If the Gauss map branches at an end then the end is flat.*

References. Our presentation of the present subsection follows [HK] where further reading can also be found.

`57. Lecture, Tuesday 18.7.06` ______________________________

4. Conjugate surface constructions

The goal of this short section is to present a further technique to establish the existence of minimal surfaces, the so-called conjugate surface construction. There are various ways to establish the existence of the surface conjugate to a given minimal surface:
1. If the minimal surface is given in terms of the Weierstrass data, the conjugate is essentially the imaginary part of the Weierstrass data. This is the route we take here.
2. The first and second fundamental form of the given minimal surface can be modified. Then Bonnet's fundamental theorem yields the existence of the surface. See, for instance, the article of Karcher in Man. Math. 64 (1989).
3. There is a first order system of differential equations, involving only the differential df of the given minimal surface f. It can be solved using the Poincaré lemma. See, for instance, the article by Grosse-Brauckmann in the Clay mathematics proceedings, vol. 2, 2005.

In any case, integrability of the conjugate data must be shown; in terms of the present exposition this is the assertion that for each minimal surface f there exist Weierstrass data g, h, a fact which we have not proven in the class.

4.1. **The associated family.** The Weierstrass representation $f = \operatorname{Re}\int\Phi$ integrates the real part of a holomorphic null curve in $\mathbb{C}^3$. It may seem reasonable to ask: Is there any meaning of the imaginary part $\operatorname{Im}\int\Phi = \operatorname{Re}\int(-i\Phi)$?

More generally, for U a simply connected domain let us consider the mapping

$$f_\vartheta\colon U\to\mathbb{R}^3, \qquad f_\vartheta(z) := \operatorname{Re}\int_{z_0}^z e^{-i\vartheta}\Phi(w)\,dw, \qquad \text{for } \vartheta\in\mathbb{R},$$

called the *associated family* of f. In particular,

$$f^* := f_{\pi/2} = \operatorname{Re}\int -i\Phi = \operatorname{Im}\int\Phi$$

is called the *conjugate surface*

Let us note a few formulas which are obvious right now: First $f_{\vartheta+2\pi} = f_\vartheta$ and moreover $f_{\vartheta+\pi} = \operatorname{Re}\int e^{-i(\vartheta+\pi)}\Phi = -\operatorname{Re}\int e^{-i\vartheta}\Phi = -f_\vartheta$. Second,

$$\begin{aligned} f_\vartheta(z) &= \operatorname{Re}\Big(\int^z \cos\vartheta\,\Phi - i\sin\vartheta\,\Phi\Big) = \cos\vartheta\,\operatorname{Re}\int^z\Phi + \sin\vartheta\,\operatorname{Im}\int^z\Phi \\ &= \cos\vartheta\, f(z) + \sin\vartheta\, f_*(z) \end{aligned} \tag{36}$$

Suppose that $\Phi\colon U\to\mathbb{C}^n$ is a null curve. Then $e^{-i\vartheta}\Phi$ is also a null curve as

$$\sum_{k=1}^n (e^{-i\vartheta}\Phi_k)^2 = \sum_{k=1}^n (e^{-i\vartheta}\Phi_k)^2 = e^{-2i\vartheta}\sum_{k=1}^n \Phi_k^2.$$

Moreover, $\partial_{\bar z}(e^{-i\vartheta}\Phi) = e^{-i\vartheta}\partial_{\bar z}\Phi$. Thus $e^{-i\vartheta}\Phi$ is a holomorphic null curve, if and only if Φ is one.

It is straightforward to determine the Weierstrass data of f_ϑ:

Theorem 30. *Let U be a simply connected domain, $g, h\colon U\to\mathbb{C}$ holomorphic or meromorphic, respectively. Then the Weierstrass data of the associated family f_ϑ of minimal surfaces are*

$$g_\vartheta = g \quad \text{and} \quad h_\vartheta := e^{-i\vartheta}h.$$

The associated surfaces are have the following geometric quantities:

$$\nu_\vartheta = \nu, \qquad ds_\vartheta^2 = ds^2, \qquad K_\vartheta = K, \qquad b_\vartheta(X,X) = -\operatorname{Re}\Big(e^{-i\vartheta}\frac{g'}{g}hX^2\Big) \tag{37}$$

In particular, all surfaces in the associated family are isometric and have the same Gauss map and Gauss curvature.

Proof. g is independent of ϑ and so is the Gauss map $\nu = \operatorname{st}\circ g$. Since $|h_\vartheta| = |h|$ the conformal factor $\lambda = \frac12\big(|g| + \frac{1}{|g|}\big)|h_\vartheta|$ is also independent of ϑ. By (12) or (25) the Gauss curvature is also independent of ϑ. Finally, (37) follows directly from (13). □

The fact that f and f_ϑ are isometric means that a physical model of the f (made from inelastic material such as metal or plastics) can be deformed into f_ϑ.

Let us now come to the second fundamental form and symmetry properties.

Corollary 31. *$f\circ\gamma$ parameterizes a straight line of $\mathbb{R}^3$ with unit tangent direction $v\in\mathbb{S}^2$ iff $f^*\circ\gamma$ parameterizes an arc of planar reflection for a plane with unit normal $v\in\mathbb{S}^2$. The same holds with f and f^* interchanged.*

Proof. By the Theorem, the first fundamental forms agrees for f and f^*. But geodesics coincide for isometric surfaces, and so γ is a geodesic for f if it is one for f_*.

Moreover, a curve $f\circ\gamma$ is an asymptote line for f precisely if $b(\gamma',\gamma') = \frac{g'}{g}h\gamma'^2 \in i\mathbb{R}$ (see Proposition 8). But this is the case if and only if $b^*(\gamma',\gamma') = -i\frac{g'}{g}h_{\pi/2}\gamma'^2\in\mathbb{R}$ which is equivalent to γ being a curvature line of $f_{\pi/2} = f^*$.

Given these two facts, it remains to refer to Lemma 10: A geodesic curvature line γ is an arc of planar reflection of f^*, and a geodesic asymptote line is straight.

Our arguments hold for any pair f_ϑ and $f_{\vartheta\pm\pi/2}$. □

Theorem 30 refers to the local Weierstrass representation. There is also a generalized global version in terms of a Riemann surface R. In that case, however, the associated family must be defined on the univeral covering $\hat R$ of R.

4.2. **The associated family of catenoid and Enneper surface.** We consider Weierstrass data which will lead to the associated family of the catenoid. They disagree with the data of the catenoid from Sect. IV 1.3. This time, we consider instead

$$U = \mathbb{C}, \qquad g(z) = -e^z, \qquad h_\vartheta(z) = e^{-i\vartheta}.$$

Since g has no zero, f will not have branch points. Then

$$\Phi(z) = e^{-i\vartheta}\Big(\frac{1}{2}(-e^{-z}+e^z), \frac{i}{2}(-e^{-z}-e^z), 1\Big) = e^{-i\vartheta}\Big(\sinh z, -i\cosh z, 1\Big)$$

and so by integration (using that $\cosh' = \sinh$ and $\sinh' = \cosh$)

$$f_\vartheta(z) = \operatorname{Re} e^{-i\vartheta}\Big(\int_0^z \sinh w, -i\int_0^z \cosh w, \int_0^z 1\Big) = \operatorname{Re}\Big(e^{-i\vartheta}\big(\cosh z - 1, -i\sinh z, z\big)\Big).$$

Now

$$\sinh z = \sinh x\cos y + i\cosh x\sin y \qquad \text{and} \qquad \cosh z = \cosh x\cos y + i\sinh x\sin y$$

and so, using real notation,

$$\begin{aligned} f_\vartheta(x,y) &= \cos\vartheta\begin{pmatrix}-1+\cosh x\cos y\\ \cosh x\sin y\\ x\end{pmatrix} + \sin\vartheta\begin{pmatrix}\sinh x\sin y\\ -\sinh x\cos y\\ y\end{pmatrix}\\ &= \cos\vartheta\, f(x,y) + \sin\vartheta\, f_*(x,y).\end{aligned}$$

Note that in this formula, f represents a translated catenoid, while f_* represents the helicoid. This family was mentioned before in Thm. I,14. Each member f_ϑ of the family is a minimal surface, and each of these surfaces can be shown to have a screw symmetry. Note also that the associated family is not defined on the topological annulus of the catenoid: It is only defined on its universal covering.

Let us exemplify Cor. 31; for simplicity we ignore the translation by $(-1,0,0)$. The curve $y \mapsto (0,y)$, maps on the catenoid f to the waist circle $(\cos y, \sin y, 0)$, which is a planar curve of reflection. On the helicoid f_* it maps to the axis $(0,0,y)$ which is a straight line. Similarly, each curve $x \mapsto (x,y)$ maps to a meridian of the catenoid f (again a planar curve of reflection), while it maps to a straight line, namely a ruling, of the helicoid f_*. The Gauss map coincides along the two surfaces; it is worthwhile to realize this fact for instance along the symmetry lines.

Let us now discuss the associated family of Enneper's surface. For Enneper's suface we have the associated family f_ϑ given by

$$U = \mathbb{C}, \qquad g(z) = z, \qquad h_\vartheta(z) = 2e^{-i\vartheta}z,$$

Thus

$$\Phi_\vartheta = e^{-i\vartheta}\Big(1-z^2, i(1+z^2), 2z\Big)$$

and so

$$f_\vartheta(z) = \operatorname{Re}\int_0^z \Phi = \operatorname{Re}\Big(e^{-i\vartheta}\Big(z - \frac{1}{3}z^3, i(z+\frac{1}{3}z^3), z^2\Big)\Big).$$

We claim that f_ϑ and f are congruent: If we rotate the image f_ϑ by an angle $-\vartheta/2$ about the z-axis, then we obtain f up to some rotation in the domain.

To see this, let us first rotate the image; we let F denote the rotated image. Since we rotate about the z-axis, $F^3 := f^3_\vartheta$. Moreover, using complex notation for the rotation in the first two coordinates, we find

$$\begin{aligned} F^1 + iF^2 &:= e^{-i\vartheta/2}(f^1_\vartheta + f^2_\vartheta)\\ &= e^{-i\vartheta/2}\Big(\operatorname{Re}(e^{-i\vartheta}z) + i\operatorname{Re}(ie^{-i\vartheta}z)\Big) + e^{-i\vartheta/2}\Big(\operatorname{Re}\big(-\frac{1}{3}e^{-i\vartheta}z^3\big) + i\operatorname{Re}\big(\frac{i}{3}e^{-i\vartheta}z^3\big)\Big).\end{aligned}$$

Now let us invoke the identities

$$\text{(38)} \qquad \operatorname{Re}\zeta + i\operatorname{Re}(i\zeta) = \overline{\zeta} \qquad \text{and} \qquad \operatorname{Re}\zeta + i\operatorname{Re}(-i\zeta) = \zeta$$

to obtain

$$(F^1+iF^2)(z) = e^{-i\vartheta/2}\,\overline{e^{-i\vartheta}z} - e^{-i\vartheta/2}\frac{1}{3}e^{-i\vartheta}z^3 \overset{(38)}{=} \overline{e^{-i\vartheta/2}z} - \frac{1}{3}\big(e^{-i\vartheta/2}z\big)^3$$

To prove our claim, we want to show that $F(z)$ agrees with the Enneper surface up to a rotation of the parameters; more precisely, the Enneper surface $f(w)$ agrees with $F(e^{i\vartheta/2}w)$.

Thus setting $w := e^{-i\vartheta/2}z$ gives

$$\big(F^1+iF^2\big)(e^{i\vartheta/2}w) = \overline{w} - \frac{1}{3}w^3 \overset{(38)}{=} \operatorname{Re} w - \frac{1}{3}\operatorname{Re} w^3 + i\operatorname{Re}(iw) + \frac{i}{3}\operatorname{Re}(iw^3).$$

Finally noting that $F^3 = e^{-i\vartheta}z^2 = w^2$ we verify

$$F(e^{i\vartheta/2}w) = \operatorname{Re}\Big(w - \frac{1}{3}w^3, i(w+\frac{1}{3}w^3), w^2\Big) = f(w).$$

This proves our claim. Thus up to reparameterization and ambient rotation, the associated family of the Enneper surface coincides with the Enneper surface. The same holds for the higher order Enneper surfaces.

Note that for Enneper's surface, the conformal factor and hence the Gauss curvature depend only on the radius $|z|$. In particular, they are invariant under the rotation $z \mapsto w = e^{-\vartheta/2}z$ and so certainly coincide on the associated family. Let us finally study the symmetry curves of Cor. 31. The arc of planar reflection $x \mapsto f(x,0)$ of Enneper's surface maps under the rotation $z \mapsto w(z) = e^{-i\pi/4}$ to the arc $t \mapsto f(t,t)$ which is a straight line. Conversely, the latter arc of f rotates to the arc of f_* corresponding to the arc of planar reflection $y \mapsto f(0,y)$.

`58. Lecture, Wednesday 19.7.06` ———————————

4.3. **Conjugate Plateau solutions.** In Sect. 5 we solved the Plateau problem for polygonal contours and then used Schwarz reflection across the bounding straight arcs to extend the Plateau solution to a complete minimal surface. A drawback of the method is that only those surfaces can be generated which contain disk-type polygons. Such surfaces are rare and have in fact been classified.

In the present section the first step also is the solution M of a Plateau problem for some polygon Γ. Then we take the conjugate M_* of M. By Cor. 31 the surface M_* is bounded by arcs of planar reflection. Then we can apply the Schwarz reflection principle III,38 mirror reflection generates a complete minimal surface.

Example. Let us go back to the D-surface from Sect. III,5.3. We consider a polygon whose vertices are six of the eight vertices of the unit cube, namely $(0,0,0)$, $(1,0,0)$, $(1,1,0)$, $(1,1,1)$, $(0,1,1)$, $(0,0,1)$. The edges take the six directions $e_1, e_2, e_3, -e_1, -e_2, -e_3$. Let M be the Plateau solution to Γ, a hexagon. By Cor. 31 the conjugate minimal hexagon M_* has its six boundary arcs planar reflection contained in planes normal to the above vectors, that is, the boundary is contained in a rectangular box.

We claim that Schwarz reflection generates from M_* the P surface, see Sect. III,5.4. To see this, consider first symmetries of the D-hexagon M: There are straight lines between opposite edge midpoints, and arcs of planar reflection between opposite vertices. Thus by Cor. 31 there are straight lines through opposite vertices of M_* and arcs of planar reflection between midpoints of opposite arcs. This proves that the rectangular box is in fact a cube. But then the straight arcs from the cube midpoint out to the endpoints of an arc of planar reflection, together with the mirror image, form a quadrilateral precisely as in Sect. III,5.4. This proves that the conjugate of the D surface is the P surface.

We would like to proceed in reverse order. Suppose a desired minimal surface M is to be generated by Schwarz reflection in planes from a fundamental piece. This means that we assume we have a set of mirror planes which are to contain arcs of planar reflection of M. Can we systematically prescribe a polygon such that its Plateau solution has a conjugate that establishes existence of the desired surface M? The following algorithm can succeed:
0. At each of the k vertices of the desired surface we read off the angles and the normal from the configuration of mirror planes.
1. The vertex angles of the polygon agree with the vertex angles of M, since the a surface and its conjugate are isometric.
2. The normals at the vertices agree, and prescribe the direction of the polygon edges.
3. Assign lengths to the edges of the polygon such that the polygon closes. In general this leaves $k-2$ parameters. Let us ignore scaling also. Then there is a $k-3$-parameter space

of geometrically different polygons

$$\mathcal{G} := \{\text{admissible polygons } \Gamma\}.$$

Certainly, by the isometry property the edge lengths of M and its conjugate M_* agree. Nevertheless from the symmetry properties alone, we cannot predict the edgelengths.

The following difficulties may arise in this construction:
Boundary regularity: While the Plateau problem is free of interior branch points, branch points can be created at the vertices. For instance, if the polygon has angle $\alpha_k \in (0, 2\pi)$ at a vertex then perhaps the Plateau solution makes an angle $2\pi - \alpha_k$ instead. Upon reflection, then a branch point will be created. In order to rule boundary branch points out, we need barriers as in Thm. III,41.
Period problem: More complicated desired surfaces M have a pair of boundary arcs in the same plane. The construction will not necessarily give this property. Instead, we will need to choose $\Gamma \in \mathcal{G}$ suitable, that is, assign lengths to the edges so that the the period vanishes. If there is only one period problem then often the intermediate value theorem can be applied. *Embeddedness*: This is a property not necessarily preserved by conjugation.

Diplomarbeiten: 1. Minimalflaechen mit hoeherer Kodimension: Variationsformel etc.

2. Minimalflaechen in $\mathbb{S}^n$: Loese Plateauprobleme

3. Isoperimetrische Probleme.

4. Klassifiziere Raumgruppen von dreifach periodischen Minimalflaechen und finde gegebenfalls weitere Kandidaten fuer interessante Gruppen.