Lecture notes for the course

Introduction to Generalized Complex Geometry

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Contents

CHAPTER 1

Generalized complex geometry

Generalized complex structures were introduced by Nigel Hitchin [37] and further developed by Gualtieri [33]. They are a simultaneous generalization of complex and symplectic structures obtained by searching for complex structures on $T \oplus T^*$, the sum of tangent and cotangent bundles of a manifold M or, more generally, on Courant algebroids over M.

Not only do generalized complex structures generalize symplectic and complex structures but also provide a unifying language for many features of these two seemingly distinct geometries. For instance, the operators ∂ and $\overline{\partial}$ and the p, q-decomposition of forms from complex geometry have their analogue in the generalized complex world as well as symplectic and Lagrangian submanifolds from the symplectic world.

This unifying property of generalized complex structures was immediately noticed by the physicists and the most immediate application was to mirror symmetry. From the generalized complex point of view, mirror symmetry should not be seen as the interchange of two different structures (complex to symplectic and vice versa) but just a transformation of the generalized complex structures in consideration. Features of mirror symmetry from the generalized complex viewpoint were studied in [26, 32, 36] and in [4, 20] from a more mathematical angle.

The relevance of generalized complex structures to string theory does not stop there. They also arise as solutions to the vacuum equations for some string theories, examples of which were given by Lindström, Minasian, Tomasiello and Zabzine [49] and Zucchini [70]. Furthermore, the generalized complex version of Kähler manifolds correspond to the bihermitian structures of Gates, Hull and Roček [29] obtained from the study of general $(2, 2)$ supersymmetric sigma models (see also [50]).

Another angle to generalized complex structures comes from the study of Dirac structures: maximal isotopic subspaces $L \subset T \oplus T^*$ together with an integrability condition. A generalized complex structure is nothing but a complex Dirac $L \subset T_{\mathbb{C}} \oplus T^*_{\mathbb{C}}$ satisfying $L \cap \overline{L} = \{0\}$. Dirac structures predate generalized complex structure by more than 20 years and due to work of Weinstein and many of his collaborators we know an awful lot about them. Many of the features of generalized complex structures are in a way results about Dirac structures, e.g., some aspects of their local structure, the d_L-cohomology, the deformation theory and reduction procedure. However due to lack of space, we will not stress the connection between generalized complex geometry and Dirac structures.

This chapter follows closely the exposition of Gualtieri's thesis [33] and includes some developments to the theory obtained thereafter. This chapter is organized as follows. In the first section we introduce linear generalized complex structures, i.e., generalized complex structures on vector spaces and then go on to show that these structures give rise to a decomposition of forms similar to the (p, q)-decomposition of forms in a complex manifold. In Section 2, we introduce the Courant bracket which furnishes the integrability condition for a generalized complex structure on a manifold, as we see in Section 3. This is a compatibility condition between the pointwise defined generalized complex structure and the differential structure, which is equivalent to saying that the pointwise decomposition of forms induced by the generalized complex structure gives rise to a decomposition of the exterior derivative $d = \partial + \overline{\partial}$. In Section 4 we state the basic result on the deformation theory of generalized complex structures and in Section 5 we study two important classes of submanifolds of a generalized complex manifold. We finish studying some interesting examples of generalized complex manifolds in the last section.

1. Linear algebra of a generalized complex structure

For any vector space V^n we define *the double of* V, $\mathcal{D}V$, to be a $2n$-dimensional vector space endowed with a nondegenerate pairing $\langle\cdot,\cdot\rangle$ and a surjective projection

$$\pi : \mathcal{D}V \longrightarrow V,$$

such that the kernel of π is isotropic. Observe that the requirement that the kernel of π is isotropic implies that the pairing has signature (n, n).

Using the pairing to identify $(\mathcal{D}V)^*$ with $\mathcal{D}V$, we get a map

$$\tfrac{1}{2}\pi^* : V^* \longrightarrow \mathcal{D}V,$$

so we can regard V^* as a subspace of $\mathcal{D}V$. By definition, $\langle \pi^*(V^*), \mathrm{Ker}\,(\pi)\rangle = 0$ and, since $\mathrm{Ker}\,(\pi)^\perp = \mathrm{Ker}\,(\pi)$, we see that $\pi^*(V^*) = \mathrm{Ker}\,(\pi)$, therefore furnishing the following exact sequence

$$0 \longrightarrow V^* \xrightarrow{\frac{1}{2}\pi^*} \mathcal{D}V \xrightarrow{\pi} V \longrightarrow 0.$$

If we choose an isotropic splitting $\nabla : V \longrightarrow \mathcal{D}V$, i.e., a splitting for which $\nabla(V)$ is isotropic, then we obtain an isomorphism $\mathcal{D}V \cong V \oplus V^*$ and the pairing is nothing but the natural pairing on $V \oplus V^*$:

$$\langle X + \xi, Y + \eta\rangle = \frac{1}{2}(\xi(Y) + \eta(X)).$$

Definition 1.1. A *generalized complex structure* on V is a linear complex structure $\mathcal{J}$ on $\mathcal{D}V$ orthogonal with respect to the pairing.

Since $\mathcal{J}^2 = -\mathrm{Id}$, it splits $\mathcal{D}V \otimes \mathbb{C}$ as a direct sum of $\pm i$-eigenspaces, L and $\overline{L}$. Further, as $\mathcal{J}$ is orthogonal, we obtain that for $v, w \in L$,

$$\langle v, w\rangle = \langle \mathcal{J}v, \mathcal{J}w\rangle = \langle iv, iw\rangle = -\langle v, w\rangle,$$

and hence L is a maximal isotropic subspace with respect to the pairing.

Conversely, prescribing such an L as the i-eigenspace determines a unique generalized complex structure on V, therefore a generalized complex structure on a vector space V^n is equivalent to a maximal isotropic subspace $L \subset \mathcal{D}V \otimes \mathbb{C}$ such that $L \cap \overline{L} = \{0\}$

This last point of view also shows that a generalized complex structure is a special case of a more general object called a *Dirac structure*, which is a maximal isotropic subspace of $\mathcal{D}V$. So a generalized complex structure is nothing but a complex Dirac structure L for which $L \cap \overline{L} = \{0\}$.

Example 1.2 (Complex structures). If we have a splitting $\mathcal{D}V = V \oplus V^*$ and V has a complex structure I, then it induces a generalized complex structure on V which can be written in matrix form using the splitting as

$$\mathcal{J}_I = \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix}.$$

The i-eigenspace of $\mathcal{J}_I$ is $L = V^{0,1} \oplus V^{*1,0} \subset (V \oplus V^*) \otimes \mathbb{C}$. It is clear that L is a maximal isotropic subspace and that $L \cap \overline{L} = \{0\}$.

EXAMPLE 1.3 (Symplectic structures). Again, if we have a splitting $\mathcal{D}V = V \oplus V^*$, then a symplectic structure ω on V also induces a generalized complex structure $\mathcal{J}_\omega$ on V by letting

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

The i-eigenspace of $\mathcal{J}_\omega$ is given by $L = \{X - i\omega(X) : X \in V\}$. The nondegeneracy of ω implies that $L \cap \overline{L} = \{0\}$ and skew symmetry implies that L is isotropic.

EXAMPLE 1.4. A real 2-form B acts naturally on $\mathcal{D}V$ by the *B-field transform*

$$e \mapsto e - B(\pi(e)).$$

If V is endowed with a generalized complex structure, $\mathcal{J}$, whose $+i$-eigenspace is L, we can consider its image under the action of a B-field: $L_B = \{e - B(\pi(e)) : e \in L\}$. Since B is real, $L_B \cap \overline{L_B} = (\mathrm{Id} - B)L \cap \overline{L} = \{0\}$. Again, skew symmetry implies that L_B is isotropic. If we have a splitting for $\mathcal{D}V$, we can write $\mathcal{J}_B$, the B-field transform of $\mathcal{J}$, in matrix form

$$\mathcal{J}_B = \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \mathcal{J} \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}.$$

One can check that any two isotropic splittings of $\mathcal{D}V$ are related by B-field transforms.

EXAMPLE 1.5. In the presence of a splitting $\mathcal{D}V = V \oplus V^*$, an element $\beta \in \wedge^2 V$ also acts in a similar fashion:

$$X + \xi \mapsto X + \xi + \xi \llcorner \beta.$$

An argument similar to the one above shows that the β-transform of a generalized complex structure is still a generalized complex structure.

1.1. Mukai pairing and pure forms. In the presence of a splitting $\mathcal{D}V = V \oplus V^*$, we have one more characterization of a generalized complex structure on V, obtained from an interpretation of forms as spinors.

The Clifford algebra of $\mathcal{D}V$ is defined using the natural form $\langle \cdot, \cdot \rangle$, i.e., for $v \in \mathcal{D}V \subset \mathrm{Cl}(\mathcal{D}V)$ we have $v^2 = \langle v, v \rangle$. Since V^* is a maximal isotropic, its exterior algebra is a subalgebra of $\mathrm{Cl}(\mathcal{D}V)$. In particular, $\wedge^n V^*$ is a distinguished line in the Clifford algebra and generates a left ideal $\mathcal{I}$. A splitting $\mathcal{D}V = V \oplus V^*$ gives an isomorphism $\mathcal{I} \cong \wedge^\bullet V \otimes \wedge^n V^* \cong \wedge^\bullet V^*$. This, in turn, determines an action of the Clifford algebra on $\wedge^\bullet V^*$ by

$$(X + \xi) \cdot \alpha = i_X \alpha + \xi \wedge \alpha.$$

If we let σ be the antiautomorphism of $\mathrm{Cl}(V \oplus V^*)$ defined on decomposables by

$$\sigma(v_1 \otimes \cdots \otimes v_k) = v_k \otimes \cdots \otimes v_1, \tag{1.1}$$

then we have the following bilinear form on $\wedge^\bullet V^* \subset \mathrm{Cl}(V \oplus V^*)$:

$$(\xi_1, \xi_2) \mapsto (\sigma(\xi_1) \wedge \xi_2)_{top},$$

where *top* indicates taking the top degree component on the form. If we decompose ξ_i by degree: $\xi_i = \sum \xi_i^j$, with $\deg(\xi_i^j) = j$, the above can be rewritten, in an n-dimensional space, as

$$(\xi_1, \xi_2) = \sum_j (-1)^j (\xi_1^{2j} \wedge \xi_2^{n-2j} + \xi_1^{2j+1} \wedge \xi_2^{n-2j-1}) \tag{1.2}$$

This bilinear form coincides in cohomology with the *Mukai pairing*, introduced in a K-theoretical framework in [56].

Now, given a form $\rho \in \wedge^\bullet V^* \otimes \mathbb{C}$ (of possibly mixed degree) we can consider its Clifford annihilator

$$L_\rho = \{v \in (V \oplus V^*) \otimes \mathbb{C} : v \cdot \rho = 0\}.$$

It is clear that $\overline{L_\rho} = L_{\overline{\rho}}$. Also, for $v \in L_\rho$,

$$0 = v^2 \cdot \rho = \langle v, v \rangle \rho,$$

thus L_ρ is always isotropic.

Definition 1.6. An element $\rho \in \wedge^\bullet V^*$ is a *pure form* if L_ρ is maximal, i.e., $\dim_{\mathbb{C}} L_\rho = \dim_{\mathbb{R}} V$.

Given a maximal isotropic subspace $L \subset V \oplus V^*$ one can always find a pure form annihilating it and conversely, if two pure forms annihilate the same maximal isotropic, then they are a multiple of each other, i.e., maximal isotropics are in one-to-one correspondence with lines of pure forms. Algebraically, the requirement that a form is pure implies that it is of the form $e^{B+i\omega}\Omega$, where B and ω are real 2-forms and Ω is a decomposable complex form. The relation between the Mukai pairing and generalized complex structures comes in the following:

Proposition 1.7. (Chevalley [21]) *Let ρ and τ be pure forms. Then $L_\rho \cap L_\tau = \{0\}$ if and only if $(\rho, \tau) \neq 0$.*

Therefore a pure form $\rho = e^{B+i\omega}\Omega$ determines a generalized complex structure if and only if $(\rho, \overline{\rho}) = \Omega \wedge \overline{\Omega} \wedge \omega^{n-k} \neq 0$, where k is the degree of Ω and V is $2n$-dimensional. This also shows that there is no generalized complex structure on odd dimensional spaces.

With this we see that, if $\mathcal{D}V$ is split, then a generalized complex structure is equivalent to a line $K \subset \wedge^\bullet V^* \otimes \mathbb{C}$ generated by a form $e^{B+i\omega}\Omega$, such that Ω is a decomposable complex form of degree, say, k, B and ω are real 2-forms and $\Omega \wedge \overline{\Omega} \wedge \omega^{n-k} \neq 0$. The degree of the form Ω is the *type* of the generalized complex structure. The line K annihilating L is the *canonical line*.

Examples 1.2 – 1.5 revised: The canonical line in $\wedge^\bullet V^* \otimes \mathbb{C}$ that gives the generalized complex structure for a complex structure is $\wedge^{n,0} V^*$, while the line for a symplectic structure ω is generated by $e^{i\omega}$. If ρ is a generator of the canonical line of a generalized complex structure $\mathcal{J}$, $e^B \wedge \rho$ is a generator of a B-field transform of $\mathcal{J}$ and $e^\beta \llcorner \rho$ is a generator for a β-field transform of $\mathcal{J}$.

1.2. The decomposition of forms. Using the same argument used before with V and V^* to the maximal isotropics L and $\overline{L}$ determining a generalized complex structure on V, we see that $\mathrm{Cl}((V \oplus V^*) \otimes \mathbb{C}) \cong \mathrm{Cl}(L \oplus \overline{L})$ acts on $\wedge^{2n} L$ and the left ideal generated is the subalgebra $\wedge^\bullet \overline{L}$. The choice of a pure form ρ for the generalized complex structure gives an isomorphism of Clifford modules:

$$\phi : \wedge^\bullet \overline{L} \longrightarrow \wedge^\bullet V^* \otimes \mathbb{C}; \qquad \phi(s) = s \cdot \rho.$$

The decomposition of $\wedge^\bullet \overline{L}$ by degree gives rise to a new decomposition of $\wedge^\bullet V^* \otimes \mathbb{C}$ and the Mukai pairing on $\wedge^\bullet V^* \otimes \mathbb{C}$ corresponds to the same pairing on $\wedge^\bullet \overline{L}$. But in $\wedge^\bullet \overline{L}$ the Mukai pairing is nondegenerate in

$$\wedge^k \overline{L} \times \wedge^{2n-k} \overline{L} \longrightarrow \wedge^{2n} \overline{L}$$

and vanishes in $\wedge^k \overline{L} \times \wedge^l \overline{L}$ for all other values of l. Therefore the same is true for the induced decomposition on forms.

Proposition 1.8. *Letting $U^k = \wedge^{n-k} \overline{L} \cdot \rho < \wedge^\bullet V^* \otimes \mathbb{C}$, the Mukai pairing on $U^k \times U^l$ is trivial unless $l = -k$, in which case it is nondegenerate.*

According to the definition above, U^n is the canonical bundle. Also, the elements of U^k are even/odd forms, according to the parity of k, n and the type of the structure. For example, if n and the type are even, the elements of U^k will be even if and only if k is even.

EXAMPLE 1.9. In the complex case, we take $\rho \in \wedge^{n,0}V^*\backslash\{0\}$ to be a form for the induced generalized complex structure. Then, from Example 1.2, we have that $\overline{L} = \wedge^{1,0}V \oplus \wedge^{0,1}V^*$, so

$$U^k = \oplus_{p-q=k} \wedge^{p,q} V^*.$$

Then, in this case, one can see Proposition 1.8 as a consequence of the fact that the top degree part of the exterior product vanishes on $\wedge^{p,q}V^* \times \wedge^{p',q'}V^*$, unless $p + p' = q + q' = n$, in which case it is a nondegenerate pairing.

EXAMPLE 1.10. The decomposition of forms into the spaces U^k for a symplectic vector space (V, ω) was worked out in [16]. In this case we have

$$U^k = \{e^{i\omega} e^{\frac{\lambda}{2i}} \wedge^{n-k} V^* \otimes \mathbb{C}\}.$$

So $\Phi : \wedge^{n-k}V^* \otimes \mathbb{C} \longrightarrow \wedge^{n-k}V^* \otimes \mathbb{C}$ defined by

$$\Phi(\alpha) = e^{i\omega} e^{\frac{\lambda}{2i}} \alpha$$

is a natural isomorphism of graded spaces: $\Phi(\wedge^k V^*) = U^{n-k}$.

EXAMPLE 1.11. If a generalized complex structure induces a decomposition of the differential forms into the spaces U^k, then the B-field transform of this structure will induce a decomposition into $U^k_B = e^B \wedge U^k$. Indeed, by Example 1.4 revised, $U^n_B = e^B \wedge U^n$, and

$$U^k_B = (\mathrm{Id} - B)(\overline{L}) \cdot U^{k+1}_B.$$

The desired expression can be obtained by induction.

1.3. The actions of $\mathcal{J}$ on forms. Recall that the group $\mathrm{Spin}(n, n)$ sits inside $\mathrm{Cl}\,(V \oplus V^*)$ as

$$\mathrm{Spin}(n, n) = \{v_1 \cdots v_{2k} : v_i \cdot v_i = \pm 1; k \in \mathbb{N}\}.$$

And $\mathrm{Spin}(n, n)$ is a double cover of $\mathrm{SO}(n, n)$:

$$\varphi : \mathrm{Spin}(n, n) \longrightarrow \mathrm{SO}(n, n); \qquad \varphi(v)X = v \cdot X \cdot \sigma(v),$$

where σ is the main antiautomorphism of the Clifford algebra as defined in (1.1).

This map identifies the Lie algebras $\mathfrak{spin}(n, n) \cong \mathfrak{so}(n, n) \cong \wedge^2 V \oplus \wedge^2 V^* \oplus \mathrm{End}(V)$:

$$\mathfrak{spin}(n, n) \longrightarrow \mathfrak{so}(n, n); \qquad d\varphi(v)(X) = [v, X] = v \cdot X - X \cdot v$$

But, as the exterior algebra of V^* is naturally the space of spinors, each element in $\mathfrak{spin}(n, n)$ acts naturally on $\wedge^\bullet V^*$.

EXAMPLE 1.12. Let $B = \sum b_{ij} e^i \wedge e^j \in \wedge^2 V^* < \mathfrak{so}(n, n)$ be a 2-form. As an element of $\mathfrak{so}(n, n)$, B acts on $V \oplus V^*$ via

$$X + \xi \mapsto X \llcorner B.$$

Then the corresponding element in $\mathfrak{spin}(n, n)$ inducing the same action on $V \oplus V^*$ is given by $\sum b_{ij} e^j e^i$, since, in $\mathfrak{so}(n, n)$, we have

$$e^i \wedge e^j : e_k \mapsto \delta_{ik} e^j - \delta_{jk} e^i.$$

And, in $\mathfrak{spin}(n, n)$,

$$d\varphi(e^j e^i) e_k = (e^j e^i) \cdot e_k - e_k \cdot (e^j e^i) = e^j \cdot (e^i \cdot e_k) - (e_k \cdot e^j) \cdot e^i = \delta_{ik} e^j - \delta_{jk} e^i.$$

Finally, the spinorial action of B on a form φ is given by

$$\sum b_{ij}e^je^i \cdot \varphi = -B \wedge \varphi.$$

EXAMPLE 1.13. Similarly, for $\beta = \sum \beta^{ij} e_i \wedge e_j \in \wedge^2 V < \mathfrak{so}(n,n)$, its action is given by

$$\beta \cdot (X + \xi) = i_\xi \beta.$$

And the corresponding element in $\mathfrak{spin}(n,n)$ with the same action is $\sum \beta^{ij} e_j e_i$. The action of this element on a form φ is given by

$$\beta \cdot \varphi = \beta \llcorner \varphi.$$

EXAMPLE 1.14. Finally, an element of $A = \sum A_i^j e^i \otimes e_j \in End(V) < \mathfrak{so}(n,n)$ acts on $V \oplus V^*$ via

$$A(X + \xi) = A(X) + A^*(\xi).$$

The element of $\mathfrak{spin}(n,n)$ with the same action is $\frac{1}{2}\sum A_i^j (e_j e^i - e^i e_j)$. And the action of this element on a form φ is given by:

$$\begin{aligned} A \cdot \varphi &= \frac{1}{2}\sum A_i^j (e_j e^i - e^i e_j) \cdot \varphi \\ &= \frac{1}{2}\sum A_i^j (e_j \llcorner (e^i \wedge \varphi) - e^i \wedge (e_j \llcorner \varphi)) \\ &= \frac{1}{2}\sum_i A_i^i \varphi - \sum_{i,j} A_i^j e^i \wedge (e_i \llcorner \varphi) \\ &= -A^*\varphi + \frac{1}{2}\mathrm{Tr} A \varphi, \end{aligned}$$

where $A^*\varphi$ is the Lie algebra adjoint of A action of φ via

$$A^*\varphi(v_1, \cdot v_p) = \sum_i \varphi(v_1, \cdots, Av_i, \cdots, v_p).$$

The reason for introducing this Lie algebra action of $\mathfrak{spin}(n,n)$ on forms is because $\mathcal{J} \in \mathfrak{spin}(n,n)$, hence we can compute its action on forms.

EXAMPLE 1.15. In the case of a generalized complex structure induced by a symplectic one, we have that $\mathcal{J}$ is the sum of a 2-form, ω, and a bivector, $-\omega^{-1}$, hence its Lie algebra action on a form φ is

$$\mathcal{J}\varphi = (-\omega \wedge \; - \omega^{-1} \llcorner)\varphi \tag{1.3}$$

EXAMPLE 1.16. If $\mathcal{J}$ is a generalized complex structure on V induced by a complex structure I, then its Lie algebra action is the one corresponding to the traceless endomorphism $-I$ (see Example 1.2). Therefore, when acting on a p,q-form α:

$$\mathcal{J} \cdot \alpha = I^*\alpha = i(p-q)\alpha.$$

From this example and Example 1.9 it is clear that in the case of a generalized complex structure induced by a complex one, the subspaces $U^k < \wedge^\bullet V^*$ are the ik-eigenspaces of the action of $\mathcal{J}$. This is general.

PROPOSITION 1.17. *The spaces U^k are the ik-eigenspaces of the Lie algebra action of $\mathcal{J}$.*

PROOF. Recall from Subsection 1.2 that the choice of a nonzero element ρ of the canonical line $K \subset \wedge^\bullet V^*$ gives an isomorphism of Clifford modules:

$$\phi : \wedge^\bullet \overline{L} \longrightarrow \wedge^\bullet V^* \otimes \mathbb{C}; \qquad \phi(s) = s \cdot \rho.$$

And the spaces U^k are defined as $U^{n-k} = \phi(\wedge^k \overline{L})$. Further, $\mathcal{J}$ acts on $L^* \cong \overline{L}$ as multiplication by $-i$. Hence, by Example 1.14, its Lie algebra action on $\gamma \in \wedge^k \overline{L}$ is:

$$\mathcal{J} \cdot \gamma = -J^* \gamma + \frac{1}{2}\mathrm{Tr} J \gamma = -ik\gamma + \frac{1}{2} 2in\gamma = i(n-k)\gamma.$$

As ϕ is an isomorphism of Clifford modules, for $\alpha \in U^{n-k}$ we have $\phi^{-1}\alpha \in \wedge^k \overline{L}$ and

$$\mathcal{J} \cdot \alpha = \mathcal{J} \cdot \phi(\phi^{-1}\alpha) = \phi(\mathcal{J}\phi^{-1} \cdot \alpha) = i(n-k)\phi(\phi^{-1}\alpha) = i(n-k)\alpha.$$

□

2. The Courant bracket and Courant algebroids

From the linear algebra developed in the previous section, it is clear that a generalized complex structure on a manifold lives naturally on the double of the tangent bundle, $\mathcal{D}T$. Similarly to the case of complex structures on a manifold, the integrability condition for a generalized complex structure is that its i-eigenspace has to be closed under a certain bracket. This bracket was originally introduced by Courant and Weinstein as an extension of the Lie bracket of vector fields to sections of $T \oplus T^*$ [**23, 22**]. One of the striking features of the Courant bracket is that it only satisfies the Jacobi identity modulo an exact element, more precisely, for all $e_1, e_2, e_3 \in C^\infty(T \oplus T^*)$ we have

$$\mathrm{Jac}(e_1, e_2, e_3) := [\![[\![e_1, e_2]\!], e_3]\!] + c.p. = \frac{1}{3} d(\langle [\![e_1, e_2]\!], e_3\rangle + c.p.).$$

Liu, Weinstein and Xu, [**51**], axiomatized the properties of the Courant bracket in the concept of a Courant algebroid, which we define next. As we will see later, *exact* Courant algebroids are the natural space where generalized complex structures live.

DEFINITION 1.18. A *Courant algebroid* over a manifold M is a vector bundle $\mathcal{E} \to M$ equipped with a skew-symmetric bracket $[\![\cdot,\cdot]\!]$ on $C^\infty(\mathcal{E})$, a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, and a bundle map $\pi : \mathcal{E} \to T$, which satisfy the following conditions for all $e_1, e_2, e_3 \in C^\infty(\mathcal{E})$ and $f, g \in C^\infty(M)$:

C1) $\pi([\![e_1, e_2]\!]) = [\pi(e_1), \pi(e_2)]$,
C2) $\mathrm{Jac}(e_1, e_2, e_3) = \frac{1}{3} d(\langle [\![e_1, e_2]\!], e_3\rangle + c.p.)$,
C3) $[\![e_1, fe_2]\!] = f[\![e_1, e_2]\!] + (\pi(e_1)f)e_2 - \langle e_1, e_2\rangle df$,
C4) $\pi \circ d = 0$, i.e. $\langle df, dg \rangle = 0$,
C5) $\pi(e_1)\langle e_2, e_3\rangle = \langle e_1 \bullet e_2, e_3\rangle + \langle e_2, e_1 \bullet e_3\rangle$,

where we consider $\Omega^1(M)$ as a subset of $C^\infty(\mathcal{E})$ via the map $\frac{1}{2}\pi^* : \Omega^1(M) \longrightarrow \mathbb{C}^\infty(\mathcal{E})$ (using $\langle \cdot, \cdot \rangle$ to identify $\mathcal{E}$ with $\mathcal{E}^*$) and $\bullet$ denotes the combination

$$e_1 \bullet e_2 = [\![e_1, e_2]\!] + d\langle e_1, e_2\rangle. \tag{1.4}$$

and is the *adjoint action* of e_1 on e_2.

DEFINITION 1.19. A Courant algebroid is *exact* if the following sequence is exact:

$$0 \longrightarrow T^* \xrightarrow{\pi^*} \mathcal{E} \xrightarrow{\pi} T \longrightarrow 0 \tag{1.5}$$

Given an exact Courant algebroid, we may always choose an isotropic right splitting $\nabla : T \to \mathcal{E}$. Such a splitting has a *curvature* 3-form $H \in \Omega^3_{cl}(M)$ defined as follows, for $X, Y \in C^\infty(TM)$:

$$H(X,Y,Z) = \tfrac{1}{2}\langle [\![\nabla(X), \nabla(Y)]\!], \nabla(Z)\rangle. \tag{1.6}$$

Using the bundle isomorphism $\nabla + \frac{1}{2}\pi^* : T \oplus T^* \to \mathcal{E}$, we transport the Courant algebroid structure onto $T \oplus T^*$. As before the pairing is nothing but the natural pairing

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\eta(X) + \xi(Y)), \tag{1.7}$$

and for $X + \xi, Y + \eta \in C^\infty(T \oplus T^*)$ the bracket becomes

$$[\![X + \xi, Y + \eta]\!] = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \tfrac{1}{2}d(i_X\eta - i_Y\xi) + i_Y i_X H, \tag{1.8}$$

which is the *H-twisted Courant bracket* on $T \oplus T^*$ [59]. Isotropic splittings of (1.5) differ by 2-forms $b \in \Omega^2(M)$, and a change of splitting modifies the curvature H by the exact form db. Hence the cohomology class $[H] \in H^3(M, \mathbb{R})$, called the *characteristic class* or *Ševera class* of $\mathcal{E}$, is independent of the splitting and determines the exact Courant algebroid structure on $\mathcal{E}$ completely.

One way to see the Courant bracket on $T \oplus T^*$ as a natural extension of the Lie bracket of vector fields is as follows. Recall that the Lie bracket satisfies (and can be defined by) the following identity when acting on a form α (see [44], Chapter 1, Proposition 3.10):

$$2i_{[v_1,v_2]}\alpha = i_{v_1\wedge v_2}d\alpha + d(i_{v_1\wedge v_2}\alpha) + 2i_{v_1}d(i_{v_2}\alpha) - 2i_{v_2}d(v_1 \llcorner \alpha).$$

Now we observe that this formula gives a natural extension of the Lie bracket to a bracket on $T \oplus T^*$, acting on forms via the Clifford action:

$$2[\![v_1, v_2]\!] \cdot \alpha = v_1 \wedge v_2 \cdot d\alpha + d(v_1 \wedge v_2 \cdot \alpha) + 2v_1 \cdot d(v_2 \cdot \alpha) - 2v_2 \cdot d(v_1 \cdot \alpha). \tag{1.9}$$

Spelling it out we obtain (see Gualtieri [33], Lemma 4.24):

$$[\![X + \xi, Y + \eta]\!] = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \tfrac{1}{2}d(i_X\eta - i_Y\xi).$$

This is the Courant bracket with $H = 0$. If we replace d by $d_H = d + H\wedge$ in (1.9) we obtain the H-twisted Courant bracket:

$$2[\![v_1, v_2]\!]_H \cdot \alpha = v_1 \wedge v_2 \cdot d_H\alpha + d_H(v_1 \wedge v_2 \cdot \alpha) + 2v_1 \cdot d_H(v_2 \cdot \alpha) - 2v_2 \cdot d_H(v_1 \cdot \alpha). \tag{1.10}$$

3. Generalized complex structures

Given the work in the previous sections it is clear that the fiber of an exact Courant algebroid $\mathcal{E}$ over a point $p \in M$ is nothing but $\mathcal{D}T_p$ and hence it is natural to define a *generalized almost complex structure* as a differentiable bundle automophism $\mathcal{J} : \mathcal{E} \longrightarrow \mathcal{E}$ which is a linear generalized complex structure on each fiber. The Courant bracket provides the integrability condition.

DEFINITION 1.20. A *generalized complex structure* on an exact Courant algebroid $\mathcal{E}$ is a generalized almost complex structure $\mathcal{J}$ on $\mathcal{E}$ whose i-eigenspace is closed with respect to the Courant bracket.

As before, $\mathcal{J}$ can be described in terms of its i-eigenspace, L, which is a maximal isotropic subspace of $\mathcal{E}_\mathbb{C}$ closed under the Courant bracket satisfying $L \cap \overline{L} = \{0\}$.

The choice of a splitting for $\mathcal{E}$ making it isomorphic to $T \oplus T^*$ with the H-Courant bracket also allows us to characterize a generalized complex structure in terms of its canonical bundle

$K \subset \wedge^\bullet T^*_{\mathbb{C}}$. If ρ is a nonvanishing local section of K then using (1.9) one can easily see that the integrability condition is equivalent to the existence of a local section $e \in C^\infty(T_{\mathbb{C}} \oplus T^*_{\mathbb{C}})$ such that

$$d_H \rho = e \cdot \rho.$$

If we let $\mathcal{U}^k$ be the space of sections of the bundle U^k, defined in Proposition 1.8, this is only the case if $d_H\rho \in \mathcal{U}^{n-1} = \overline{L} \cdot \mathcal{U}^n$.

EXAMPLE 1.21. An almost complex structure on a manifold M induces a generalized almost complex structure with i-eigenspace $T^{0,1} \oplus T^{*1,0}$. If this generalized almost complex structure is integrable, then $T^{0,1}$ has to be closed with respect to the Lie bracket and hence the almost complex structure is actually a complex structure. Conversely, any complex structure gives rise to an integrable generalized complex structure.

EXAMPLE 1.22. If M has a nondegenerate 2-form ω, then the induced generalized almost complex structure will be integrable if for some $X + \xi$ we have

$$de^{i\omega} = (X + \xi) \cdot e^{i\omega}.$$

The degree 1 part gives that $X \llcorner \omega + \xi = 0$ and the degree 3 part, that $d\omega = 0$ and hence M is a symplectic manifold.

EXAMPLE 1.23. The action of a real closed 2-form B by B-field transforms on an exact Courant algebroid $\mathcal{E}$ is a symmetry of the bracket. In fact, B-field transforms together with diffeomorphisms of the manifold, form the group of orthogonal symmetries of the Courant bracket. Therefore we can always transform a given a generalized complex structure by B-fields to obtain a new generalized complex structure which should be considered equivalent to the first one.

Assume we have a splitting for $\mathcal{E}$ rendering it isomorphic to $T \oplus T^*$ with the H-Courant bracket. If the 2-form B is not closed, then it induces an isomophism between the H-Courant bracket and the $H + dB$-Courant bracket. In particular, if $[H] = 0 \in H^3(M, \mathbb{R})$, the bracket $[,]_H$ is isomorphic to $[,]_0$ by the action of a nonclosed 2-form.

EXAMPLE 1.24. Consider $\mathbb{C}^2$ with complex coordinates z_1, z_2. The differential form

$$\rho = z_1 + dz_1 \wedge dz_2$$

is equal to $dz_1 \wedge dz_2$ along the locus $z_1 = 0$, while away from this locus it can be written as

$$\rho = z_1 \exp\left(\tfrac{dz_1 \wedge dz_2}{z_1}\right). \tag{1.11}$$

Since it also satisfies $d\rho = -\partial_2 \cdot \rho$, we see that it generates a canonical bundle K for a generalized complex structure which has type 2 along $z_1 = 0$ and type 0 elsewhere, showing that a generalized complex structure does not necessarily have constant type.

In order to obtain a compact type-change locus we observe that this structure is invariant under translations in the z_2 direction, hence we can take a quotient by the standard $\mathbb{Z}^2$ action to obtain a generalized complex structure on the torus fibration $D^2 \times T^2$, where D^2 is the unit disc in the z_1-plane. Using polar coordinates, $z_1 = re^{2\pi i\theta_1}$, the canonical bundle is generated, away from the central fibre, by

$$\begin{aligned}\exp(B + i\omega) &= \exp(d\log r + id\theta_1) \wedge (d\theta_2 + id\theta_3)\\ &= \exp(d\log r \wedge d\theta_2 - d\theta_1 \wedge d\theta_3 + i(d\log r \wedge d\theta_3 + d\theta_1 \wedge d\theta_2)),\end{aligned}$$

where θ_2 and θ_3 are coordinates for the 2-torus with unit periods. Away from $r = 0$, therefore, the structure is a B-field transform of a symplectic structure ω, where

$$\begin{aligned} B &= d\log r \wedge d\theta_2 - d\theta_1 \wedge d\theta_3 \\ \omega &= d\log r \wedge d\theta_3 + d\theta_1 \wedge d\theta_2. \end{aligned} \tag{1.12}$$

The type jumps from 0 to 2 along the central fibre $r = 0$, inducing a complex structure on the restricted tangent bundle, for which the tangent bundle to the fibre is a complex sub-bundle. Hence the type change locus inherits the structure of a smooth elliptic curve with Teichmüller parameter $\tau = i$.

Similarly to the complex case, the integrability condition places restrictions on $d_H(\mathcal{U}^k)$ for every k and hence allows us to define operators ∂ and $\overline{\partial}$.

Theorem 1.25 (Gualtieri [33], Theorem 4.3). *A generalized almost complex structure is integrable if and only if* $d_H : \mathcal{U}^k \longrightarrow \mathcal{U}^{k+1} \oplus \mathcal{U}^{k-1}$.

So, on a generalized complex manifold M we can define the operators

$$\partial : \mathcal{U}^k \longrightarrow \mathcal{U}^{k+1};$$

$$\overline{\partial} : \mathcal{U}^k \longrightarrow \mathcal{U}^{k-1};$$

as the projections of d_H onto each of these factors. We also define $d^{\mathcal{J}} = -i(\partial - \overline{\partial})$.

Similarly to the operator d^c from complex geometry, we can find other expressions for $d^{\mathcal{J}}$ based on the action of the generalized complex structure on forms. One can easily check that if we consider the Lie group action of $\mathcal{J}$, i.e, $\mathcal{J}$ acts on $\mathcal{U}^k$ as multiplication by i^k, then

$$d^{\mathcal{J}} = \mathcal{J}^{-1} d \mathcal{J}.$$

And if one considers the Lie algebra action, then

$$d^{\mathcal{J}} = [d, \mathcal{J}].$$

As a consequence of Example 1.21 it is clear that in the complex case, ∂ and $\overline{\partial}$ are just the standard operators denoted by the same symbols and $d^{\mathcal{J}} = d^c$. In the symplectic case, $d^{\mathcal{J}}$ corresponds to Kosul's *canonical homology derivative* [46] introduced in the context of Poisson manifolds and studied by Brylinski [8], Mathieu [53],Yan [69], Merkulov [55] and others [25, 41] in the symplectic setting.

In the symplectic case, the operators ∂ and $\overline{\partial}$ are related to d and $d^{\mathcal{J}}$ also in a more subtle way. Recall from Example 1.10 that for a symplectic structure we have a map $\Phi : \wedge^\bullet T^* \longrightarrow \wedge^\bullet T^*$ such that $\Phi(\wedge^k T^*) = U^{n-k}$. The operators ∂, $\overline{\partial}$, d and $d^{\mathcal{J}}$ and the map Φ are related by (see [16])

$$\overline{\partial}\Phi(\alpha) = \Phi(d\alpha) \qquad 2i\partial\Phi(\alpha) = \Phi(d^{\mathcal{J}}\alpha).$$

Example 1.26. If a generalized complex structure induces a splitting of $\wedge^\bullet T^*$ into subspaces U^k, then, according to Example 1.11, a B-field transform of this structure will induce a decomposition into $e^B U^k$. As B is a closed form, for $v \in \mathcal{U}^k$ we have:

$$d(e^B v) = e^B dv = e^B \partial v + e^B \overline{\partial} v \in e^B \mathcal{U}^{k+1} + e^B \mathcal{U}^{k-1},$$

hence the corresponding operators for the B-field transform, ∂_B and $\overline{\partial}_B$, are given by

$$\partial_B = e^B \partial e^{-B}; \qquad \overline{\partial}_B = e^B \overline{\partial} e^{-B}.$$

3.1. The differential graded algebra $(\Omega^\bullet(\overline{L}), d_L)$. A peculiar characteristic of the operators ∂ and $\overline{\partial}$ introduced last section is that they are not derivations, i.e., they do not satisfy the Leibniz rule. There is, however, another differential complex associated to a generalized complex structure for which the differential is a derivation. We explain this in this section.

As we mentioned before, the Courant bracket does not satisfy the Jacoby identity, and instead we have

$$\mathrm{Jac}(e_1, e_2, e_3) = \frac{1}{3} d(\langle [\![e_1, e_2]\!], e_3\rangle + c.p.).$$

However, the identity above also shows that the Courant bracket induces a Lie bracket when restricted to sections of any involutive isotropic space L. This Lie bracket together with the projection $\pi_T : L \longrightarrow TM$, makes L into a Lie algebroid and allows us to define a differential d_L on $\Omega^\bullet(L^*) = \mathbb{C}^\infty(\wedge^\bullet L^*)$ making it into a differential graded algebra (DGA). If L is the i-eigenspace of a generalized complex structure, then the natural pairing gives an isomorphism $L^* \cong \overline{L}$ and with this identification $(\Omega^\bullet(\overline{L}), d_L)$ is a DGA.

If a generalized complex structure has type zero over M, i.e., is of symplectic type, then $\pi : L \xrightarrow{\cong} T_{\mathbb{C}}$ is an isomorphism and the Courant bracket on $C^\infty(L)$ is mapped to the Lie bracket on $C^\infty(T_{\mathbb{C}})$. Therefore, in this particular case, $(\Omega^\bullet(\overline{L}), d_L)$ and $(\Omega^\bullet_{\mathbb{C}}(M), d)$ are isomorphic DGAs.

Further, recall that the choice of a nonvanishing local section ρ of the canonical bundle gives an isomorphism of Clifford modules:

$$\phi : \Omega^\bullet(\overline{L}) \longrightarrow \Omega^\bullet_{\mathbb{C}}(M); \qquad \phi(s \cdot \sigma) = s \cdot \rho.$$

With these choices, the operators $\overline{\partial}$ and d_L are related by

$$\overline{\partial}\phi(\alpha) = \phi(d_L\alpha) + (-1)^{|\alpha|}\alpha \cdot d_H\rho.$$

In the particular case when there is a nonvanishing global holomorphic section ρ we can define ϕ globally and have

$$\overline{\partial}\phi(\alpha) = \phi(d_L\alpha).$$

4. Deformations of generalized complex structures

In this section we state part of Gualtieri's deformation theorem for generalized complex structures. The space of infinitesimal deformations is naturally a subspace of the space of sections of $\wedge^2\overline{L}$ and we want to know which sections of $\wedge^2\overline{L}$ give rise to deformations of the generalized complex structure. We shall not discuss when such deformations are trivial and instead refer to Gualtieri's thesis. Drawing on a result of Liu, Weinstein and Xu [51], Gualtieri established the following deformation theorem.

THEOREM 1.27 (Gualtieri [33], Theorem 5.4). *An element $\varepsilon \in \wedge^2\overline{L}$ gives rise to a deformation of generalized complex structures if and only if ε is small enough and satisfies the Maurer–Cartan equation*

$$d_L\varepsilon + \frac{1}{2}[\![\varepsilon, \varepsilon]\!] = 0.$$

The deformed generalized complex structure is given by

$$L^\varepsilon = (\mathrm{Id} + \varepsilon)L \qquad \overline{L}^\varepsilon = (\mathrm{Id} + \overline{\varepsilon})\overline{L}.$$

In the complex case, a bivector $\varepsilon \in \wedge^{2,0}TM < \wedge^2\overline{L}$ gives rise to a deformation only if each of the summands vanish, i.e., $\overline{\partial}\varepsilon = 0$ (ε is holomorphic) and $[\![\varepsilon, \varepsilon]\!] = 0$ (ε is Poisson).

EXAMPLE 1.28. Consider $\mathbb{C}^2$ with its standard complex structure and let $\varepsilon = z_1\partial_{z_1} \wedge \partial_{z_2}$. One can easily check that ε is a holomorphic Poisson bivector, hence we can use ε to deform the complex structure on $\mathbb{C}^2$. According to Example 1.5, the canonical bundle of the deformed structure is given by

$$e^{\varepsilon} \cdot dz_1 \wedge dz_2 = z_1 + dz_1 \wedge dz_2.$$

One can readily recognize this as the generalized complex structure from Example 1.24. This illustrates the fact that in 2 complex dimensions the zeros of the holomorphic bivector correspond to the type-change points in the deformed structure.

EXAMPLE 1.29. Any holomorphic bivector ε on a complex surface M is also Poisson, as $[\varepsilon, \varepsilon] \in \wedge^{3,0}TM = \{0\}$, and hence gives rise to a deformation of generalized complex structures. The deformed generalized complex structure will be symplectic outside the divisor representing $c_1(M)$ where the bivector vanishes. At the points where $\varepsilon = 0$ the deformed structure agrees with the original complex structure.

EXAMPLE 1.30. Let M^{4n} be a hyperkähler manifold with Kähler forms ω_I, ω_J and ω_K. According to the Kähler structure (ω_I, I), $(\omega_J + i\omega_K)$ is a closed holomorphic 2-form and $(\omega_J + i\omega_K)^n$ is a holomorphic volume form. Therefore these generate a holomorphic Poisson bivector $\Lambda \in \wedge^{2,0}TM$ by

$$\Lambda \cdot (\omega_J + i\omega_K)^n = n(\omega_J + i\omega_K)^{n-1}.$$

The deformation of the complex structure I by $t\Lambda$ is given by

$$e^{t\Lambda}(\omega_J + i\omega_K)^n = t^n e^{\frac{\omega_J + i\omega_K}{t}}.$$

which interpolates between the complex structure I and the B-field transform of the symplectic structure ω_K as t varies from 0 to 1.

5. Submanifolds and the restricted Courant algebroid

In this section we introduce two special types of submanifolds of a generalized complex manifold. The first type are the *generalized Lagrangians* introduced by Gualtieri [33]. This class of submanifolds comprises complex submanifolds from complex geometry and Lagrangian submanifolds from symplectic geometry. Generalized Lagrangians are intimately related to branes [33, 42, 71]. The second type of submanifolds consists of those which inherit a generalized complex structure from the original manifold. These correspond to one of the definitions of submanifolds introduced by Ben-Bassat and Boyarchenko [5] and are the analogue of symplectic submanifolds from symplectic geometry.

In order to understand generalized complex submanifolds it is desirable to understand how to restrict Courant algebroids to submanifolds. Given a Courant algebroid $\mathcal{E}$ over a manifold M and a submanifold $\iota : N \hookrightarrow M$ there are two natural bundles one can form over N. The first is $\iota^*\mathcal{E}$, the pull back of $\mathcal{E}$ to N. The pairing on $\mathcal{E}$ induces a pairing on $\iota^*\mathcal{E}$, however the same is not true about the Courant bracket: even if a section vanishes over N, it may bracket nonzero with a nonvanishing section.

EXERCISE 1.31. Show that if $e_1, e_2 \in C^\infty(\mathcal{E})$, $\pi(\iota^* e_1) \in C^\infty(TN)$ and $\iota^* e_2 = 0$, then $\iota^*[\![e_1, e_2]\!]$ is not necessarily zero but lies in $\mathcal{N}^* = \mathrm{Ann}(TN) \subset T^*M$, the conormal bundle of N.

The second bundle, called the *restricted Courant algebroid*, and denoted by $\mathcal{E}|_N$, is a Courant algebroid, as the name suggests. It is defined by

$$\mathcal{E}|_N = \frac{\mathcal{N}^{*\perp}}{\mathcal{N}^*} = \frac{\{e \in \iota^*\mathcal{E} : \pi(e) \in TN\}}{\mathrm{Ann}(TN)}.$$

The bracket is defined using the Courant bracket on $\mathcal{E}$: according to Exercise 1.31, the ambiguity of the bracket on $\mathcal{N}^{*\perp}$ lies in $\mathcal{N}^*$, hence the bracket on $\mathcal{E}|_N$ is well defined. If $\mathcal{E}$ is split and has curvature H, $\mathcal{E}|_N$ is naturally isomorphic to $TN \oplus T^*N$ endowed with the ι^*H-Courant bracket.

Using these two bundles we can define two types of generalized complex submanifolds.

DEFINITION 1.32. Given a generalized complex structure $\mathcal{J}$ on a Courant algebroid $\mathcal{E}$ over a manifold M, a *generalized Lagrangian* is a submanifold $\iota : N \longrightarrow M$ together with a maximal isotropic subbundle $\tau_N \subset \iota^*\mathcal{E}$ invariant under $\mathcal{J}$ such that $\pi(\tau_N) = TN$ and if $\iota^*e_1, \iota^*e_2 \in C^\infty(\tau_N)$ then $\iota^*[\![e_1, e_2]\!] \in \tau_N$.

Since τ_N is maximal isotropic and $\pi(\tau_N) = TN$, it follows that τ_N sits in an exact sequence

$$0 \longrightarrow \mathcal{N}^* \longrightarrow \tau_N \longrightarrow TN \longrightarrow 0.$$

Since $\mathcal{N}^* \subset \tau_N$, it makes sense to ask for τ_N to be closed under the bracket on $\iota^*\mathcal{E}$, even though this bracket is not well defined, since the indeterminacy lies in $\mathcal{N}^*$.

EXERCISE 1.33. Show that if $\mathcal{E}$ is split, then there is $F \in \Omega^2(N)$ such that

$$\tau_N = (Id + F) \cdot TN \oplus \mathcal{N}^* = \{X + \xi \in TN \oplus T^*M : \xi|_{TN} = i_X F\}. \tag{1.13}$$

Show that τ_N is closed under the bracket if and only if $dF = \iota^*H$, where H is the curvature of the splitting. Therefore a necessary condition for a manifold to be a submanifold is that $\iota^*[H] = 0$.

Using this exercise we obtain an alternative description of a generalized Lagrangian for a split Courant algebroid: it is a submanifold N with a 2-form $F \in \Omega^2(N)$ such that $dF = \iota^*H$ and τ_N, as defined in (1.13) is invariant under $\mathcal{J}$.

For the second definition of submanifold we observe that there is a natural way to transport a Dirac structure $D \subset \mathcal{E}$ to a Dirac structure on $\mathcal{E}|_N$:

$$D_{red} = \frac{D \cap \mathcal{N}^{*\perp} + \mathcal{N}^*}{\mathcal{N}^*}.$$

The distribution D_{red} is a Dirac structure whenever it is smooth and it is called the *pull-back* of D. So, if L is the i-eigenspace of a generalized complex structure $\mathcal{J}$ on $\mathcal{E}$, L_{red}, the pull back of L, is a Dirac structure on $\mathcal{E}|_N$.

DEFINITION 1.34. A *generalized complex submanifold* is a submanifold $\iota : N \longrightarrow M$ for which L_{red} determines a generalized complex structure on $\mathcal{E}|_N$, i.e., $L_{red} \cap \overline{L_{red}} = \{0\}$.

One can check that L_{red} is a generalized complex structure if and only if it is smooth and the following algebraic condition is satisfied

$$\mathcal{J}\mathcal{N}^* \cap \mathcal{N}^{*\perp} \subset \mathcal{N}^*.$$

Some particular cases when the above holds are when $\mathcal{N}^*$ is $\mathcal{J}$ invariant, i.e, $\mathcal{J}\mathcal{N}^* = \mathcal{N}^*$ or when the natural pairing is nondegenerate on $\mathcal{J}\mathcal{N}^* \times \mathcal{N}^*$.

EXAMPLE 1.35 (Gualtieri [33], Example 7.8). In this example we describe generalized Lagrangians of a symplectic manifold. So, the starting point is the split Courant algebroid $T \oplus T^*$ with $H = 0$ and generalized complex structure given by Example 1.3. Since the Courant algebroid is split, we can use the description of τ_N in terms of TN and a 2-form F. In this case, as was observed by Gualtieri, the definition of generalized Lagrangian agrees with the A-branes of Kapustin and Orlov [42].

We claim that if (N, F) is a generalized Lagrangian, then N is a coisotropic submanifold. Also, both F and, obviously, $\omega|_N$ are annihilated by the distribution $TN^\omega = \omega^{-1}(\mathcal{N}^*)$. In the quotient $V = TN/TN^\omega$, there is a complex structure induced by $\omega^{-1}F$ and $(F + i\omega)$ is a $(2,0)$-form whose top power is a volume element in $\wedge^{k,0}V$. Finally, if $F = 0$, then N is just a Lagrangian submanifold of M.

To prove that N is coisotropic, we have to prove that $\omega^{-1}(\mathcal{N}^*) \subset TN$. This is a simple consequence of the fact that $\mathcal{N}^* \subset \tau_N$ and τ_N is $\mathcal{J}$ invariant, hence

$$\mathcal{J}\mathcal{N}^* = -\omega^{-1}\mathcal{N}^* \in \tau_N,$$

showing that N is coisotropic.

To show that F is annihilated by TN^ω, we choose a local extension, B, of F to $\Omega^2(M)$, so that for $X \in TN$, $X + B(X) \in \tau_N$. Since τ_N is $\mathcal{J}$ invariant,

$$\mathcal{J}(X + B(X)) = -\omega^{-1}B(X) + \omega(X) \in \tau_N,$$

and hence $\omega^{-1}B(X) \in TN$, which implies that $B(X)$ vanishes on $TN^\omega = \omega^{-1}(\mathcal{N}^*)$ since

$$0 = \langle \omega^{-1}B(X), \mathcal{N}^* \rangle = \langle B(X), -omega^{-1}(\mathcal{N}^*) \rangle.$$

Hence F is also annihilated by the distribution TN^ω.

To find the complex structure on the quotient TN/TN^ω we take $X \in TN$ and apply $\mathcal{J}$ to $X + B(X) \in \tau_N$. Invariance implies that

$$-\omega^{-1}B(X) + \omega(X) \in \tau_N,$$

which is the same as $-F \circ \omega^{-1} \circ F(X)|_N = \omega(X)|_N$. In TN/TN^ω, there is an inverse ω^{-1} and hence, in the quotient, we have the identity

$$-X = (\omega^{-1}F)^2(X),$$

showing that $\omega^{-1}F$ induces a complex structure on TN/TN^ω.

For an $X \in \wedge^{0,1}(TN/TN^\omega)$ we have $\omega^{-1}F(X) = -iX$. Applying ω we get $F(X,\cdot)+i\omega(X,\cdot) = 0$ and hence $F + i\omega$ is annihilated by $\wedge^{0,1}(TN/TN^\omega)$ and thus is a $(2,0)$-form. Finally, for $X = X_1 + iX_2 \in \wedge^{1,0}(TN/TN^\omega)$, as before we obtain $(F - i\omega)(X,\cdot) = 0$, which spells out as

$$F(X_1,\cdot) = -\omega(X_2,\cdot) \qquad \text{and} \qquad F(X_2,\cdot) = \omega(X_1,\cdot),$$

and therefore $(F + i\omega)(X,\cdot) = -2\omega(X_2,\cdot) + 2\omega(X_1,\cdot) \neq 0$, as ω is nondegenerate in TN/TN^ω. Thus $F + i\omega$ is a nondegenerate 2,0-form.

If F vanishes, 0 is a complex structure in TN/TN^ω which must therefore be the trivial vector space and hence N is Lagrangian.

Exercise 1.36. Show that a generalized Lagrangian of a complex manifold is a complex submanifold with a closed form $F \in \Omega^{1,1}(M)$.

Exercise 1.37. Show that generalized complex submanifolds of a complex manifold are complex while those of a symplectic manifold are symplectic submanifolds.

6. Examples

In this section we give some interesting examples of generalized complex structures. In the first example we find generalized complex structures on symplectic fibrations and, in the second, on Lie algebras. The last example consists of a surgery procedure for symplectic manifolds which produce generalized complex structures on Courant algebroids over a topologically distinct manifold and whose characteristic class is not necessarily trivial.

6.1. Symplectic fibrations. The differential form description of a generalized complex structure furnishes also a very pictorial one around *regular points*. Indeed, in this case one can choose locally a closed form ρ defining the structure. Then the integrability condition tells us that $\Omega\wedge\overline{\Omega}$ is a real closed form and therefore the distribution $\mathrm{Ann}(\Omega\wedge\overline{\Omega})\subset TM$ is an integrable distribution. The algebraic condition $\Omega\wedge\overline{\Omega}\wedge\omega^{n-k}\neq 0$ implies that ω is nondegenerate on the leaves and the integrability condition $\Omega\wedge d\omega=0$, that ω is closed when restricted to these leaves. Therefore, around a regular point, the generalized complex structure furnishes a natural symplectic foliation, and further, the space of leaves has a natural complex structure given by Ω.

This suggests that symplectic fibrations should be a way to construct nontrivial examples of generalized complex structures. Next we see that Thurston's argument for symplectic fibrations [**63, 54**] can also be used in the generalized complex setting.

THEOREM 1.38 (Cavalcanti [**15**]). *Let M be a generalized complex manifold with the H-Courant bracket. Let $\pi : P\longrightarrow M$ be a symplectic fibration with compact fiber (F,ω). Assume that there is $a\in H^2(P)$ which restricts to the cohomology class of ω on each fiber. Then P admits a generalized complex structure with the π^*H-Courant bracket.*

PROOF. The usual argument using partitions of unit shows that we can find a closed 2-form τ representing the cohomology class a such that $\tau|_F=\omega$ on each fiber. If K is the canonical bundle of the generalized complex structure on M, then we claim that, for ε small enough, the subbundle $K_P=e^{i\varepsilon\tau}\wedge\pi^*K\subset\Omega^\bullet_{\mathbb{C}}(P)$ determines a generalized complex structure on P with the π^*H-Courant bracket.

Let U_α be a covering of B where we have trivializations ρ_α of K. Then the forms $e^{i\varepsilon\tau}\wedge\pi^*\rho_\alpha$ are nonvanishing local sections of K_P. Since τ is nondegenerate on the vertical subspaces Ker π_* it determines a field of horizontal subspaces

$$Hor_x=\{X\in T_xP:\tau(X,Y)=0,\forall Y\in T_xF\}.$$

The subspace Hor_x is a complement to T_xF and isomorphic to $T_{\pi(x)}M$ via π_*. Also, denoting by $(\cdot,\cdot)_M$ the Mukai pairing on M, $(\rho_\alpha,\overline{\rho_\alpha})_M\neq 0$ and hence pulls back to a volume form on Hor. Therefore, for ε small enough,

$$(e^{i\varepsilon\tau}\wedge\pi^*\rho_\alpha,e^{-i\varepsilon\tau}\wedge\pi^*\overline{\rho}_\alpha)_P=(\varepsilon\tau)^{\dim(F)}\wedge\pi^*(\rho_\alpha,\overline{\rho_\alpha})_M\neq 0,$$

and $e^{i\varepsilon\tau}\wedge\pi^*\rho_\alpha$ is of the right algebraic type. Finally, from the integrability condition for ρ_α, there are X_α,ξ_α such that

$$d_H\rho_\alpha=(X_\alpha+\xi_\alpha)\rho_\alpha.$$

If we let $X_\alpha^{hor}\in Hor$ be the horizontal vector projecting down to X_α, then the following holds on $\pi^{-1}(U_\alpha)$:

$$d_{\pi^*H}(e^{i\varepsilon\tau}\wedge\pi^*\rho_\alpha)=(X_\alpha^{hor}+\pi^*\xi_\alpha-X_\alpha^{hor}\llcorner i\varepsilon\tau)e^{i\varepsilon\tau}\wedge\pi^*\rho_\alpha,$$

showing that the induced generalized complex structure is integrable. □

Several cases for when the conditions of the theorem are fulfilled have been studied for symplectic manifolds and many times purely topological conditions on the base and on the fiber are enough to ensure that the hypotheses hold. We give the following two examples adapted from McDuff and Salamon [**54**], Chapter 6.

THEOREM 1.39. *Let $\pi : X\longrightarrow M$ be a symplectic fibration over a compact generalized complex base with fiber (F,ω). If the first Chern class of TF is a nonzero multiple of $[\omega]$, then the conditions of Theorem 1.38 hold. In particular, any oriented surface bundle can be given a symplectic fibration structure and, if the fibers are not tori, the total space has a generalized complex structure.*

THEOREM 1.40. *A symplectic fibration with compact and 1-connected fiber and a compact twisted generalized complex base admits a twisted generalized complex structure.*

6.2. Nilpotent Lie algebras. Another source of examples comes from considering left invariant generalized complex structures on Lie groups, which is equivalent to consider integrable linear generalized complex structures on their Lie algebra $\mathfrak{g}$. In this case, one of the terms in the H-Courant bracket always vanishes and we have

$$[\![X+\xi, Y+\eta]\!] = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi + i_Y i_X H,$$

which is a Lie bracket on $\mathfrak{g}\oplus\mathfrak{g}^*$.

Hence the search for a generalized complex structure on a Lie algebra $\mathfrak{g}$ amounts to finding a complex structure on $\mathfrak{g}\oplus\mathfrak{g}^*$ orthogonal with respect to the natural pairing. Therefore, given a fixed Lie algebra, finding a generalized complex structure on it or proving it does not admit any such structure is only a matter of perseverance.

In joint work with Gualtieri, the author carried out this task studying generalized complex structures on nilpotent Lie algebras. There we proved

THEOREM 1.41 (Cavalcanti–Gualtieri [18]). *Every 6-dimensional nilpotent Lie algebra has a generalized complex structure.*

A classification of which of those algebras had complex or symplectic structures had been carried out before by Salamon [57] and Goze and Khakimdjanov [31] and 5 nilpotent Lie algebras don't have either. Therefore these results gave the first nontrivial instances of generalized complex structures on spaces which admitted neither (left invariant) complex or symplectic structures.

6.3. A surgery. One of the features of generalized complex structures is that they don't necessarily have constant type along the manifold, as we saw in Example 1.24. In four dimensions, this is one of the main features distinguishing generalized complex structures from complex or symplectic structures and was used to produce some interesting examples in [19] by means of a surgery.

The idea of the surgery is to replace a neighborhood U of a symplectic 2-torus T with trivial normal bundle on a symplectic manifold (M,σ) by $D^2\times T^2$ with the generalized complex structure from Example 1.24 using a symplectomorphism which is a nontrivial diffeomeorphism of $\partial U \cong T^3$. This surgery is a particular case a C^∞ *logarithmic transformation*, a surgery introduced and studied by Gompf and Mrowka in [30].

THEOREM 1.42 (Cavalcanti–Gualtieri [19]). *Let (M,σ) be a symplectic 4-manifold, $T\hookrightarrow M$ be a symplectic 2-torus with trivial normal bundle and tubular neighbourhood U. Let $\psi : S^1\times T^2 \longrightarrow \partial U \cong S^1\times T^2$ be the map given on standard coordinates by*

$$\psi(\theta_1,\theta_2,\theta_3) = (\theta_3,\theta_2,-\theta_1).$$

Then

$$\tilde{M} = M\backslash U \cup_\psi D^2\times T^2,$$

admits a generalized complex structure with type change along a 2-torus, and which is integrable with respect to a 3-form H, such that $[H]$ is the Poincaré dual to the circle in $S^1\times T^2$ preserved by ψ. If M is simply connected and $[T]\in H^2(M,\mathbb{Z})$ is k times a primitive class, then $\pi_1(\tilde{M}) = \mathbb{Z}_k$.

PROOF. By Moser's theorem, symplectic structures with the same volume on an oriented compact surface are isomorphic. Hence, after rescaling, we can assume that T is endowed with its standard symplectic structure. Therefore, by Weinstein's neighbourhood theorem [65], the neighborhood U is symplectomorphic to $D^2 \times T^2$ with standard sympletic form:

$$\sigma = \frac{1}{2} d\tilde{r}^2 \wedge d\tilde{\theta}_1 + d\tilde{\theta}_2 \wedge d\tilde{\theta}_3.$$

Now consider the symplectic form ω on $D^2 \backslash \{0\} \times T^2$ from Example 1.24:

$$\omega = d \log r \wedge d\theta_3 + d\theta_1 \wedge d\theta_2.$$

The map $\psi : (D^2 \backslash D^2_{1/\sqrt{e}} \times T^2, \omega \longrightarrow (D^2 \backslash \{0\} \times T^2, \sigma)$ given by

$$\psi(r, \theta_1, \theta_2, \theta_3) = (\sqrt{\log er^2}, \theta_3, \theta_2, -\theta_1)$$

is a symplectomorphism.

Let B be the 2-form defined by (1.12) on $D^2 \backslash D^2_{1/\sqrt{e}} \times T^2$, and choose an extension $\tilde{B}$ of $\psi^{-1*}B$ to $M \backslash T$. Therefore $(M \backslash T, \tilde{B} + i\sigma)$ is a generalized complex manifold of type 0, integrable with respect to the $d\tilde{B}$-Courant bracket.

Now the surgery $\tilde{M} = M \backslash T \cup_\psi D^2 \times T^2$ obtains a generalized complex structure since the gluing map ψ satisfies $\psi^*(\tilde{B} + i\sigma) = B + i\omega$, therefore identifying the generalized complex structures on $M \backslash T$ and $D^2 \times T^2$ over the annulus where they are glued together. Therefore, the resulting generalized complex structure exhibits type change along the 2-torus coming from the central fibre of $D^2 \times T^2$. This structure is integrable with respect to $H = d\tilde{B}$, which is a globally defined closed 3-form on $\tilde{M}$.

The 2-form $\tilde{B}$ can be chosen so that it vanishes outside a larger tubular neighbourhood U' of T, so that $H = d\tilde{B}$ has support in $U' \backslash U$ and has the form

$$H = f'(r) dr \wedge d\theta_1 \wedge d\theta_3,$$

for a smooth bump function f such that $f|_U = 1$ and vanishes outside U'. Therefore, we see that H represents the Poincaré dual of the circle parametrized by θ_2, as required.

The last claim is a consequence of van Kampen's theorem and that $H^2(M, \mathbb{Z})$ is spherical, as M is simply connected. □

Example 1.43. Given an elliptic K3 surface, one can perform the surgery above along one of the T^2 fibers. In [30], Gompf and Mrowka show that the resulting manifold is diffeomorphic to $3\mathbb{C}P^2 \# 19\overline{\mathbb{C}P^2}$. Due to Taubes's results on Seiberg–Witten invariants [62] and Kodaira's classification of complex surfaces [45], we know this manifold does not admit symplectic or complex structures therefore providing the first example of generalized complex manifold without complex or symplectic structures.

CHAPTER 2

Generalized metric structures

In this chapter we present metrics on Courant algebroids as introduced in [33, 68] and further developed in [34]. Since a Courant algebroid is endowed with a natural pairing, one has to place a compatibility condition between metric and pairing. This is done by defining that a *generalized metric* is a self adjoint, orthogonal endomorphism $\mathcal{G}: \mathcal{E} \longrightarrow \mathcal{E}$ such that, for $v \in \mathcal{E} \backslash \{0\}$,

$$\langle \mathcal{G}v, v \rangle > 0.$$

A generalized metric on a split Courant algebroid gives rise to a Hodge star-like operator on forms. Further, for a given generalized metric, there is a natural splitting of any exact Courant algebroid as the sum of T^* with its metric orthogonal complement, $(T^*)^{\mathcal{G}}$. If this splitting is chosen, the star operator coincides with the usual Hodge star, while in general it differs from it by nonclosed B-field tranforms.

Similarly to the complex case, one can ask for a generalized metric to be compatible with a given generalized complex structure. There always are such metrics. Whenever a metric compatible with a generalized complex structure is chosen, we automatically get a second generalized complex structure which is not integrable in general. By studying Hodge theory on generalized complex manifolds we obtain Serre duality for the operator $\overline{\partial}$. When both of the generalized complex structures are integrable, we obtain a generalized Kähler structure.

The compatibility between a metric and a generalized complex structure is used to the full in the case of a generalized Kähler structure, for Gualtieri proved that in a generalized Kähler manifold a number of Laplacians coincide [34, 35] furnishing Hodge identities for those manifolds. These are powerful results which have implications for a generalized Kähler manifold similar to the formality theorem for Kähler manifolds [24], as made explicit in [17]. Moreover these identities are a key fact for the proof of smoothness of the moduli space of generalized Kähler structures [47].

After introducing the generalized metric, this chapter follows closely [34, 35] and gives some applications of the results therein. In the first section we introduce the concept of generalized metric on a vector space and investigate the consequences of the compatibility of this metric with a linear generalized complex structure. In Section 2 we do things over a manifold with the requirement that the generalized complex structure involved is integrable. We also state Gualtieri's theorem relating generalized Kähler structures to bihermitian structures. In Section 3 we study Hodge theory on a generalized Kähler manifold and in Section 4 we give an application of those results.

1. Linear algebra of the metric

1.1. Generalized metric. The concept of generalized metric on $\mathcal{D}V$, the double of a vector space V, was introduced by Gualtieri [33] and Witt [68] and was further studied by Gualtieri in connection with generalized complex structures in [34]. Following Gualtieri's exposition, a *generalized metric* is an orthogonal, self adjoint operator $\mathcal{G} : \mathcal{D}V \longrightarrow \mathcal{D}V$ such that

$$\langle \mathcal{G}e, e \rangle > 0 \text{ for } e \in \mathcal{D}V \backslash \{0\}.$$

Using symmetry and orthogonality we see that

$$\mathcal{G}^2 = \mathcal{G}\mathcal{G}^t = \mathcal{G}\mathcal{G}^{-1} = \mathrm{Id}.$$

Hence $\mathcal{G}$ splits $\mathcal{D}V$ into its ± 1-eigenspaces, $C_\pm$, which are orthogonal subspaces of $\mathcal{D}V$ where the pairing is $\pm$-definite. Therefore $C_\pm$ are maximal and, since $V^* \subset \mathcal{D}V$ is isotropic, the projections $\pi : C_\pm \longrightarrow V$ are isomorphisms. Conversely, prescribing orthogonal spaces $C_\pm$ where the pairing is $\pm$-definite defines a metric $\mathcal{G}$ by letting $\mathcal{G} = \pm 1$ on $C_\pm$.

A generalized metric induces a metric on the underlying vector space. This is obtained using the isomorphism $\pi : C_+ \longrightarrow V$ and defining

$$g(X, Y) = \langle \pi^{-1}(X), \pi^{-1}(Y) \rangle.$$

One can alternatively define a metric using C_-, but this renders the same metric on V.

If we have a splitting $\mathcal{D}V = V \oplus V^*$, then a generalized metric can be described in terms of forms. Indeed, in this case C_+ can be described as a graph over V of an element in $\otimes^2 V^* = \mathrm{Sym}^2 V^* \oplus \wedge^2 V^*$, i.e., there is $g \in \mathrm{Sym}^2 V^*$ and $b \in \wedge^2 V^*$ such that

$$C_+ = \{X + (b + g)(X) : X \in V\}.$$

It is clear that g above is nothing but the metric induced by $\mathcal{G}$ on V. The subspace C_- is also a graph over V. Since it is orthogonal to C_+ with respect to the natural pairing we see that

$$C_- = \{X + (b - g)(X) : X \in V\}.$$

Conversely, a metric g and a 2-form b define a pair of orthogonal spaces $C_\pm \subset V \oplus V^*$ where the pairing is $\pm$-definite, so, on $V \oplus V^*$, a generalized metric is equivalent to a metric and a 2-form.

A generalized metric on $V \oplus V^*$ allows us to define a Hodge star operator [**34**]:

Definition 2.1. Fix an orientation for C_+ and let $e_1, \cdots, e_n$ be an oriented orthonormal basis for this space. Denoting by τ the product $e_1 \cdots e_n \in \mathrm{Cl}(V \oplus V^*)$, the *generalized Hodge star* is defined by $\star\alpha = (-1)^{|\alpha|(n-1)} \tau \cdot \alpha$, where $\cdot$ is the Clifford action of $\mathrm{Cl}(V \oplus V^*)$ on forms.

If we denote by $\star_g$ the usual Hodge star associated to the metric g, the Mukai pairing gives the following relation, if $b = 0$:

$$(\alpha, \star\beta) = \alpha \wedge \star_g \beta.$$

In the presence of a b-field, if we let $\alpha = e^{-b}\tilde{\alpha}$ and $\beta = e^{-b}\tilde{\beta}$, then the relation becomes

$$(\alpha, \star\beta) = \tilde{\alpha} \wedge \star_g \tilde{\beta}.$$

Hence $(\alpha, \star\alpha)$ is a nonvanishing volume form whenever $\alpha \neq 0$.

1.2. Hermitian structures. Given a generalized complex structure $\mathcal{J}_1$ on a vector space V, we say a generalized metric $\mathcal{G}$ is *compatible* with $\mathcal{J}_1$ if they commute. In this situation, $\mathcal{J}_2 = \mathcal{G}\mathcal{J}_1$ is automatically a generalized complex structure which commutes with $\mathcal{G}$ and $\mathcal{J}_1$. Given a generalized complex structure on V, one can always find a generalized metric compatible with it.

If we let $C_\pm \subset \mathcal{D}V$ be the ± 1-eigenspaces of $\mathcal{G}$, then, using the fact that $\mathcal{J}_1$ and $\mathcal{G}$ commute, we see that $\mathcal{J}_1 : C_\pm \longrightarrow C_\pm$. Since $\pi : C_\pm \longrightarrow V$ are isomorphisms, we can transport the complex structures on $C_\pm$ to complex structures $I_\pm$ on V. Furthermore, as $\mathcal{J}_1$ is orthogonal with respect to the natural pairing, $I_\pm$ are orthogonal with respect to the induced metric on V.

If the double of V is split as $V \oplus V^*$, then both $\mathcal{J}_1$ and $\mathcal{J}_2$ give rise to a decomposition of $\wedge^\bullet V^* \otimes \mathbb{C}$ into their ik-eigenvalues. Since $\mathcal{J}_1$ and $\mathcal{J}_2$ commute, they can be diagonalized simultaneously:

$$U^{p,q} = U^p_{\mathcal{J}_1} \cap U^q_{\mathcal{J}_2}; \qquad \oplus_{p,q} U^{p,q} = \wedge^\bullet V^* \otimes \mathbb{C}.$$

So, given a metric compatible with a generalized complex structure on $V \oplus V^*$, we obtain $\mathbb{Z}^2$ grading on forms. As we will see later, this bigrading will be compatible with the differentiable structure on a manifold only if $\mathcal{J}_i$ are integrable, which corresponds to the generalized Kähler case.

EXAMPLE 2.2. Let V be a vector space endowed with a complex structure I compatible with a metric g. Then the induced generalized complex structure $\mathcal{J}_I$ is compatible with the generalized metric $\mathcal{G}$ on $V \oplus V^*$ induced by g with $b = 0$. The generalized complex structure defined by $\mathcal{G}\mathcal{J}_I$ is nothing but the generalized complex structure $\mathcal{J}_\omega$ defined by the symplectic form $\omega = g(I\cdot, \cdot)$.

In this case, the decomposition of forms into $U^{p,q}$ induced by this hermitian structure corresponds to the intersection of U^p_I and U^q_ω, as determined in Examples 1.9 and 1.10, i.e.,

$$U^{p-q,n-p-q} = U^{p-q}_{\mathcal{J}_I} \cap U^{n-p-q}_{\mathcal{J}_\omega} = \Phi(\wedge^{p,q} V^*), \tag{2.1}$$

where $\Phi(\alpha) = e^{\frac{\Lambda}{2i}} e^{i\omega}\alpha$ is the map defined in Example 1.10. In particular, the decomposition of forms into $U^{p,q}$ is not just the decomposition into $\wedge^{p,q} V^*$, which only depends on I, but isomorphic to it via an isomorphism which depends on the symplectic form.

Also, as we saw in the previous section, a metric on $V \oplus V^*$ gives rise to a Hodge star operator. If the metric is compatible with a generalized complex structure, then this star operator can be expressed in terms of the Lie group action of $\mathcal{J}_1$ and $\mathcal{J}_2$ on forms:

LEMMA 2.3. (Gualtieri [35]) *If a metric $\mathcal{G}$ is compatible with a generalized complex structure $\mathcal{J}_1$ and we let $\mathcal{J}_2 = \mathcal{G}\mathcal{J}_1$, then $\star\alpha = \mathcal{J}_1\mathcal{J}_2\alpha$.*

COROLLARY 2.4. *If $V \oplus V^*$ has a generalized complex structure $\mathcal{J}_1$and a compatible metric $\mathcal{G}$, then the Hodge star operator preserves the bigrading of forms.*

PROOF. Indeed if $\alpha \in U^{p,q}$, then $\star\alpha = \mathcal{J}_1\mathcal{J}_2\alpha = i^{p+q}\alpha$. □

For a generalized complex structure with compatible metric the operator $\overline{\star}$ defined by $\overline{\star}\alpha = \star\overline{\alpha}$ is also important, as it furnishes a definite, hemitian, bilinear functional,

$$h(\alpha, \beta) = \int_M (\alpha, \overline{\star}\beta), \qquad \alpha, \beta \in \wedge^\bullet V^* \otimes \mathbb{C}. \tag{2.2}$$

We finish this section with a remark. The fact that a pair of commuting generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ on V gives rise to a metric places algebraic restrictions on $\mathcal{J}_1$ and $\mathcal{J}_2$. For example, one can check that,

$$\mathit{type}(\mathcal{J}_1) + \mathit{type}(\mathcal{J}_2) \leq n,$$

where $\dim(V) = 2n$.

2. Generalized Kähler structures

2.1. Hermitian structures. A Hermitian structure on an exact Courant algebroid $\mathcal{E}$ over a manifold M is a generalized complex structure $\mathcal{J}$ with compatible metric $\mathcal{G}$ on $\mathcal{E}$. Given a generalized complex structure on $\mathcal{E}$ one can always find a metric compatible with it, therefore obtaining a Hermitian structure. From a general point of view, this is true because it corresponds to a reduction of the structure group of the Courant algebroid from $U(n,n)$ to its maximal compact subgroup $U(n)\times U(n)$ and such reduction is unobstructed.

Incidentally, the existence of such a metric implies that every generalized almost complex manifold has an almost complex structure. Indeed, from the work in the previous section, if $\mathcal{J}$ is a generalized almost complex structure, then a metric compatible with $\mathcal{J}$ allows one to define an almost complex structure on M using the projection $\pi: C_+ \longrightarrow T$.

Given a splitting for $\mathcal{E}$, $\mathcal{J}$ induces a decomposition of differential forms, $\mathcal{G}$ furnishes a Hodge star operator preserving this decomposition and the operator h from equation (2.2) is a definite hermitian bilinear functional.

LEMMA 2.5. *Let $(M,\mathcal{J},\mathcal{G})$ be a generalized complex manifold with compatible metric. Then the h-adjoint of $\overline{\partial}$ is given by $\overline{\partial}^* = -\overline{\star}\overline{\partial}\overline{\star}^{-1}$*

PROOF. We start observing that $(d_H\alpha,\beta)+(\alpha,d_H\beta) = (d(\sigma(\alpha)\wedge\beta))_{top}$ is an exact form. Now, let $\alpha\in\mathcal{U}^{k+1}$ and $\beta\in\mathcal{U}^{-k}$, then

$$(d(\sigma(\alpha)\wedge\beta))_{top} = (d_H\alpha,\beta)+(\alpha,d_H\beta) = (\partial\alpha,\beta)+(\overline{\partial}\alpha,\beta)+(\alpha,\partial\beta)+(\alpha,\overline{\partial}\beta), \tag{2.3}$$

and according to Proposition 1.8, the terms $(\partial\alpha,\beta)$ and $(\alpha,\partial\beta)$ vanish. Therefore

$$\begin{aligned} h(\overline{\partial}\alpha,\beta) &= \int_M(\overline{\partial}\alpha,\star\overline{\beta}) = -\int_M(\alpha,\overline{\partial}\star\overline{\beta})\\ &= -\int_M(\alpha,\overline{\star}\overline{\star}^{-1}\overline{\partial}\star\overline{\beta})\\ &= h(\alpha,-\overline{\star}^{-1}\overline{\partial}\overline{\star}\beta)\end{aligned}$$

□

Now, the Laplacian $\overline{\partial}\overline{\partial}^*+\overline{\partial}^*\overline{\partial}$ is an elliptic operator and in a compact generalized complex manifold every $\overline{\partial}$-cohomology class has a unique harmonic representative, which is a $\overline{\partial}$ and $\overline{\partial}^*$-closed form. Also, from the expression above for $\overline{\partial}^*$, we see that $\overline{\star}$ maps harmonic forms to harmonic forms.

THEOREM 2.6 (Serre duality; Cavalcanti [16]). *In a compact generalized complex manifold $(M^{2n},\mathcal{J})$, the Mukai pairing gives rise to a pairing in cohomology $H^k_{\overline{\partial}}\times H^l_{\overline{\partial}}\longrightarrow H^{2n}(M)$ which vanishes if $k\neq -l$ and is nondegenerate if $k=-l$.*

PROOF. Given cohomology classes $a\in H^k_{\overline{\partial}}(M)$ and $b\in H^l_{\overline{\partial}}(M)$, choose representative $\alpha\in\mathcal{U}^k$ and $\beta\in\mathcal{U}^l$. According to Lemma 1.8, (α,β) vanishes if $k\neq -l$, therefore proving the first claim.

If $k=-l$ and $b=0$, so that $\beta=\overline{\partial}\gamma$ is a $\overline{\partial}$-exact form, then, according to (2.3),

$$[(\alpha,\overline{\partial}\gamma)] = [(\overline{\partial}\alpha,\gamma)] = 0,$$

Showing that the pairing is well defined.

Finally, if we let α be the harmonic representative of the class a, then $\overline{\star}\overline{\alpha}$ is $\overline{\partial}$ closed form in $\mathcal{U}^{-k}$ which pairs nontrivially with α, showing nondegeneracy. □

2.2. Generalized Kähler structures. Given a Hermitian structure $(\mathcal{J},\mathcal{G})$ on a Courant algebroid $\mathcal{E}$, we can always define the generalized almost complex structure $\mathcal{J}_2=\mathcal{J}\mathcal{G}$. This structure is not integrable in general. Particular examples are given by an almost complex structure taming a symplectic structure or by the structure associated to a nondegenerate 2-form of type $(1,1)$ on a complex manifold. So the integrability of $\mathcal{J}_2$ is the analogue of the Kähler condition.

DEFINITION 2.7. A *generalized Kähler structure* on an exact Courant algebroid $\mathcal{E}$ is a pair of commuting generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ such that $\mathcal{G} = -\mathcal{J}_1\mathcal{J}_2$ is a generalized metric.

As we have seen, given a generalized complex structure compatible with a generalized metric one can define almost complex structures $I_\pm$ on T using the projections $\pi : C_\pm \longrightarrow T$. The integrability of $\mathcal{J}_1$ and $\mathcal{J}_2$ imply that $I_\pm$ are integrable complex structures and also that

$$dd^c_-\omega_- = 0 \qquad d^c_-\omega_- = -d^c_+\omega_+,$$

where $\omega_\pm = g(I_\pm\cdot,\cdot)$ are the Kähler forms associated to $I_\pm$ and $d^c_\pm = -I_\pm dI_\pm$ (see [33], Proposition 6.16).

The converse also holds and is the heart of Gualtieri's theorem relating generalized Kähler structure to bihermitian structures:

THEOREM 2.8 (Gualtieri [33], Theorem 6.37). *A manifold M admits a bihermitian structure (g, I_+, I_-), such that*

$$dd^c_-\omega_- = 0 \qquad d^c_-\omega_- = -d^c_+\omega_+$$

if and only if the Courant algebroid $\mathcal{E}$ with characteristic class $[d^c_-\omega_-]$ has a generalized Kähler structure which induces the bihermitian structure (g, I_+, I_-) on M.

In the case of a generalized Kähler induced by a Kähler structure, $I_+ = \pm I_-$ and the equations above hold trivially as $d^c\omega = 0$. Now we use the bihermitian characterization of a generalized Kähler manifold to give nontrivial examples of generalized Kähler structures.

EXAMPLE 2.9 (Gualtieri [33], Example 6.30). Let (M, g, I, J, K) be a hyperkähler manifold. Then it automatically has an S^2 worth of Kähler structures which automatically furnish generalized Kähler structures. However there are other generalized Kähler structures on M. For example, if we let $I_+ = I$ and $I_- = J$, then all the conditions of Theorem 2.8 hold hence providing M with a generalized Kähler structure (with $H = 0$). The forms generating the canonical bundles of this generalized Kähler structure are

$$\rho_1 = \exp(\omega_K + \frac{i}{2}(\omega_I - \omega_J));$$

$$\rho_2 = \exp(-\omega_K + \frac{i}{2}(\omega_I + \omega_J)).$$

showing that a generalized Kähler structure can be determined by two generalized complex structures of symplectic type.

EXAMPLE 2.10 (Gualtieri [33], Example 6.39). Every compact even dimensional Lie group G admits left and right invariant complex structures [58, 64]. If G is semi-simple, we can choose such complex structures to be Hermitian with respect to the invariant metric induced by the Killing form K. This bihermitian structure furnishes a generalized Kähler structure on the Courant algebroid over G with characteristic class given by the bi-invariant Cartan 3-form: $H(X, Y, Z) = K([X, Y], Z)$. To prove this we let J_L and J_R be left and right invariant complex structures as above and compute $d^c_L\omega_L$:

$$\begin{aligned} A = d^c_L\omega_L(X,Y,Z) &= d_L\omega_L(J_LX, J_LY, J_LZ)\\ &= -\omega_L([J_LX,J_LY],J_LZ) + c.p.\\ &= -K([J_LX,J_LY],Z) + c.p.\\ &= -K(J_L[J_LX,Y] + J_L[X,J_LY] + [X,Y],Z) + c.p.\\ &= (2K([J_LX,J_LY],Z) + c.p.) - 3H(X,Y,Z)\\ &= -2A - 3H, \end{aligned}$$

where *c.p.* stands for cyclic permutations. This proves that $d^c_L\omega_L = -H$. Since the right Lie algebra is antiholomorphic to the left, the same calculation yields $d^c_R\omega_R = H$ and by Theorem 2.8, this bihermitian structure induces a generalized Kähler structure on the Courant algebroid with characteristic class $[H]$.

More recently, using a classification theorem for bihermitian structures on 4-manifolds by Apostolov *et al* [1], Apostolov and Gualtieri managed to classify all 4-manifolds admiting generalized Kähler structures [2].

3. Hodge identities

Assume that an exact Courant algebroid $\mathcal{E}$ has a generalized Kähler structure $(\mathcal{J}_1,\mathcal{J}_2)$. Once a splitting for $\mathcal{E}$ is chosen, we obtain a bigrading of forms into $U^{p,q} = U^p_{\mathcal{J}_1}\cap U^q_{\mathcal{J}_2}$. Since for generalized complex structure $d_H : \mathcal{U}^k \longrightarrow \mathcal{U}^{k+1} + \mathcal{U}^{k-1}$, we see that for a generalized Kähler structure d_H decomposes in four components:

$$d_H : \mathcal{U}^{p,q} \longrightarrow \mathcal{U}^{p+1,q+1} + \mathcal{U}^{p+1,q-1} + \mathcal{U}^{p-1,q+1}, \mathcal{U}^{p-1,q-1}.$$

We denote these components by

$$\delta_+ : \mathcal{U}^{p,q} \longrightarrow \mathcal{U}^{p+1,q+1} \qquad \delta_- : \mathcal{U}^{p,q} \longrightarrow \mathcal{U}^{p+1,q-1}$$

and their conjugates

$$\overline{\delta_+} : \mathcal{U}^{p,q} \longrightarrow \mathcal{U}^{p-1,q-1} \qquad \overline{\delta_-} : \mathcal{U}^{p,q} \longrightarrow \mathcal{U}^{p-1,q+1}.$$

So that, for example, $\partial_{\mathcal{J}_1} = \delta_+ + \delta_-$ and $\partial_{\mathcal{J}_2} = \delta_+ + \overline{\delta_-}$. One can easily show that h-adjoints of $\delta_\pm$ are given by $\delta^*_\pm = -\overline{\star}\delta_\pm\overline{\star}^{-1}$ and similarly for δ_-, $\overline{\delta_+}$ and $\overline{\delta_-}$. Given the description of $\star$ in terms of the Lie algebra action of $\mathcal{J}_1$ and $\mathcal{J}_2$ given in Lemma 2.3, we have

Theorem 2.11 (Gualtieri [34, 35]). *The following relations hold in a generalized Kähler manifold*

$$\delta^*_+ = \overline{\delta_+} \qquad \textit{and} \qquad \delta^*_- = -\overline{\delta_-}; \tag{2.4}$$

$$4\triangle_{d_H} = 2\triangle_{\partial_{\mathcal{J}_1}} = 2\triangle_{\overline{\partial}_{\mathcal{J}_1}} = 2\triangle_{\partial_{\mathcal{J}_2}} = 2\triangle_{\overline{\partial}_{\mathcal{J}_2}} = \triangle_{\delta_+} = \triangle_{\delta_-} = \triangle_{\overline{\delta_+}} = \triangle_{\overline{\delta_-}}. \tag{2.5}$$

Proof. To prove that $\delta^*_+ = -\overline{\delta_+}$, one only has to prove these operators agree when action on $\mathcal{U}^{p,q}$. So let $\alpha \in \mathcal{U}^{p,q}$. Then, according to Lemma 2.3, we get $\overline{\star}\alpha = i^{-p-q}\overline{\alpha}$ and

$$\delta^*_+ = -\overline{\star}\delta_+\overline{\star}^{-1}\alpha = -i^{p+q}\overline{\star}\delta_+\overline{\alpha} = -i^{p+q}i^{-p+1-q+1}\overline{\delta_+\overline{\alpha}} = \overline{\delta_+}\alpha$$

And similarly one can check the identity $\delta^*_- = -\overline{\delta_-}$.

To prove the equality of the Laplacians now one only has to observe that due to (2.4)

$$\triangle_{\delta_+} = \delta_+\overline{\delta}_+ + \overline{\delta}_+\delta_+ \text{ and } \triangle_{\delta_-} = -(\delta_-\overline{\delta}_- + \overline{\delta}_-\delta_-).$$

On the other hand, if $\alpha \in \mathcal{U}^{p,q}$ then $d_H^2\alpha = 0$ implies

$$(\delta_+\overline{\delta}_+ + \overline{\delta}_+\delta_+ + \delta_-\overline{\delta}_- + \overline{\delta}_-\delta_-)\alpha = 0$$

for all α and hence $\triangle_{\delta_+} = \triangle_{\delta_-}$.

Since $\partial^2_{\mathcal{J}_1} = 0$, δ_+ and δ_- anticommute, so, e.g.,

$$\begin{aligned}\triangle_{d_H} &= d_H d_H^* + d_H^* d_H = (\delta_+ + \delta_- + \overline{\delta_+} + \overline{\delta_-})(\delta_+^* + \delta_-^* + \overline{\delta}_+^* + \overline{\delta}_-^*)\\ &+ (\delta_+^* + \delta_-^* + \overline{\delta}_+^* + \overline{\delta}_-^*)(\delta_+ + \delta_- + \overline{\delta_+} + \overline{\delta_-})\\ &= (\delta_+ + \delta_- + \overline{\delta_+} + \overline{\delta_-})(\overline{\delta}_+ - \overline{\delta}_- + \delta_+ - \delta_-) + (\overline{\delta}_+ - \overline{\delta}_- + \delta_+ - \delta_-)(\delta_+ + \delta_- + \overline{\delta_+} + \overline{\delta_-})\\ &= 2(\delta_+\overline{\delta}_+ + \overline{\delta}_+\delta_+) + 2(\delta_-\overline{\delta}_- + \overline{\delta}_-\delta_-)\\ &= 2\triangle_{\delta_+} + 2\triangle_{\delta_-} = 4\triangle_{\delta_+}.\end{aligned}$$

And similar arguments prove all the other identities. □

Here we mention a couple of standard consequences of this theorem whose proof follows the same argument given in the classical Kähler case.

COROLLARY 2.12 (Gualtieri [34, 35]). *In a compact generalized Kähler manifold the decomposition of forms into $\mathcal{U}^{p,q}$ gives rise to a p,q-decomposition of the d_H-cohomology.*

COROLLARY 2.13 (Gualtieri [34, 35]). **$\delta_+\delta_-$-Lemma** *In a compact generalized Kähler manifold*

$$\mathrm{Im}\,(\delta_+) \cap \mathrm{Ker}\,(\delta_-) = \mathrm{Im}\,(\delta_-) \cap \mathrm{Ker}\,(\delta_+) = \mathrm{Im}\,(\delta_+\delta_-).$$

4. Formality in generalized Kähler geometry

A differential graded algebra $(\mathcal{A}, d)$ is *formal* if there is a finite sequence of differential graded algebras $(\mathcal{A}_i, d)$ with quasi isomorphisms φ_i between $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$ such that $(\mathcal{A}_1, d) = (\mathcal{A}, d)$ and $(\mathcal{A}_n, d) = (H^\bullet(A), 0)$:

$$\mathcal{A}_1 \xrightarrow{\varphi_1} \mathcal{A}_2 \xleftarrow{\varphi_2} \mathcal{A}_3 \xrightarrow{\varphi_3} \cdots \xrightarrow{\varphi_{n-2}} \mathcal{A}_{n-1} \xleftarrow{\varphi_{n-1}} \mathcal{A}_n.$$

And a manifold is *formal* if the algebra of differential forms, $(\Omega^\bullet(M), d)$, is formal.

One of the most striking uses of the $\partial\overline{\partial}$-lemma for a complex structure appears in the proof that it implies formality [24], therefore providing fine topological obstructions for a manifold to admit a Kähler structure [61]. A very important fact used in this proof is that for a complex structure ∂ and $\overline{\partial}$ are derivations. As we mentioned before, in the generalized complex world the operators ∂ and $\overline{\partial}$ are not derivations and indeed there are examples of nonformal generalized complex manifolds for which the $\partial\overline{\partial}$-lemma holds [14].

However, as we saw in Section 3.1, given a generalized complex structure we can form the differential graded algebra $(\Omega^\bullet(\overline{L}), d_L)$ and if the generalized complex structure has holomorphically trivial canonical bundle, trivialized by a form ρ, we get a map

$$\phi : \Omega^\bullet(\overline{L}) \longrightarrow \Omega^\bullet_{\mathbb{C}}(M); \qquad \phi(s \cdot \sigma) = s \cdot \rho$$

such that

$$\overline{\partial}\phi(\alpha) = \phi(d_L\alpha).$$

In the case of a generalized Kähler structure, since $\mathcal{J}_1$ and $\mathcal{J}_2$ commute, $\mathcal{J}_2$ furnishes an integrable complex structure on L_1, the i-eigenspace of $\mathcal{J}_1$. With that we obtain a p,q-decomposition of $\wedge^\bullet\overline{L_1}$ and decomposition $d_{L_1} = \partial_{L_1} + \overline{\partial}_{L_1}$ of the differential. Since this is nothing but the decomposition of d_{L_1} by an underlying complex structure, the operators ∂_{L_1} and $\overline{\partial}_{L_1}$ are derivations. And if further $\mathcal{J}_1$ has holomorphically trivial canonical bundle, then the correspondence between d_L and ∂ gives an identifications between ∂_{L_1} and δ_+ and $\overline{\partial}_{L_1}$ and δ_-:

$$\delta_+\phi(\alpha) = \phi(\partial_{L_1}\alpha) \qquad \delta_-\phi(\alpha) = \phi(\overline{\partial}_{L_1}\alpha).$$

So, if we let $d^c_{L_1} = -i(\partial_{L_1} - \overline{\partial}_{L_1})$, Corollary 2.13 implies

LEMMA 2.14. *If $(\mathcal{J}_1, \mathcal{J}_2)$ is a generalized Kähler structure on a compact manifold and $\mathcal{J}_1$ has holomorphically trivial canonical bundle then*

$$\mathrm{Im}\,(d_{L_1}) \cap \mathrm{Ker}\,(d^c_{L_1}) = \mathrm{Im}\,(d^c_{L_1}) \cap \mathrm{Ker}\,(d_{L_1}) = \mathrm{Im}\,(d_{L_1}d^c_{L_1}).$$

And hence the same argument from [24] gives

THEOREM 2.15 (Cavalcanti [17]). *If $(\mathcal{J}_1, \mathcal{J}_2)$ is a generalized Kähler structure on a compact manifold and $\mathcal{J}_1$ has holomorphically trivial canonical, then $(\Omega^\bullet(\overline{L_1}), d_{L_1})$ is a formal differential graded algebra.*

PROOF. Let $(\Omega^\bullet_c(\overline{L_1}), d_{L_1})$ be the algebra of $d^c_{L_1}$-closed element of $\Omega^\bullet(\overline{L_1})$ endowed with differential d_{L_1} and $(H_{d^c_{L_1}}(\overline{L_1}), d_{L_1})$ be the cohomology of $\Omega^\bullet(\overline{L_1})$ with respect to $d^c_{L_1}$, also endowed with differential d_{L_1}. Then we have maps

$$(\Omega(\overline{L_1}), d_{L_1}) \xleftarrow{\ i\ } (\Omega_c(\overline{L_1}), d_{L_1}) \xrightarrow{\ \pi\ } (H_{d^c_{L_1}}(\overline{L_1}), d_{L_1}),$$

and, as we are going to see, these maps are quasi-isomorphisms and the differential of $(H_{d^c_{L_1}}(M), d_{L_1})$ is zero, therefore showing that $\Omega(\overline{L_1})$ is formal.

i) i^* is surjective:
Given a d_{L_1}-closed form α, let $\beta = d^c_{L_1}\alpha$. Then $d_{L_1}\beta = d_{L_1}d^c_{L_1}\alpha = -d^c_{L_1}d_{L_1}\alpha = 0$, so β satisfies the conditions of the $d_{L_1}d^c_{L_1}$-lemma, hence $\beta = d^c_{L_1}d_{L_1}\gamma$. Let $\tilde{\alpha} = \alpha - d_{L_1}\gamma$, then $d^c_{L_1}\tilde{\alpha} = d^c_{L_1}\alpha - d^c_{L_1}d_{L_1}\gamma = \beta - \beta = 0$, so $[\alpha] \in \mathrm{Im}\,(i^*)$.

ii) i^* is injective:
If $i^*\alpha$ is exact, then α is $d^c_{L_1}$-closed and exact, hence by the $d_{L_1}d^c_{L_1}$-lemma $\alpha = d_{L_1}d^c_{L_1}\beta$, so α is the derivative of a $d^c_{L_1}$-closed form and hence its cohomology class in Ω_c is also zero.

iii) The differential of $(H_{d^c_{L_1}}(M), d)$ is zero:
Let α be $d^c_{L_1}$-closed, then $d_{L_1}\alpha$ is exact and $d^c_{L_1}$ closed so, by the $d_{L_1}d^c_{L_1}$-lemma, $d_{L_1}\alpha = d^c_{L_1}d_{L_1}\beta$ and so it is zero in $d^c_{L_1}$-cohomology.

iv) π^* is onto:
Let α be $d^c_{L_1}$-closed. Then, as above, $d_{L_1}\alpha = d_{L_1}d^c_{L_1}\beta$. Let $\tilde{\alpha} = \alpha - d^c_{L_1}\beta$, and so $d_{L_1}\tilde{\alpha} = 0$ and $[\tilde{\alpha}]_{d^c_{L_1}} = [\alpha]_{d^c_{L_1}}$, so $\pi^*([\tilde{\alpha}]_{d_{L_1}}) = [\alpha]_{d^c_{L_1}}$.

v) π^* is injective:
Let α be closed and $d^c_{L_1}$-exact, then the $d_{L_1}d^c_{L_1}$-lemma implies that α is exact and hence $[\alpha] = 0$ in $\mathcal{E}_c$.

□

If $\mathcal{J}_1$ is a structure of type 0, i.e., is of symplectic type, then not only does it have holomorphically trivial canonical bundle but $\pi : L \longrightarrow T_{\mathbb{C}}$ is an isomorphism and the bracket on L is mapped to the Lie bracket of vector fields. Therefore, in this case, $(\Omega(\overline{L_1}), d_{L_1})$ is isomorphic to $(\Omega_{\mathbb{C}}(M), d)$. So the previous theorem gives:

Corollary 2.16 (Cavalcanti [17]). *If* $(\mathcal{J}_1, \mathcal{J}_2)$ *is a generalized Kähler structure on a compact manifold* M *and* $\mathcal{J}_1$ *is of symplectic type, then* M *is formal.*

Similarly to the original theorem of formality of Kähler manifolds, Theorem 2.15 furnishes a nontrivial obstruction for a given generalized complex structure to be part of a generalized Kähler structure. As an application of this result one can prove that no generalized complex structure on a nilpotent Lie algebra is part of generalized Kähler pair [17].

CHAPTER 3

Reduction of Courant algebroids

Given a structure on a manifold M and a group G acting on M by symmetries of that structure, one can ask what kind of conditions have to be imposed on the group action in order for that structure on M to descend to a similar type of structure on M/G. Examples include the quotient of metrics when a Riemannian manifold is acted on by Killing fields, quotient of complex manifolds by holomorphic actions of a complex group and the *reduction* of symplectic manifolds acted on by a group of symplectomorphisms.

The latter example is particularly interesting as it shows that sometimes it may be necessary to take submanifolds as well as the quotient by the G action in order to find a manifold with the desired structure. This is going to be a central feature in theory developed in this chapter and one of the tasks ahead is to define a notion of action which includes the choice of submanifolds on it. Although we are primarily concerned about how to take quotients of generalized complex structures, there is a more basic question which needs to be answered first: how can we quotient a Courant algebroid $\mathcal{E}$?

To answer this question we recall that the action of G on M can be fully described by the infinitesimal action of its Lie algebra, $\psi : \mathfrak{g} \longrightarrow C^\infty(T)$, which is a morphism of Lie algebras. We want to have a similar picture for a Courant algebroid $\mathcal{E}$ over M, i.e., we want to describe a G-action on $\mathcal{E}$ covering the G-action on M by a map $\Psi : \mathfrak{a} \longrightarrow C^\infty(\mathcal{E})$. Here we encounter our first problem which is to determine what kind of object $\mathfrak{a}$ is. It is natural to ask that it has the same sort of structure that $\mathcal{E}$ has, i.e., instead of being a Lie algebra, it has to encode the properties of a Courant algebroid. This leads us to the concept of a *Courant algebra*. Another point is that we still want to have the same group G acting on both $\mathcal{E}$ and M, thus restricting the Courant algebra morphisms Ψ one is allowed to consider: these are the *extended actions*.

We will see that once an extended action on a Courant algebroid is chosen, it determines a foliation on M whose leaves are invariant under the G action. The 'quotient' algebroid is an algebroid defined over the quotient of a leaf of this distribution by G and the algebroid itself is obtained by considering the quotient of a subspace of $\mathcal{E}$ and hence we call these the *reduced manifold* and the *reduced Courant algebroid*. Some of the formalism we introduce appeared before in the physics literature in the context of gauging the Wess–Zumino term in a sigma model [**40**, **28**, **27**].

Once the reduced Courant algebroid $\mathcal{E}_{red}$ is understood, it is relatively easy to reduce structures from $\mathcal{E}$. Here we will deal only with Dirac and generalized complex structures. However, even by just considering these we see that this theory is actually quite strong and includes as subcases pull back and push forward of Dirac structures (as studied in [**11**]), quotient of complex manifolds by holomorphic actions by complex groups, symplectic reduction. Other interesting and new cases consist for type changing reductions: for example, the reduction of a symplectic structure can lead to a generalized complex structure of nonzero type (see Example 3.22).

The material of this chapter is mostly an extraction from the collaboration with Bursztyn and Gualtieri [**10**], whose reading I highly recommend if you are interested in these topics. There we also deal with the reduction of generalized Kähler structures and present the concept of Hamiltonian actions. Other independent work dealing with reduction of generalized complex structures with different degrees of generality are [38, 48, 60].

This chapter is organized as follows. In Section 1 we give a precise description of the group of symmetries of an exact Courant algebroid, define Courant algebras and extended actions. In Section 2 we explain how to reduce manifolds and exact Courant algebroid over them given an extended action. In Section 3 we show how to transport Dirac structures from the original Courant algebroid to the reduced Courant algebroid and then we finish in Section 4 applying these results to reduction of generalized complex structures.

1. Courant algebras and extended actions

1.1. Symmetries of Courant algebroids. As we have mentioned before, the group of symmetries, $\mathcal{C}$, of an exact Courant algebroid is formed by diffeomorphisms and B-fields. A more precise description of $\mathcal{C}$ can be given once an isotropic splitting is chosen [33]: it consists of the group of ordered pairs $(\varphi, B) \in \mathrm{Diff}(M) \times \Omega^2(M)$ such that $\varphi^* H - H = dB$, where H is the curvature of the splitting. Diffeomorphisms act in the usual way on $T \oplus T^*$, while 2-forms act via B-field transforms. As a result we see that $\mathcal{C}$ is an extension

$$0 \longrightarrow \Omega^2_{cl}(M) \longrightarrow \mathcal{C} \longrightarrow \mathrm{Diff}_{[H]}(M) \longrightarrow 0\ ,$$

where $\mathrm{Diff}_{[H]}(M)$ is the group of diffeomorphisms preserving the cohomology class $[H]$.

Therefore, the Lie algebra $\mathfrak{c}$ of symmetries consists of pairs $(X, B) \in C^\infty(T) \oplus \Omega^2(M)$ such that $\mathcal{L}_X H = dB$. For this reason, it is an extension of the form

$$0 \longrightarrow \Omega^2_{cl}(M) \longrightarrow \mathfrak{c} \longrightarrow C^\infty(T) \longrightarrow 0\ .$$

We have mentioned before that there is a natural adjoint action of a section e_1 of $\mathcal{E}$ on $C^\infty(\mathcal{E})$ by $e_1 \bullet e_2 = [\![e_1, e_2]\!] + d\langle e_1, e_2\rangle$. It follows from the definition of a Courant algebroid that the adjoint action of e_1 is an infinitesimal symmetry of $\mathcal{E}$, i.e.,

$$\pi(e_1)\langle e_2, e_3\rangle = \langle e_1 \bullet e_2, e_3\rangle + \langle e_2, e_1 \bullet e_3\rangle \tag{3.1}$$

$$e_1 \bullet [\![e_2, e_3]\!] = [\![e_1 \bullet e_2, e_3]\!] + [\![e_2, e_1 \bullet e_3]\!], \tag{3.2}$$

and hence we have a map $\mathrm{ad} : C^\infty(\mathcal{E}) \longrightarrow \mathfrak{c}$. However, unlike the usual adjoint action of vector fields on the tangent bundle, ad is neither surjective nor injective; instead, in an exact Courant algebroid, the Lie algebra $\mathfrak{c}$ fits into the following exact sequence:

$$0 \longrightarrow \Omega^1_{cl}(M) \longrightarrow C^\infty(\mathcal{E}) \longrightarrow \mathfrak{c} \longrightarrow H^2(M, \mathbb{R}) \longrightarrow 0\ ,$$

where the map to cohomology can be written as $(X, B) \mapsto [i_X H - B]$ in a given splitting.

1.2. Extended actions. In the same way a Lie group action is described in terms of a morphism between a Lie algebra $\mathfrak{g}$ and the Lie algebra $C^\infty(T)$ we want to describe an action on M together with an action on an exact Courant algebroid $\mathcal{E}$ over M in terms of a map $\Psi : \mathfrak{a} \longrightarrow C^\infty(\mathcal{E})$. Since $C^\infty(\mathcal{E})$ is not a Lie algebra, but has a structure coming from the Courant algebroid structure, it is natural to ask that $\mathfrak{a}$ has a similar kind of structure and Ψ to be a morphism. The structure we want on $\mathfrak{a}$ is that of a Courant algebra, as introduced in [10].

Definition 3.1. A *Courant algebra* over the Lie algebra $\mathfrak{g}$ is a vector space $\mathfrak{a}$ equipped with a skew-symmetric bracket $[\cdot,\cdot] : \mathfrak{a} \times \mathfrak{a} \longrightarrow \mathfrak{a}$, a symmetric bilinear operation $\theta : \mathfrak{a} \times \mathfrak{a} \longrightarrow \mathfrak{a}$, and a map $\pi : \mathfrak{a} \longrightarrow \mathfrak{g}$, which satisfy the following conditions for all $a_1, a_2, a_3 \in \mathfrak{a}$:

c1) $\pi([a_1, a_2]) = [\pi(a_1), \pi(a_2)]$,
c2) $\mathrm{Jac}(a_1, a_2, a_3) = \frac{1}{3}(\theta([a_1, a_2], a_3) + c.p.)$,
c3) $\theta(a_1, a_2) \bullet a_3 = 0$,
c4) $\pi \circ \theta = 0$,
c5) $a_1 \bullet \theta(a_2, a_3) = \theta(a_1 \bullet a_2, a_3) + \theta(a_2, a_1 \bullet a_3)$,

where $\bullet$ denotes the combination

$$a_1 \bullet a_2 = [a_1, a_2] + \theta(a_1, a_2).$$

Definition 3.2. An *exact* Courant algebra is one for which

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{a} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

is an exact sequence and such that $[h_1, h_2] = \theta(h_1, h_2) = 0$ for all $h_i \in \mathfrak{h} = \ker \pi$.

A Courant algebroid $\mathcal{E}$ gives an example of a Courant algebra over $\mathfrak{g} = C^\infty(TM)$, taking $\mathfrak{a} = C^\infty(\mathcal{E})$ and $\theta(e_1, e_2) = d\langle e_1, e_2\rangle$. If $\mathcal{E}$ is exact, then $C^\infty(\mathcal{E})$ is an exact Courant algebra.

For an exact Courant algebra, one obtains immediately an action of $\mathfrak{g}$ on $\mathfrak{h}$: $g \in \mathfrak{g}$ acts on $h \in \mathfrak{h}$ via $g \cdot h = a \bullet h$, for any a such that $\pi(a) = g$. This is well defined, and it determines an action because of the Leibniz property of $\bullet$: for all $a_i \in \mathfrak{a}$,

$$a_1 \bullet (a_2 \bullet a_3) = (a_1 \bullet a_2) \bullet a_3 + a_2 \bullet (a_1 \bullet a_3),$$

which implies that, given $g_i \in \mathfrak{g}$ and $a_i \in \mathfrak{a}$ such that $\pi(a_i) = g_i$,

$$\begin{aligned} g_1 \cdot (g_2 \cdot h) - g_2 \cdot (g_1 \cdot h) &= a_1 \bullet (a_2 \bullet h) - a_2 \bullet (a_1 \bullet h) \\ &= ([a_1, a_2] + \theta(a_1, a_2)) \bullet h \\ &= [a_1, a_2] \bullet h = [g_1, g_2] \cdot h, \end{aligned}$$

for all $h \in \mathfrak{h}$, proving that $\mathfrak{g}$ acts on $\mathfrak{h}$. In fact there is a natural nontrivial exact Courant algebra associated with any $\mathfrak{g}$-module, as we now explain.

Example 3.3 (Demisemidirect product). Let $\mathfrak{g}$ be a Lie algebra acting on the vector space $\mathfrak{h}$. Then $\mathfrak{a} = \mathfrak{g} \oplus \mathfrak{h}$ becomes a Courant algebra over $\mathfrak{g}$ via the bracket

$$[(g_1, h_1), (g_2, h_2)] = ([g_1, g_2], \tfrac{1}{2}(g_1 \cdot h_2 - g_2 \cdot h_1)), \tag{3.3}$$

and the bilinear operation

$$\theta((g_1, h_1), (g_2, h_2)) = (0, \tfrac{1}{2}(g_1 \cdot h_2 + g_2 \cdot h_1)), \tag{3.4}$$

where here $g \cdot h$ denotes the $\mathfrak{g}$-action. This bracket has appeared before in the context of Leibniz algebras [43], where it was called the *demisemidirect* product, due to the factor of $\frac{1}{2}$. Note that in [67], Weinstein studied the case where $\mathfrak{g} = \mathfrak{gl}(V)$ and $\mathfrak{h} = V$, and called it an *omni-Lie algebra* due to the fact that, when $\dim V = n$, any n-dimensional Lie algebra can be embedded inside $\mathfrak{g} \oplus \mathfrak{h}$ as an involutive subspace.

Definition 3.4. A morphism of Courant algebras from $(\mathfrak{a} \xrightarrow{\pi} \mathfrak{g}, [\cdot,\cdot], \theta)$ to $(\mathfrak{a}' \xrightarrow{\pi'} \mathfrak{g}', [\cdot,\cdot]', \theta')$ is a commutative square

$$\begin{array}{ccc} \mathfrak{a} & \xrightarrow{\pi} & \mathfrak{g} \\ \Psi \downarrow & & \downarrow \psi \\ \mathfrak{a}' & \xrightarrow{\pi'} & \mathfrak{g}' \end{array}$$

where ψ is a Lie algebra homomorphism, $\Psi([a_1, a_2]) = [\Psi(a_1), \Psi(a_2)]'$ and $\Psi(\theta(a_1, a_2)) = \theta'(\Psi(a_1), \Psi(a_2))$ for all $a_i \in \mathfrak{a}$. Note that a morphism of Courant algebras induces a chain homomorphism of associated chain complexes $\mathfrak{h} \longrightarrow \mathfrak{a} \xrightarrow{\pi} \mathfrak{g}$.

We now have all we need to define the extension of a G-action to a Courant algebroid $\mathcal{E}$.

Definition 3.5 (Extended action). Let G be a connected Lie group acting on a manifold M with infinitesimal action $\psi : \mathfrak{g} \longrightarrow C^\infty(T)$. An extension of this action to an exact Courant algebroid $\mathcal{E}$ over M is an exact Courant algebra $\mathfrak{a}$ over $\mathfrak{g}$ together with a Courant morphism $\Psi : \mathfrak{a} \longrightarrow C^\infty(\mathcal{E})$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{h} & \longrightarrow & \mathfrak{a} & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & C^\infty(T^*) & \longrightarrow & C^\infty(\mathcal{E}) & \longrightarrow & C^\infty(T) & \longrightarrow & 0 \end{array}$$

which is such that $\mathfrak{h}$ acts trivially, i.e. $(\mathrm{ad} \circ \Psi)(\mathfrak{h}) = 0$, and the induced action of $\mathfrak{g} = \mathfrak{a}/\mathfrak{h}$ on $C^\infty(\mathcal{E})$ integrates to a G-action on the total space of $\mathcal{E}$.

Requiring that $\mathfrak{h}$ acts trivially means that $\mathfrak{h}$ acts via *closed* 1-forms, i.e. $\nu(\mathfrak{h}) \subset \Omega^1_{cl}(M)$. Furthermore the induced $\mathfrak{g}$-action on $\mathcal{E}$ must integrate to a G-action (a priori, one has only the action of the universal cover of G). In order to make this condition more concrete, we observe that since we already know that the $\mathfrak{g}$-action on T integrates to a G-action, one needs only to find a $\mathfrak{g}$-invariant splitting of $\mathcal{E}$ to guarantee that it is a G-bundle, as the splitting $\mathcal{E} = T \oplus T^*$ carries a canonical G-equivariant structure.

Proposition 3.6 (Busztyn–Cavalcanti–Gualtieri [10]). *Let the Lie group G act on the manifold M, and let $\mathfrak{a} \xrightarrow{\pi} \mathfrak{g}$ be an exact Courant algebra with a morphism Ψ to an exact Courant algebroid $\mathcal{E}$ over M such that $\nu(\mathfrak{h}) \subset \Omega^1_{cl}(M)$.*

If $\mathcal{E}$ has a $\mathfrak{g}$-invariant splitting, then the $\mathfrak{g}$-action on $\mathcal{E}$ integrates to an action of G, and hence Ψ is an extended action of G on $\mathcal{E}$. Conversely, if G is compact and Ψ is an extended action, then by averaging splittings one can always find a $\mathfrak{g}$-invariant splitting of $\mathcal{E}$.

The condition that a splitting is $\mathfrak{g}$-invariant can be expressed more concretely as follows. A split exact Courant algebroid is isomorphic to $T \oplus T^*$ with the H-twisted Courant bracket for a closed 3-form H. In this splitting, therefore, for each $\alpha \in \mathfrak{a}$ the section $\Psi(\alpha)$ decomposes as $\Psi(\alpha) = X_\alpha + \xi_\alpha$, and it acts via $(X_\alpha + \xi_\alpha) \bullet (Y + \eta) = [X_\alpha, Y] + \mathcal{L}_{X_\alpha}\eta - i_Y d\xi_\alpha + i_Y i_{X_\alpha} H$, or as a matrix,

$$ad_{\Psi(\alpha)} = \begin{pmatrix} \mathcal{L}_{X_\alpha} & 0 \\ i_{X_\alpha} H - d\xi_\alpha & \mathcal{L}_{X_\alpha} \end{pmatrix}$$

We see immediately from this that the splitting is preserved by this action if and only if for each $a \in \mathfrak{a}$,

$$i_{X_\alpha} H - d\xi_\alpha = 0. \tag{3.5}$$

1.3. Moment maps. Suppose that we have an extended G-action on an exact Courant algebroid as in the previous section, so that we have the map $\nu : \mathfrak{h} \longrightarrow \Omega^1_{cl}(M)$. Because the action is a Courant algebra morphism, this map is $\mathfrak{g}$-equivariant in the sense

$$\nu(g \cdot h) = \mathcal{L}_{\psi(g)} \nu(h). \tag{3.6}$$

Therefore we are led naturally to the definition of a moment map for this extended action, as an equivariant factorization of μ through the smooth functions.

DEFINITION 3.7. A *moment map* for an extended $\mathfrak{g}$-action on an exact Courant algebroid is a $\mathfrak{g}$-equivariant map $\mu : \mathfrak{h} \longrightarrow C^\infty(M, \mathbb{R})$ satisfying $d \circ \mu = \nu$, i.e. such that the following diagram commutes:

$$\begin{array}{ccc} & & \mathfrak{h} \\ & \overset{\mu}{\swarrow} & \downarrow \nu \\ C^\infty(M) & \xrightarrow{d} & C^\infty(T^*M) \end{array}$$

Note that μ may be alternatively viewed as an equivariant map $\mu : M \longrightarrow \mathfrak{h}^*$.

As we see next, the usual notions of symplectic and Hamiltonian actions fit into the framework of extended actions of Courant algebras.

EXAMPLE 3.8 (Symplectic actions). Let G be a Lie group acting on a symplectic manifold (M, ω) preserving the symplectic form, and let $\psi : \mathfrak{g} \longrightarrow C^\infty(T)$ denote the infinitesimal action. We now show that there is a natural extended action of the Courant algebra associated to the adjoint action on the standard Courant algebroid $T \oplus T^*$ with zero twist $H = 0$. As described in Example 3.3, the Courant algebra is described by the sequence

$$0 \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g} \oplus \mathfrak{g} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

and is equipped with the bracket

$$[(g_1, h_1), (g_2, h_2)] = ([g_1, g_2], \tfrac{1}{2}([g_1, h_2] - [g_2, h_1])), \tag{3.7}$$

and the bilinear operation

$$\theta((g_1, h_1), (g_2, h_2)) = (0, \tfrac{1}{2}([g_1, h_2] + [g_2, h_1])). \tag{3.8}$$

We now claim that this Courant algebra acts naturally on $T \oplus T^*$. Let $X_g = \psi(g)$, for $g \in \mathfrak{g}$, denote the symplectic vector fields. Then we define the action $\Psi : \mathfrak{g} \oplus \mathfrak{g} \longrightarrow C^\infty(T \oplus T^*)$ by

$$\Psi(g, h) = X_g + i_{X_h}\omega,$$

where ω is the symplectic form. It is enough to verify that the pairing $\bullet$ is preserved; on the Courant algebra it is simply

$$(g_1, h_1) \bullet (g_2, h_2) = ([g_1, g_2], [g_1, h_2]),$$

whereas in $T \oplus T^*$ we have

$$(X_{g_1} + i_{X_{h_1}}\omega) \bullet (X_{g_2} + i_{X_{h_2}}\omega) = [X_{g_1}, X_{g_2}] + \mathcal{L}_{X_{g_1}} i_{X_{h_2}}\omega = X_{[g_1,g_2]} + i_{X_{[g_1,h_2]}}\omega,$$

showing that Ψ is a Courant morphism.

The question of finding a moment map for this extended action then becomes one of finding an equivariant map $\mu : \mathfrak{g} \longrightarrow C^\infty(M)$ such that

$$d(\mu_g) = i_{X_g}\omega.$$

Hence we recover the usual notion of moment map for a Hamiltonian action on a symplectic manifold.

2. Reduction of Courant algebroids

In the previous section we saw how a G-action on a manifold M could be extended to a Courant algebroid $\mathcal{E}$, making it an equivariant G-bundle in such a way that the Courant structure is preserved by the G-action. In this section we will see that an extended action determines the reduced Courant algebroid in a more subtle way, as not only does it furnish us the G-action but also an equivariant subbundle whose quotient is the reduced Courant algebroid. This reduced Courant algebroid is defined over a *reduced manifold* which is the quotient by G of a submanifold $P \hookrightarrow M$, which is also determined by the extended action.

The starting point is to consider the two natural distributions in $\mathcal{E}$ determined by the extended action, which may be viewed as a bundle map $\Psi : \mathfrak{a} \times M \longrightarrow \mathcal{E}$. The image of this map is a distribution $K \subset \mathcal{E}$, and its orthogonal complement is a second distribution $K^{\perp} \subset \mathcal{E}$. The basic idea behind reduction relies on the following facts:

(1) The distributions K and $K^{\perp}$ are G-invariant and $\pi(K+K^{\perp})$ is a G-invariant distribution Δ on M;
(2) The set of G-invariant sections of $K + K^{\perp}$ is closed under the Courant bracket;
(3) The invariant sections of K, $C^{\infty}(K)^G$, form an ideal of $C^{\infty}(K + K^{\perp})^G$ suggesting that

$$\mathcal{F} = \frac{(K + K^{\perp})^G}{K^G},$$

potentially has a structure of Courant algebroid;

There are two problems with $\mathcal{F}$ defined above: first, in general, it is not a bundle, as its rank can jump even if K has constant rank, and second the anchor $\pi : \mathcal{F} \longrightarrow T(M/G)$ is not surjective, instead $\pi(\mathcal{F}) = \Delta/G$. These two problems cancel each other magically:

(4) If P is a leaf of Δ where G acts freely and properly and where K has constant rank, then the distribution

$$\mathcal{E}_{red} = \left.\frac{(K + K^{\perp})^G}{K^G}\right|_{P/G}$$

defined over $M_{red} = P/G$ is a bundle and the bracket from $\mathcal{E}$ gives rise to a bracket on $\mathcal{E}_{red}$, making it into a Courant algebroid.

The manifold $M_{red} = P/G$ is a *reduced manifold* and the Courant algebroid $\mathcal{E}_{red}$ the *reduced Courant algebroid*. The reduced Courant algebroid is not necessarily *exact*. This will be the case if and only if

(5) The distribution $\widetilde{K} \subset \mathcal{E}$ given by

$$\widetilde{K} = K \cap (K^{\perp} + T^*),$$

is isotropic. This is the case, for example if K is isotropic.

THEOREM 3.9 (Bursztyn, Cavalcanti and Gualtieri [10]). *Let $P \subset M$ be a leaf of $\Delta = \pi(K + K^{\perp})$ on which G acts freely and properly, and over which $\Psi(\mathfrak{h})$ and $\Psi(\mathfrak{a})$ have constant rank. Then*

$$\mathcal{E}_{red} = \left.\frac{(K + K^{\perp})^G}{K^G}\right|_{P/G}$$

is a bundle over $M_{red} = P/G$ which inherits a structure of Courant algebroid from $\mathcal{E}$. If K is isotropic then $\mathcal{E}_{red}$ is an exact Courant algebroid. In general, $\mathcal{E}_{red}$ is exact if and only if along P:

$$\tilde{K} = K \cap (K^{\perp} + T^*) \tag{3.9}$$

is isotropic.

Instead of dwelling on the proof of this theorem, we will see how to concretely use it in some examples. But before we do so, we remark that under the hypothesis of the theorem above, one can describe $\mathcal{E}_{red}$ and M_{red} in alternative ways.

For example, if $\widetilde{K}$, as defined in equation (3.9), is isotropic, one can check that the facts (1) – (5) still hold using $\widetilde{K}$ in place of K and hence we have

$$\mathcal{E}_{red} = \left.\frac{\widetilde{K}^{\perp G}}{\widetilde{K}^{G}}\right|_{P/G}.$$

This way of describing the reduced algebroid as a reduction by an *isotropic* subspace is theoretically useful as we will see later when we try to transport generalized complex structures from $\mathcal{E}$ to $\mathcal{E}_{red}$ (see section 3.1). In practice nearly all our worked examples will use K isotropic.

Another way of describing $\mathcal{E}_{red}$ arises if instead of considering the distributions $K + K^{\perp}$ and K we use $K^{\perp}$ and $K \cap K^{\perp}$. For the point of view of linear algebra, it is obvious that

$$\frac{K + K^{\perp}}{K} = \frac{K^{\perp}}{K \cap K^{\perp}}.$$

However, now the corresponding distribution on M is given by $\Delta_s = \pi(K^{\perp}) \subset \pi(K + K^{\perp}) = \Delta$, or alternatively, $\Delta_s = \mathrm{Ann}(\mathfrak{h})$. A leaf S of Δ_s is not acted on by the whole of G as $\pi(K) = \psi(\mathfrak{g})$ is not necessarily a subspace of $\Delta_s = \pi(K^{\perp})$, nonetheless for any fixed leaf S of Δ_s the set of G orbits passing through S forms a leaf P of Δ, so if we let $G_s \subset G$ be the stabilizer of S we have

$$M_{red} = \frac{S}{G_s} = \frac{P}{G}.$$

And, under the hypothesis of Theorem 3.9, the reduced algebroid can be described as

$$\mathcal{E}_{red} = \left.\frac{K^{\perp G_s}}{(K \cap K^{\perp})^{G_s}}\right|_{S/G_s}.$$

Using this description, it is clear that the pairing on $\mathcal{E}$ gives rise to a pairing on $\mathcal{E}_{red}$.

2.1. Examples. In this section we will provide some examples of Courant algebroid reduction. Since Courant algebroids are often given together with a splitting, we describe the behaviour of splittings under reduction. This is then related to the way in which the Ševera class $[H]$ of an exact Courant algebroid is transported to the reduced space.

Example 3.10. Even a trivial group action may be extended by 1-forms; consider the extended action $\Psi : \mathbb{R} \longrightarrow \mathbb{C}^{\infty}(\mathcal{E})$ given by $\Psi(1) = \xi$ for some closed 1-form ξ. Then $K = \langle \xi \rangle$ and $K^{\perp} = \{v \in \mathcal{E} : \pi(v) \in \mathrm{Ann}(\xi)\}$ which induces the distribution $\Delta = \Delta_s = \mathrm{Ann}(\xi) \subset T$, which is integrable wherever ξ is nonzero. Since the group action is trivial, a reduced manifold M_{red} is simply leaf of the distribution $\Delta = \mathrm{Ann}(\xi)$, so that the conormal bundle of M_{red} is $\mathcal{N}^* = \langle \xi \rangle$. The reduced Courant algebroid is

$$\mathcal{E}_{red} = \left.\frac{K^{\perp}}{K}\right|_{M_{red}} = \left.\frac{\mathcal{N}^{*\perp}}{\mathcal{N}^{*}}\right|_{M_{red}}.$$

So, in this case the reduced Courant algebroid is nothing but the restricted algebroid, as defined in Chapter 1, Section 5.

EXAMPLE 3.11. Let G act freely and properly on M with infinitesimal action $\psi : \mathfrak{g} \longrightarrow C^\infty(T)$, and consider the split Courant algebroid $(T \oplus T^*, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!]_H)$ over M. Using the inclusion $\nabla : T \hookrightarrow T \oplus T^*$, we obtain a natural lift of ψ to a map $\Psi : \mathfrak{g} \longrightarrow C^\infty(T \oplus T^*)$, $\Psi = \nabla \circ \psi$, and hence we can try and see Ψ as an extended action of the Courant algebra $\mathfrak{a} = \mathfrak{g}$ on the split Courant algebroid. One way to ensure that Ψ is an extended action is by requiring that it preserves the splitting as in Proposition 3.6. By Equation (3.5), this is equivalent to the requirement that H is an invariant basic form.

In these conditions, $K = \nabla \circ \psi(\mathfrak{g}) \subset T \subset T \oplus T^*$ and $K^\perp = T \oplus \mathrm{Ann}(K)$ so that $\Delta = \Delta_s = T$. Therefore the only leaf of Δ is M itself and $M_{red} = M/G$. The reduced Courant algebroid is

$$T/K \oplus \mathrm{Ann}K = TM_{red} \oplus T^*M_{red},$$

and the 3-form twisting the Courant bracket on $\mathcal{E}_{red}$ is the push-down of the basic form H.

In the preceding examples, the reduced Courant algebroid inherited a natural splitting; this is not always the case. The next example demonstrates this as well as the phenomenon by which a trivial twisting $[H] = 0$ may give rise to a reduced Courant algebroid with nontrivial curvature.

EXAMPLE 3.12. Assume that S^1 acts freely and properly on M with infinitesimal action $\psi : \mathfrak{s}^1 \longrightarrow C^\infty(T)$, $\psi(1) = \partial_\theta$, and let $\Psi : \mathfrak{s}^1 \longrightarrow C^\infty(\mathcal{E})$ be a trivial extension of this action (i.e., $\mathfrak{h} = \{0\}$) such that $\Psi(\mathfrak{s}^1) = K$ is isotropic. In these conditions $\Delta_s = \mathrm{Ann}(\mathfrak{h}) = T$, so the only leaf of Δ_s is M itself and $M_{red} = M/S^1$.

By Proposition 3.6, we may choose an invariant splitting so that $\mathcal{E} = (T \oplus T^*, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!]_H)$, with $\Psi(1) = \partial_\theta + \xi$ and

$$i_{\partial_\theta} H = d\xi. \tag{3.10}$$

The form ξ is basic as isotropy implies that $\xi(\partial_\theta) = 0$ and condition (3.10) tells us it is invariant:

$$\mathcal{L}_{\partial_\theta}\xi = d\xi(\partial_\theta) + i_{\partial_\theta} d\xi = i_{\partial_\theta} i_{\partial_\theta} H = 0.$$

Equation (3.10) also implies that H is invariant under the circle action, hence, if we choose a connection θ for the circle bundle M, we have

$$H = d\xi \wedge \theta + h,$$

where $d\xi$ and h are basic forms.

Further, the connection θ furnishes an identification

$$TM_{red} \oplus T^*M_{red} \longrightarrow (K^\perp/K)/S^1,$$

$$X + \eta \mapsto X^h + i_{X^h}(\theta \wedge \xi) + q^*\eta + K,$$

where the superscript of X^h denotes horizontal lift. To compute the reduced Courant bracket in this splitting, we use the decomposition $H = d\xi \wedge \theta + h$, and let $F = d\theta$ be the curvature of the connection. Then we obtain the following expression for the curvature 3-form $\tilde{H}$ associated to the splitting of $\mathcal{E}_{red}$:

$$\begin{aligned}\tilde{H}(X,Y,Z) &= 2\langle [\![X^h + i_{X^h}(\theta\wedge\xi), Y^h + i_{Y^h}(\theta\wedge\xi)]\!]_H, Z^h + i_{Z^h}(\theta\wedge\xi)\rangle \\ &= 2\langle [\![X^h, Y^h]\!]_{H+d(\theta\wedge\xi)}, Z^h\rangle \\ &= (h + F\wedge\xi)(X,Y,Z),\end{aligned}$$

i.e.,

$$\tilde{H} = H - d(\xi \wedge \theta) \tag{3.11}$$

Observe that the splitting of the reduced Courant algebroid over M_{red} obtained above is not determined by the original splitting of $\mathcal{E}$ alone, but also the choice of connection θ. Also, if $H = 0$, the curvature of the reduce algebroid is given by $F \wedge \xi$, which may be a nontrivial cohomology class on M_{red}, therefore showing that even if $\mathcal{E}$ has trivial characteristic class $\mathcal{E}_{red}$ can have nonvanishing characteristic class.

EXERCISE 3.13. Assume that $\mathcal{E}$ is equipped with a G-invariant splitting ∇ and *the action* Ψ *is split*, in the sense that there is a splitting s for $\pi : \mathfrak{a} \longrightarrow \mathfrak{g}$ making the diagram commutative:

$$\begin{array}{ccc} \mathfrak{a} & \xleftarrow{\;s\;} & \mathfrak{g} \\ \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \psi} \\ C^\infty(\mathcal{E}) & \xleftarrow[\nabla]{} & C^\infty(T) \end{array} \tag{3.12}$$

Show that in this case $\mathcal{E}_{red}$ is exact and has a natural splitting.

EXAMPLE 3.14. Let (M, ω) be a symplectic manifold and consider the extended G-action $\Psi : \mathfrak{g} \oplus \mathfrak{g} \longrightarrow C^\infty(T \oplus T^*)$ with curvature $H = 0$ defined in Example 3.8.

Let $\psi : \mathfrak{g} \longrightarrow C^\infty(T)$ be the infinitesimal action and $\psi(\mathfrak{g})^\omega$ denote the symplectic orthogonal of the image distribution $\psi(\mathfrak{g})$. Then the extended action has image

$$K = \psi(\mathfrak{g}) \oplus \omega(\psi(\mathfrak{g})),$$

so that the orthogonal complement is

$$K^\perp = \psi(\mathfrak{g})^\omega \oplus \mathrm{Ann}(\psi(\mathfrak{g})).$$

Then the distributions Δ and Δ_s on M are

$$\Delta = \psi(\mathfrak{g})^\omega + \psi(\mathfrak{g}),$$
$$\Delta_s = \psi(\mathfrak{g})^\omega.$$

If the action is Hamiltonian, with moment map $\mu : M \longrightarrow \mathfrak{g}^*$, then Δ_s is the tangent distribution to the level sets $\mu^{-1}(\lambda)$ while Δ is the tangent distribution to the sets $\mu^{-1}(\mathcal{O}_\lambda)$, for $\mathcal{O}_\lambda$ a coadjoint orbit containing λ. Therefore we see that the reduced Courant algebroid is simply $TM_{red} \oplus T^*M_{red}$ with $H = 0$, for the usual symplectic reduced space $M_{red} = \mu^{-1}(\mathcal{O}_\lambda)/G = \mu^{-1}(\lambda)/G_\lambda$.

Finally, we present an example of a reduced Courant algebroid which is not exact.

EXAMPLE 3.15. Let $\Psi : \mathfrak{s}^1 \longrightarrow \mathcal{E}$ be a trivially extended S^1 action (i.e., $\mathfrak{h} = 0$) which is not isotropic, i.e. $\langle \Psi(1), \Psi(1) \rangle \neq 0$. Hence the reduced manifold for this action is just M/S^1 and the reduced algebroid is $\mathcal{E}_{red} = (K^\perp/(K \cap K^\perp))/S^1$. However, $K \cap K^\perp = \{0\}$ and so $\mathcal{E}_{red}$ is odd dimensional; hence it is not an exact Courant algebroid.

3. Reduction of Dirac and generalized complex structures

In this section we study how to transport Dirac structures invariant under an extended action from $\mathcal{E}$ to $\mathcal{E}_{red}$.

3.1. Odd symplectic category. Let $\mathcal{E}, \mathcal{F}$ be real vector spaces with nondegenerate, symmetric bilinear forms of split signature. Linear Dirac structures on these are simply maximal isotropic subspaces, and they may be transported between $\mathcal{E}$ and $\mathcal{F}$ if there is a morphism between them in the sense of the odd symplectic category [11],[66]. Here "odd" indicates a parity reversal, whereby the symmetric inner product is viewed as an odd symplectic form and maximal isotropic subspaces are odd Lagrangians. Therefore, a morphism $Q : \mathcal{E} \longrightarrow \mathcal{F}$ is a maximal isotropic subspace

$$Q \subset \overline{\mathcal{E}} \times \mathcal{F},$$

where $\overline{\mathcal{E}}$ is obtained from $\mathcal{E}$ by multiplying the inner product by -1. This means that a Dirac structure $D \subset \mathcal{E}$ may itself be viewed as a morphism $D : \{0\} \longrightarrow \mathcal{E}$, which may then be composed as a relation with Q to yield $Q \circ D : \{0\} \longrightarrow \mathcal{F}$, a Dirac structure in $\mathcal{F}$. In this way, we obtain a map of linear Dirac structures:

$$Q : \mathrm{Dir}(\mathcal{E}) \to \mathrm{Dir}(\mathcal{F}).$$

An isotropic subspace $\widetilde{K} \subset \mathcal{E}$ determines not only another split-signature space $\widetilde{K}^{\perp}/\widetilde{K}$, but also a morphism

$$\varphi_{\widetilde{K}} : \mathcal{E} \longrightarrow \widetilde{K}^{\perp}/\widetilde{K},$$

given by the following maximal isotropic:

$$\varphi_{\widetilde{K}} = \left\{(x, [x]) \in \overline{\mathcal{E}} \times \widetilde{K}^{\perp}/\widetilde{K} \ : \ x \in \widetilde{K}^{\perp}\right\}$$

Given a Dirac structure $D \subset \mathcal{E}$, one obtains by composition with $\varphi_{\widetilde{K}}$ the Dirac structure

$$\varphi_{\widetilde{K}} \circ D = \frac{D \cap \widetilde{K}^{\perp} + \widetilde{K}}{\widetilde{K}} \subset \widetilde{K}^{\perp}/\widetilde{K}.$$

If one recalls that the reduced Courant algebroid is given by $\mathcal{E}_{red} = \widetilde{K}^{\perp}/\widetilde{K}$, with $\widetilde{K}$ isotropic, as defined in equation (3.9), this point of view shows that one can transport Dirac structures from $\mathcal{E}$ to $\mathcal{E}_{red}$. This is what we do next.

3.2. Reduction procedure. Let $\Psi : \mathfrak{a} \longrightarrow C^{\infty}(\mathcal{E})$ be an extended action for which the reduced Courant algebroid over a reduced manifold M_{red} is exact. According to the previous section, if a Dirac structure D is preserved by Ψ, i.e., $\Psi(\mathfrak{a}) \bullet C^{\infty}(D) \subset C^{\infty}(D)$, then we have a natural candidate for a Dirac structure on the reduced algebroid:

$$D_{red} = \left.\frac{(D \cap \widetilde{K}^{\perp} + \widetilde{K})^G}{\widetilde{K}^G}\right|_{M_{red}} \subset \mathcal{E}_{red}. \tag{3.13}$$

The distribution D_{red} is certainly maximal isotropic, however it is not necessarily smooth. One case when D_{red} is smooth is if $D \cap \widetilde{K}^{\perp}$ is a bundle, or equivalently if $D \cap \widetilde{K}$ is a bundle, as in this case D_{red} is just the smooth quotient of two bundles. If D_{red} is smooth, its sections are closed under the Courant bracket on $\mathcal{E}_{red}$, since D is closed under the bracket on $\mathcal{E}$. These are the main arguments needed to prove the following theorem.

THEOREM 3.16 (Bursztyn–Cavalcanti–Gualtieri [10]). *Let $\rho : \mathfrak{a} \longrightarrow C^{\infty}(\mathcal{E})$ be an extended action preserving a Dirac structure $D \subset \mathcal{E}$, and such that $\mathcal{E}_{red}$ is exact over $M_{red} = P/G$, i.e., the subbundle $\widetilde{K} = K \cap (K^{\perp} + T^*)$ is isotropic along P. If $D \cap \widetilde{K}$ is a smooth bundle, then*

$$D_{red} = \left.\frac{(D \cap \widetilde{K}^{\perp} + \widetilde{K})^G}{(\widetilde{K})^G}\right|_{M_{red}} \subset \mathcal{E}_{red}. \tag{3.14}$$

defines a Dirac structure on $\mathcal{E}_{red}$.

The reduction of Dirac structures works in the same way for *complex* Dirac structures, provided one replaces K by its complexification.

4. Reduction of generalized complex structures

4.1. Reduction procedure. As we know, a generalized complex structure is a complex Dirac structure $L \subset \mathcal{E}_{\mathbb{C}}$ satisfying $L \cap \overline{L} = \{0\}$. So, if an extended action Ψ preserves a generalized complex structure $\mathcal{J}$ with i-eigenspace L we can try and transport L, as a Dirac structure, to the reduced Courant algebroid over a reduced manifold:

$$L_{red} = \left. \frac{(L \cap \widetilde{K}_{\mathbb{C}}^{\perp} + \widetilde{K}_{\mathbb{C}})^G}{\widetilde{K}_{\mathbb{C}}^G} \right|_{M_{red}} \tag{3.15}$$

This reduced Dirac structure L_{red} is not necessarily a generalized complex structure as it may not satisfy $L_{red} \cap \overline{L_{red}} = \{0\}$. Whenever it does it determines a generalized complex structure.

The condition $L_{red} \cap \overline{L_{red}} = \{0\}$ is a simple linear algebraic condition which can be rephrased in the following way (compare with the condition for a manifold to be generalized complex).

LEMMA 3.17. *The distribution L_{red} satisfies $L_{red} \cap \overline{L}_{red} = \{0\}$ if and only if*

$$\mathcal{J}\widetilde{K} \cap \widetilde{K}^{\perp} \subset \widetilde{K} \text{ over } P. \tag{3.16}$$

So, this lemma tells us precisely when a generalized complex structure can be reduced. However (3.16) may be hard to check in real examples, so we settle with more meaningful conditions in the following theorems.

THEOREM 3.18 (Bursztyn–Cavalcanti–Gualtieri [10]). *Let Ψ be an extended G-action on the exact Courant algebroid $\mathcal{E}$. Let P be a leaf of the distribution Δ where G acts freely and properly with exact quotient $\mathcal{E}_{red}$. If the action preserves a generalized complex structure $\mathcal{J}$ on $\mathcal{E}$ and $\mathcal{J}K = K$ over P then $\mathcal{J}$ reduces to $\mathcal{E}_{red}$.*

PROOF. We start with a general observation: given a complex Dirac structure D invariant under an extended action, let us consider in the reduced Courant algebroid the isotropic distribution

$$D' := \left. \frac{(D \cap K_{\mathbb{C}}^{\perp} + K_{\mathbb{C}} \cap K_{\mathbb{C}}^{\perp})^G}{(K_{\mathbb{C}} \cap K_{\mathbb{C}}^{\perp})^G} \right|_{M_{red}}.$$

One can check that $D' \subset D_{red}$, so, if D' is maximal isotropic, then it agrees with D_{red}.

In our case, we have

$$L' = \frac{L \cap K_{\mathbb{C}}^{\perp} + K_{\mathbb{C}} \cap K_{\mathbb{C}}^{\perp}}{K_{\mathbb{C}} \cap K_{\mathbb{C}}^{\perp}}. \tag{3.17}$$

Since $\mathcal{J}K^{\perp} = K^{\perp}$, it follows that $K_{\mathbb{C}}^{\perp} = L \cap K_{\mathbb{C}}^{\perp} + \overline{L} \cap K_{\mathbb{C}}^{\perp}$. Hence

$$L' + \overline{L'} = \frac{L \cap K_{\mathbb{C}}^{\perp} + \overline{L} \cap K_{\mathbb{C}}^{\perp} + K_{\mathbb{C}} \cap K_{\mathbb{C}}^{\perp}}{K_{\mathbb{C}} \cap K_{\mathbb{C}}^{\perp}} = \frac{K_{\mathbb{C}}^{\perp}}{K_{\mathbb{C}} \cap K_{\mathbb{C}}^{\perp}} = \mathcal{E}_{red} \otimes \mathbb{C},$$

showing that L' is maximal and therefore agrees with L_{red}. The argument above also shows that $L \cap K_{\mathbb{C}}^{\perp}$ is a bundle and, since $K \cap K^{\perp}$ is a bundle over P, this implies that L' as defined in (3.17) is smooth.

Finally, in order to conclude that L_{red} induces a generalized complex structure we must check that condition (3.16) in Lemma 3.17 holds:

$$\mathcal{J}\widetilde{K} \cap \widetilde{K}^{\perp} = K \cap (K^{\perp} + \mathcal{J}T^*) \cap (K^{\perp} + K \cap T^*) \subset K \cap (K^{\perp} + K \cap T^*) = \widetilde{K},$$

as desired. $\square$

COROLLARY 3.19. *If the hypotheses of the previous theorem hold and the extended action has a moment map $\mu : M \to \mathfrak{h}^*$, then the reduced Courant algebroid over $\mu^{-1}(O_\lambda)/G$ has a reduced generalized complex structure.*

It is easy to check that the reduced generalized complex structure $\mathcal{J}^{red}$ constructed in Theorem 3.18 is characterized by the following commutative diagram:

$$\begin{array}{ccc} K^\perp & \xrightarrow{\mathcal{J}} & K^\perp \\ \downarrow & & \downarrow \\ \frac{K^\perp}{K\cap K^\perp} & \xrightarrow{\mathcal{J}^{red}} & \frac{K^\perp}{K\cap K^\perp} \end{array} \tag{3.18}$$

Theorem 3.18 uses the compatibility condition $\mathcal{J}K = K$ for the reduction of $\mathcal{J}$. We now observe that the reduction procedure also works in an extreme opposite situation.

THEOREM 3.20 (Bursztyn–Cavalcanti–Gualtieri [10]). *Consider an extended G-action Ψ on an exact Courant algebroid $\mathcal{E}$. Let P be a leaf of the distribution Δ where G acts freely and properly. If K is isotropic over P and $\langle\cdot,\cdot\rangle : K \times \mathcal{J}K \longrightarrow \mathbb{R}$ is nondegenerate then $\mathcal{J}$ reduces.*

PROOF. As K is isotropic over P, the reduced Courant algebroid is exact and $\widetilde{K} = K$. The nondegeneracy assumption implies that $\mathcal{J}K \cap K^\perp = \{0\}$, and it follows that $L \cap K^\perp_{\mathbb{C}}$ is a bundle and the Dirac reduction of L is smooth. Finally, (3.16) holds trivially. $\square$

4.2. Symplectic structures. We now present two examples of reduction obtained from a symplectic manifold (M,ω): First, we show that ordinary symplectic reduction is a particular case of our construction; the second example illustrates how one can obtain a type 1 generalized complex structure as the reduction of an ordinary symplectic structure. In both examples, the initial Courant algebroid is just $T \oplus T^*$ with $H = 0$.

EXAMPLE 3.21 (Ordinary symplectic reduction). Let (M,ω) be a symplectic manifold, and let $\mathcal{J}_\omega$ be the generalized complex structure associated with ω. Following Example 3.8 and keeping the same notation, consider a symplectic G-action on M, regarded as an extended action. It is clear that $\mathcal{J}_\omega K = K$, so we are in the situation of Theorem 3.18.

Following Example 3.14, let S be a leaf of the distribution $\Delta_s = \psi(\mathfrak{g})^\omega$. Since K splits as $K_T \oplus K_{T^*}$, the reduction procedure of Theorem 3.16 in this case amounts to the usual pull-back of ω to S, followed by a Dirac push-forward to $S/G_s = M_{red}$. If the symplectic action admits a moment map $\mu : M \to \mathfrak{g}^*$, then the leaves of Δ_s are level sets $\mu^{-1}(\lambda)$, and Theorem 3.18 simply reproduces the usual Marsden-Weinstein quotient $\mu^{-1}(\lambda)/G_\lambda$.

Next, we show that by allowing the projection $\pi : K \longrightarrow T$ to be injective, one can reduce a symplectic structure (type 0) to a generalized complex structure with nonzero type.

EXAMPLE 3.22. Assume that X and Y are linearly independent symplectic vector fields generating a T^2-action on M. Assume further that $\omega(X,Y) = 0$ and consider the extended T^2-action on $T \oplus T^*$ defined by

$$\Psi(\alpha_1) = X + \omega(Y); \qquad \Psi(\alpha_2) = -Y + \omega(X),$$

where $\{\alpha_1, \alpha_2\}$ is the standard basis of $\mathfrak{t}^2 = \mathbb{R}^2$. It follows from $\omega(X,Y) = 0$ and the fact that the vector fields X and Y are symplectic that this is an extended action with isotropic K.

Since $\mathcal{J}_\omega K = K$, Theorem 3.18 implies that the quotient M/T^2 has an induced generalized complex structure. Note that

$$L \cap K_{\mathbb{C}}^{\perp} = \{Z - i\omega(Z) : Z \in \mathrm{Ann}(\omega(X) \wedge \omega(Y))\},$$

and it is simple to check that $X - i\omega(X) \in L \cap K_{\mathbb{C}}^{\perp}$ represents a nonzero element in $L_{red} = ((L \cap K_{\mathbb{C}}^{\perp} + K_{\mathbb{C}})/K_{\mathbb{C}})/G$ which lies in the kernel of the projection $L_{red} \to T(M/T^2)$. As a result, this reduced generalized complex structure has type 1.

One can find concrete examples illustrating this construction by considering symplectic manifolds which are T^2-principal bundles with lagrangian fibres, such as $T^2 \times T^2$, or the Kodaira–Thurston manifold. In these cases, the reduced generalized complex structure determines a complex structure on the base 2-torus.

4.3. Complex structures. In this section we show how a complex manifold (M, I) may have different types of generalized complex reductions.

EXAMPLE 3.23 (Holomorphic quotient). Let G be a complex Lie group acting holomorphically on (M, I), so that the induced infinitesimal map $\Psi : \mathfrak{g} \longrightarrow C^\infty(T)$ is a holomorphic map. Since $K = \Psi(\mathfrak{g}) \subset T$, it is clear that K is isotropic and the reduced Courant algebroid is exact. Furthermore, as Ψ is holomorphic, it follows that $\mathcal{J}_I K = K$. By Theorem 3.18, the complex structure descends to a generalized complex structure in the reduced manifold M/G. The reduced generalized complex structure is nothing but the quotient complex structure obtained from holomorphic quotient.

EXERCISE 3.24. Let (M, I) be a complex manifold, $\Psi : \mathfrak{a} \longrightarrow C^\infty(T \oplus T^*)$ be an extended action and $K = \Psi(\mathfrak{a})$. If $\mathcal{J}_I$ is the generalized complex structure induced by I, show that if $\mathcal{J}_I K = K$, then reduction of $\mathcal{J}_I$ is of complex type.

EXAMPLE 3.25. Consider $\mathbb{C}^2$ equipped with its standard holomorphic coordinates $(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2)$, and let Ψ be the extended $\mathbb{R}^2$-action on $\mathbb{C}^2$ defined by

$$\Psi(\alpha_1) = \partial_{x_1} + dx_2, \qquad \Psi(\alpha_2) = \partial_{y_2} + dy_1,$$

where $\{\alpha_1, \alpha_2\}$ is the standard basis for $\mathbb{R}^2$. Note that $K = \Psi(\mathbb{R}^2)$ is isotropic, so the reduced Courant algebroid over $\mathbb{C}/\mathbb{R}^2$ is exact. Since the natural pairing between K and $\mathcal{J}_I K$ is nondegenerate, Proposition 3.20 implies that one can reduce $\mathcal{J}_I$ by this extended action. In this example, one computes

$$K_{\mathbb{C}}^{\perp} \cap L = \mathrm{span}\{\partial_{x_1} - i\partial_{x_2} - dy_1 + idx_1, \partial_{y_1} - i\partial_{y_2} - dy_2 + idx_2\}$$

and $K_{\mathbb{C}}^{\perp} \cap L \cap K_{\mathbb{C}} = \{0\}$. As a result, $L_{red} \cong K_{\mathbb{C}}^{\perp} \cap L$. So $\pi : L_{red} \longrightarrow \mathbb{C}^2/\mathbb{R}^2$ is an injection, and $\mathcal{J}^{red}$ has zero type, i.e., it is of symplectic type.

CHAPTER 4

T-duality with NS-flux and generalized complex structures

T-duality in physics is a symmetry which relates IIA and IIB string theory and T-duality transformations act on spaces in which at least one direction has the topology of a circle. In this chapter, we consider a mathematical version of T-duality introduced by Bouwknegt, Evslin and Mathai for principal circle bundles with nonzero twisting 3-form H [**6, 7**].

From the point of view adopted in these notes, the relation between two T-dual spaces can be best described using the language of Courant algebroids. Two T-dual spaces are principal circle bundles E and $\tilde{E}$ over a common base M and hence the space of invariant sections of $(TE \oplus T^*E, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!]_H)$ and $(T\tilde{E} \oplus T^*\tilde{E}, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!]_{\tilde{H}})$ can both be identified with nonexact Courant algebroids over M. With that said, the T-duality condition is nothing but requiring that these Courant algebroids are isomorphic:

$$\begin{array}{ccc} ((TE \oplus T^*E)^{S^1}, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!]_H) & \xrightarrow{\cong} & ((T\tilde{E} \oplus T^*\tilde{E})^{S^1}, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!]_{\tilde{H}}) \\ & {\scriptstyle\pi} \searrow \quad \swarrow {\scriptstyle\tilde{\pi}} & \\ & TM & \end{array}$$

Therefore any invariant structure on $TE \oplus T^*E$ can be transported to an invariant structure on $T\tilde{E} \oplus T^*\tilde{E}$. This is particularly interesting since E and $\tilde{E}$ have different topologies, in general. Further, when using the map above to transport generalized complex structures, the type changes by ± 1. This means that even if E is endowed with a symplectic or complex structure, the corresponding structure on $\tilde{E}$ will not be either complex or symplectic, but just generalized complex.

Another structure which can be transported by the isomorphism above is a generalized metric invariant under the circle action. Since a generalized metric can be described in terms of a metric g on E and a 2-form b, studying the way the generalized metric transforms is equivalent to studying the transformations rules for g and b. As we will, these rules are nothing but the Buscher rules [**12, 13**], which are obtained in a geometrical way, using this point of view.

A final interesting point is that given two principal circle bundles E and $\tilde{E}$ over M, we can always for the fiber product, or *correspondence space*, $E \times_M \tilde{E}$, which we can endow with, say, the 3-form H from E. The condition that E and $\tilde{E}$ are T-dual can then be stated by saying that $(TE \oplus T^*E, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!]_H)$ and $(T\tilde{E} \oplus T^*\tilde{E}, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!]_{\tilde{H}})$ are different reductions of the Courant algebroid over the correspondence space with curvature H:

$$\begin{array}{ccc} & (E \times_M \tilde{E}, p^*H) & \\ {\scriptstyle /\frac{\partial}{\partial\tilde{\theta}}} \swarrow & & \searrow {\scriptstyle /\frac{\partial}{\partial\theta}-\tilde{\theta}} \\ (E, H) & & (\tilde{E}, \tilde{H}) \end{array}$$

This chapter is heavily based ona collaborative work with Gualtieri [20] and organized in the following way. In the first section we introduce T-duality for principal circle bundles as presented in [6] and prove the main result in that paper stating that T-dual manifolds have isomorphic twisted cohomologies. In Section 2 we prove that T-duality can be expressed as an isomorphism of Courant algebroids and hence Dirac and generalized complex structures can be transported via T-duality as well as a generalized metric. In the last section we show that T-duality can be seen in the light of reduction of Courant algebroids.

1. T-duality with NS-flux

In this section we review the definition of T-duality for principal circle bundles as expressed by Bouwknegt, Evslin and Mathai [6] and some of their results regarding T-dual spaces.

Given a principal circle bundle $E \stackrel{\pi}{\longrightarrow} M$, with an invariant closed integral 3-form $H \in \Omega^3(E)$ and a connection θ, we can always write $H = \tilde{F} \wedge \theta + h$, where $\tilde{F}$ and h are basic forms. We denote by $F = d\theta$ the curvature of θ. Bouwknegt *et al* define the *T-dual space* to be another principal circle bundle $\tilde{E}$ over M with a connection $\tilde{\theta}$ whose curvature is the pushforward of H to M, $d\tilde{\theta} = \pi_! H = \tilde{F}$, (and this determines $\tilde{E}$) with associated 3-form $\tilde{H} = F \wedge \tilde{\theta} + h$.

$$\begin{array}{ccc} (E,H) & & (\tilde{E},\tilde{H}) \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \tilde{\pi}} & \\ & M & \end{array}$$

In this setting another space which is important is the *correspondence space* which is the fiber product of the bundles E and $\tilde{E}$. The correspondence space projects over each of the T-dual spaces and has a natural 3-form on it: $H - \tilde{H} = dA$, where $A = -\theta \wedge \tilde{\theta}$.

$$\begin{array}{ccccc} & & (E \times_M \tilde{E}, p^*H - \tilde{p}^*\tilde{H}) & & \\ & {\scriptstyle p}\swarrow & & \searrow{\scriptstyle \tilde{p}} & \\ (E,H) & & & & (\tilde{E},\tilde{H}) \\ & {\scriptstyle \pi}\searrow & & \swarrow{\scriptstyle \tilde{\pi}} & \\ & & M & & \end{array} \tag{4.1}$$

We remark that although the space $\tilde{E}$ is well defined from the data (E, H, θ), the same is not true about $[\tilde{H}]$, which is well defined up to an element in the ideal $[F] \wedge H^1(\tilde{E}) \subset H^3(\tilde{E})$.

Example 4.1. The Hopf fibration makes the 3-sphere, S^3, a principal S^1 bundle over S^2. The curvature of this bundle is a volume form of S^2, σ. So S^3 equipped with the zero 3-form is T-dual to $(S^2 \times S^1, \sigma \wedge \theta)$. On the other hand, still considering the Hopf fibration, the 3-sphere endowed with the 3-form $H = \theta \wedge \sigma$ is self T-dual.

Example 4.2 (Lie Groups). Let (G, H) be a semi-simple Lie group with 3-form $H(X, Y, Z) = K([X, Y], Z)$, the Cartan form generating $H^3(G, \mathbb{Z})$, where K is the Killing form.

With a choice of an $S^1 < G$, we can think of G as a principal circle bundle. For $X = \partial/\partial\theta \in \mathfrak{g}$ tangent to S^1 and of length -1 according to the Killing form, a natural connection on G is given by $-K(X, \cdot)$. The curvature of this connection is given by

$$d(-K(X, \cdot))(Y, Z) = K(X, [Y, Z]) = H(X, Y, Z),$$

hence c_1 and $\tilde{c}_1$ are related by

$$c_1 = H(X, \cdot, \cdot) = X\lfloor H = \tilde{c}_1.$$

Which shows that semi-simple Lie groups with the Cartan 3-form are self T-dual.

If E and $\tilde{E}$ are T-dual spaces, we can define a map of invariant forms $\tau : \Omega^\bullet_{S^1}(E) \longrightarrow \Omega^\bullet_{S^1}(\tilde{E})$ by $\tau(\rho) = \tilde{p}_* e^A p^* \rho$, where $H - \tilde{H} = dA$, or more explicitly

$$\tau(\rho) = \frac{1}{2\pi}\int_{S_1} e^A \rho. \tag{4.2}$$

If we decompose $\rho = \rho = \theta\rho_1 + \rho_0$, with ρ_i pull back from M, then one can check that τ is given by

$$\tau(\theta\rho_1 + \rho_0) = \rho_1 - \tilde{\theta}\rho_0. \tag{4.3}$$

It is clear from (4.3) and that if we T-dualize twice and choose $\theta = \tilde{\tilde{\theta}}$ for the second T-duality, we get (E, H) back and $\tau^2 = -\mathrm{Id}$.

The main theorem from [6] concerning us is:

THEOREM 4.3 (Bouwknegt, Evslin and Mathai [6]). *The map*

$$\tau : (\Omega^\bullet_{S^1}(E), d_H) \longrightarrow (\Omega^\bullet_{S^1}(\tilde{E}), -d_{\tilde{H}})$$

is an isomorphism of differential complexes.

PROOF. Given that τ has an inverse, obtained by T-dualizing again, we only have to check that τ preserves the differentials, i.e., $-d_{\tilde{H}} \circ \tau = \tau \circ d_H$. To obtain this relation we use equation (4.2):

$$\begin{aligned} -d_{\tilde{H}}\tau(\rho) &= \frac{1}{2\pi}\int_{S^1} d_{\tilde{H}}(e^{-\theta\tilde{\theta}}\rho) \\ &= \frac{1}{2\pi}\int_{S^1} (H - \tilde{H})e^{-\theta\tilde{\theta}}\rho + e^{-\theta\tilde{\theta}}d\rho + \tilde{H}e^{-\theta\tilde{\theta}}\rho \\ &= \frac{1}{2\pi}\int_{S^1} He^{-\theta\tilde{\theta}}\rho + e^{-\theta\tilde{\theta}}d\rho \\ &= \tau(d_H\rho) \end{aligned}$$

□

Remark: If one considers τ as a map of the complexes of differential forms (no invariance required), it will not be invertible. Nonetheless, every d_H-cohomology class has an invariant representative, hence τ is a quasi-isomorphism.

1.1. Principal torus bundles. The construction of the T-dual described above can also be used to construct T-duals of principal torus bundles. What one has to do is just to split the torus into a product of circles and use the previous construction with "a circle at a time" (see [7]). However, this is only possible if

$$H(X, Y, \cdot) = 0 \qquad \text{if } X, Y \text{ are vertical.} \tag{4.4}$$

Mathai and Rosenberg studied the case when (4.4) fails in [52]. There they propose that the T-dual is a bundle of noncommutative tori.

Another important difference between the circle bundle case and the torus bundle case is that in the former $\tilde{E}$ is determined by E and $[H]$ while in the latter this is not true [9]. One can see why this is the case if we recall that for circle bundles, even though $\tilde{E}$ is well defined from E and $[H]$, the same is not true about $[\tilde{H}]$. So if one wants to T-dualize again, along a different circle direction, the topology of the next T-dual, $\tilde{\tilde{E}}$, will depend on $\tilde{E}$ and $[\tilde{H}]$ and hence is not well defined.

EXAMPLE 4.4. A simple example to illustrate this fact is given by a 2-torus bundle with nonvanishing Chern classes but with $[H] = 0$. Taking the 3-form $H = 0$ as a representative, a T-dual will be a flat torus bundle. Taking $H = d(\theta_1 \wedge \theta_2) = c_1\theta_2 - c_2\theta_1$ as a representative of the zero cohomology class, a T-dual will be the torus bundle with (nonzero) Chern classes $[c_1]$ and $[-c_2]$.

This fact leads us to define T-duality as a relation.

DEFINITION 4.5. Let (E, H) and $(\tilde{E}, \tilde{H})$ be principal n-torus bundles over a base M. We say that E and $\tilde{E}$ are *T-dual* if on the correspondence space $E \times_M \tilde{E}$ we have $H - \tilde{H} = dA$, where

$$[A]|_{T^n \times \widetilde{T^n}} = \sum \theta_i \wedge \tilde{\theta}_i \in H^2(T^n \times \widetilde{T^n})/H^2(T^n) \times H^2(\widetilde{T^n})$$

Clearly Theorem 4.3 still hods in this case with the same proof.

2. T-duality as a map of Courant algebroids

In this section we state our main result for T-dual circle bundles. The case of torus bundles can be dealt with similar techniques. We have seen that given two T-dual circle bundles we have a map of differential algebras τ which is an isomorphism of the invariant differential exterior algebras. Now we introduce a map on invariant sections of generalized tangent spaces:

$$\varphi : T_{S^1}E \oplus T^*_{S^1}E \longrightarrow T_{S^1}\tilde{E} \oplus T^*_{S^1}\tilde{E}.$$

Any invariant section of $TE \oplus TE^*$ can be written as $X + f\partial/\partial\theta + \xi + g\theta$, where X is a horizontal vector and ξ is pull-back from the base. We define φ by:

$$\varphi(X + f\frac{\partial}{\partial\theta} + \xi + g\theta) = -X - g\frac{\partial}{\partial\tilde{\theta}} - \xi - f\tilde{\theta}. \tag{4.5}$$

The relevance of this map comes from our main result.

THEOREM 4.6 (Cavalcanti and Gualtieri [20]). *The map φ defined in (4.5) is an orthogonal isomorphism of Courant algebroids and relates to τ acting on invariant forms via*

$$\tau(V \cdot \rho) = \varphi(V) \cdot \tau(\rho). \tag{4.6}$$

PROOF. It is obvious from equation (4.5) that φ is orthogonal with respect to the natural pairing. To prove equation (4.6) we split an invariant form $\rho = \theta\rho_1 + \rho_0$ and $V = X + f\partial/\partial\theta + \xi + g\theta$. Then a direct computation using equation (4.3) gives:

$$\begin{aligned}\tau(V \cdot \rho) &= \tau(\theta(-X\lfloor\rho_1 - \xi\rho_1 + g\rho_0) + X\lfloor\rho_0 + f\rho_1 + \xi\rho_0)\\ &= -X\lfloor\rho_1 - \xi\rho_1 + g\rho_0 + \tilde{\theta}(-X\lfloor\rho_0 - f\rho_1 - \xi\rho_0).\end{aligned}$$

While

$$\begin{aligned}\varphi(V) \cdot \tau(\rho) &= (-X - g\partial/\partial\theta - \xi - f\theta)(\rho_1 - \tilde{\theta}\rho_0)\\ &= -X\lfloor\rho_1 - \xi\rho_1 + g\rho_0 + \tilde{\theta}(-X\lfloor\rho_0 - \xi\rho_0 - f\rho_1).\end{aligned}$$

Finally, we have established that under the isomorphisms φ of Clifford algebras and τ of Clifford modules, d_H corresponds to $-d_{\tilde{H}}$, hence the induced brackets (according to equation 1.9) are the same. □

Remark: As E is the total space of a circle bundle, its invariant tangent bundle sits in the Atiyah sequence:

$$0 \longrightarrow 1 = T_1 S^1 \longrightarrow T_{S^1}E \longrightarrow TM \longrightarrow 0$$

or, taking duals,

$$0 \longrightarrow T^*M \longrightarrow T^*_{S^1}E \longrightarrow T^*_1 S^1 = 1^* \longrightarrow 0.$$

The choice of a connection on E induces a splitting of the sequences above and an isomorphism

$$T_{S^1}E \oplus T^*_{S^1}E \cong TM \oplus T^*M \oplus 1 \oplus 1^*,$$

The argument also applies to $\tilde{E}$:

$$T_{S^1}\tilde{E} \oplus T^*_{S^1}\tilde{E} \cong TM \oplus T^*M \oplus 1 \oplus 1^*.$$

The map φ can be seen in this light as the permutation of the terms 1 and 1^*. This is Ben-Bassat's starting point for the study of mirror symmetry and generalized complex structures in [4].

Since the Courant algebroids $(T_{S^1}E \oplus T^*_{S^1}E, [\![,]\!]_H)$ and $(T_{S^1}\tilde{E} \oplus T^*_{S^1}\tilde{E}, [\![,]\!]_{\tilde{H}})$ are isomorphic, according to Theorem 4.6, we see that any invariant structure on $(TE \oplus T^*E, [\![,]\!]_H)$ defined in terms of the Courant bracket and natural pairing correspond to a similar structure on $(T\tilde{E} \oplus T^*\tilde{E}, [\![,]\!]_{\tilde{H}})$.

Theorem 4.7 (Cavalcanti and Gualtieri [20]). *Any invariant Dirac, generalized complex, generalized Kähler on $(TE \oplus T^*E, [\![,]\!]_H)$ is transformed into a similar one via φ.*

Exercise 4.8. What happens with the generalized Kähler structure on Lie groups described in Example 2.10 under T-duality?

The decomposition of $\wedge^\bullet T^*_{\mathbb{C}}M$ into subbundles U^k is also preserved from T-duality.

Corollary 4.9. *If two generalized complex manifolds $(E, \mathcal{J}_1)$ and $(\tilde{E}, \mathcal{J}_2)$ correspond via T-duality, then $\tau(\mathcal{U}^k_E) = \mathcal{U}^k_{\tilde{E}}$ and also*

$$\tau(\partial_E \psi) = -\partial_{\tilde{E}}\tau(\psi) \qquad \tau(\overline{\partial}_E \psi) = -\overline{\partial}_{\tilde{E}}\tau(\psi).$$

Proof. The T-dual generalized complex structure in $\tilde{E}$ is determined by $\tilde{L} = \varphi(L)$, where L is the $+i$-eigenspace of the generalized complex structure on E. Since φ is real, $\overline{\tilde{L}} = \varphi(\overline{L})$, and hence

$$\mathcal{U}^{n-k}_{\tilde{E}} = \Omega^k(\overline{\tilde{L}}) \cdot \tau(\rho) = \tau(\Omega^k(\overline{L}) \cdot \rho) = \tau(\mathcal{U}^k_E).$$

Finally, if $\alpha \in \mathcal{U}^k$, then

$$\partial_{\tilde{E}}\tau(\alpha) - \overline{\partial}_{\tilde{E}}\tau(\alpha) = d_{\tilde{H}}\tau(\alpha) = -\tau(d_H\alpha) = -\tau(\partial_E\alpha) + \tau(\overline{\partial}_E\alpha).$$

Since $\tau(\mathcal{U}^k) = \mathcal{U}^k_{\tilde{E}}$, we obtain the identities for the operators $\partial_{\tilde{E}}$ and $\overline{\partial}_{\tilde{E}}$. □

Example 4.10 (Change of type of generalized complex structures). As even and odd forms get swapped with T-duality along a circle, the type of a generalized complex structure is not preserved. However, it can only change, at a point, by ± 1. Indeed, if $\rho = e^{B+i\omega}\Omega$ is an invariant form determining a generalized complex structure there are two possibilities: If Ω is a pull back from the base, the type will increase by 1, otherwise will decrease by 1.

For a principal n-torus bundle, the rule is not so simple. If we let T^n be the fiber, $\rho = e^{B+i\omega}\Omega$ be a local trivialization of the canonical bundle and define

$$l = \max\{i : \wedge^i TT \cdot \Omega \neq 0\}$$

and

$$r = rank\omega|_V, \text{ where } V = \text{Ann}(\Omega) \cap TT,$$

then the type, $\tilde{t}$ of the T-dual structure relates to the type, t, of the original structure by

$$\tilde{t} = t + n - 2l - r. \tag{4.7}$$

The following table sumarizes different ways the type changes for generalized complex structures in E^{2n} induced by complex and symplectic structures if the fibers are n-tori of some special types:

Struture on E	Fibers of E	Structure on $\tilde{E}$	Fibers of $\tilde{E}$
Complex	Complex	Complex	Complex
Complex	Real ($TT \cap J(TT) = \{0\}$)	Symplectic	Lagrangian
Symplectic	Symplectic	Symplectic	Symplectic
Symplectic	Lagrangian	Complex	Real

Table 1: Change of type of generalized complex structures under T-duality according to the type of fiber.

Example 4.11 (Hopf surfaces). Given two complex numbers a_1 and a_2, with $|a_1|, |a_2| > 1$, the quotient of $\mathbb{C}^2$ by the action $(z_1, z_2) \mapsto (a_1 z_1, a_2 z_2)$ is a primary Hopf surface (with the induced complex structure). Of all primary Hopf surfaces, these are the only ones admiting a T^2 action preserving the complex structure (see [3]). If $a_1 = a_2$, the orbits of the 2-torus action are elliptic surfaces and hence, according to Example 4.10, the T-dual will still be a complex manifold. If $a_1 \neq a_2$, then the orbits of the torus action are real except for the orbits passing through $(1, 0)$ and $(0, 1)$, which are elliptic. In this case, the T-dual will be generically symplectic except for the two special fibers corresponding to the elliptic curves, where there is type change. This example also shows that even if the initial structure on E has constant type, the same does not need to be true in the T-dual.

Example 4.12 (Mirror symmetry of Betti numbers). Consider the case of the mirror of a Calabi-Yau manifold along a special Lagrangian fibration. We have seen that the bundles $U^k_{\omega,J}$ induced by both the complex and symplectic structure are preserved by T-duality. Hence $U^{p,q} = U^p_\omega \cap U^q_J$ is also preserved, but, $U^{p,q}$ will be associated in the mirror to $U^p_{\tilde{J}} \cap U^q_{\tilde{\omega}}$, as complex and symplectic structure get swaped. Finally, as remarked Chapter 2, example 2.2, we have an isomorphism between $\Omega^{p,q}$ and $\mathcal{U}^{n-p-q,p-q}$. Making these identifications, we have

$$\Omega^{p,q}(E) \cong \mathcal{U}^{n-p-q,p-q}(E) \cong \tilde{\mathcal{U}}^{n-p-q,p-q}(\tilde{E}) \cong \Omega^{n-p,q}(\tilde{E}).$$

Which, in cohomology, gives the usual 'mirror symmetry' of the Hodge diamond.

2.1. The metric and the Buscher rules. Another geometric structure that can be transported via T-duality is the generalized metric. Assume that a principal circle bundle E is endowed with an invariant generalized metric $\mathcal{G}$. Then, since φ is orthogonal, $\tilde{\mathcal{G}} = \varphi\mathcal{G}\varphi^{-1}$ is a generalized metric on $\tilde{E}$ and with these metrics φ is an isometry between $TE \oplus T^*E$ and $T\tilde{E} \oplus T^*\tilde{E}$.

Since a generalized metric in a split Courant algebroid is defined by a metric and a 2-form, $\mathcal{G}$ is equivalent to an invariant metric g and an invariant 2-form b which we can write as

$$g = g_0 \theta \odot \theta + g_1 \odot \theta + g_2$$
$$b = b_1 \wedge \theta + b_2.$$

If one wants to determine the corresponding metric $\tilde{g}$ and 2-form b on $\tilde{E}$ we just have to recall that the 1-eigenspace of $\tilde{\mathcal{G}}$, $\tilde{C_+} = \varphi(C_+)$, is the graph of $\tilde{g} + \tilde{b}$. One can check that $\tilde{C_+}$ is the graph of:

$$\tilde{g} = \frac{1}{g_0}\tilde{\theta}\odot\tilde{\theta} - \frac{b_1}{g_0}\odot\tilde{\theta} + g_2 + \frac{b_1\odot b_1 - g_1\odot g_1}{g_0} \tag{4.8}$$
$$\tilde{b} = -\frac{g_1}{g_0}\wedge\tilde{\theta} + b_2 + \frac{g_1\wedge b_1}{g_0}$$

Of course, in the generalized Kähler case, this is how the g and b induced by the structure transform. These equations, however, are not new. They had been encountered before by the physicists [12, 13], independently of generalized complex geometry and are called *Buscher rules!*

2.2. The bihermitian structure. The choice of a generalized metric (g, b) gives us two orthogonal spaces

$$C_\pm = \{X + b(X,\cdot) \pm g(X,\cdot) : X \in TM\},$$

and the projections $\pi_\pm : C_\pm \longrightarrow TM$ are isomorphisms. Hence, any endomorphism $A \in End(TM)$ induces endomorphisms $A_\pm$ on $C_\pm$. Using the map φ we can transport this structure to a T-dual:

$$\begin{array}{ccc}
A_+ \in End(C_+) & \xrightarrow{\quad\varphi\quad} & \tilde{A}_+ \in End(\tilde{C}_+) \\
{\scriptstyle\pi_+}\big\downarrow & & {\scriptstyle\tilde{\pi}_+}\big\downarrow \\
A \in End(TE) & & \tilde{A}_\pm \in End(T\tilde{E}) \\
{\scriptstyle\pi_-}\big\uparrow & & {\scriptstyle\tilde{\pi}_-}\big\uparrow \\
A_- \in End(C_-) & \xrightarrow{\quad\varphi\quad} & \tilde{A}_- \in End(\tilde{C}_-)
\end{array}$$

As we are using the generalized metric to transport A and the maps $\pi_\pm$ and φ are orthogonal, the properties shared by A and $A_\pm$ will be metric related ones, e.g., self-adjointness, skew-adjointness and orthogonality. In the generalized Kähler case, it is clear that if we transport $J_\pm$ via $C_\pm$ we obtain the corresponding complex structures of the induced generalized Kähler structure in the dual:

$$\tilde{J}_\pm = \tilde{\pi}_\pm\varphi\pi_\pm^{-1}J_\pm(\tilde{\pi}_\pm\varphi\pi_\pm^{-1})^{-1}.$$

In the case of a metric connexion, $\theta = g(\frac{\partial}{\partial\theta},\cdot)/g(\frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta})$, we can give a very concrete description of $\tilde{J}_\pm$. We start describing the maps $\tilde{\pi}_\pm\varphi\pi_\pm^{-1}$. If V is orthogonal do $\partial/\partial\theta$, then $g_1(V) = 0$ and

$$\begin{aligned}
\tilde{\pi}_\pm\varphi\pi_\pm^{-1}(V) &= \tilde{\pi}_\pm\varphi(V + b_1(V)\theta + b_2(V) \pm g_2(V,\cdot)) = \tilde{\pi}_\pm(V + b_1(V)\frac{\partial}{\partial\tilde{\theta}} + b_2(V) \pm g_2(V,\cdot)) \\
&= V + b_1(V)\frac{\partial}{\partial\tilde{\theta}}.
\end{aligned}$$

And for $\partial/\partial\theta$ we have

$$\tilde{\pi}_\pm\varphi\pi_\pm^{-1}(\partial/\partial\theta) = \tilde{\pi}_\pm\varphi(\partial/\partial\theta + b_1 \pm (\frac{1}{g_0}\theta + g_1)) = \tilde{\pi}_\pm(\frac{1}{g_0}\partial/\partial\tilde{\theta} + \tilde{\theta})) = \pm\frac{1}{g_0}\frac{\partial}{\partial\tilde{\theta}}.$$

Remark: The T-dual connection is *not* the metric connection for the T-dual metric. This is particularly clear in this case, as the vector $\tilde{\pi}_\pm\varphi\pi_\pm^{-1}(V) = V + b_1(V)\partial/\partial\tilde{\theta}$, although not horizontal for the T-dual connection, is perpendicular to $\partial/\partial\tilde{\theta}$ according to the dual metric. This means that if we use the metric connections of both sides, the map $\tilde{\pi}_\pm \varphi \pi_\pm^{-1}$ is the identity from the orthogonal complement of $\partial/\partial\theta$ to the orthogonal complement of $\partial/\partial\tilde{\theta}$.

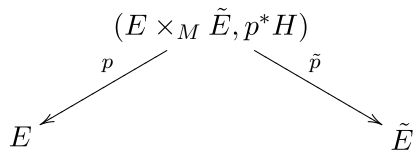
Now, if we let $V_\pm$ be the orthogonal complement to $\mathrm{span}\{\partial/\partial\theta, J_\pm \partial/\partial\theta\}$ we can describe $\tilde{J}_\pm$ by

$$\tilde{J}_\pm w = \begin{cases} J_\pm w, & \text{if } w \in V_\pm \\ \pm \frac{1}{g_0} J_\pm \partial/\partial\theta & \text{if } w = \frac{\partial}{\partial\tilde{\theta}} \\ \mp g_0 \frac{\partial}{\partial\tilde{\theta}} & \text{if } w = J_\pm \frac{\partial}{\partial\theta} \end{cases} \tag{4.9}$$

Therefore, if we identify $\partial/\partial\theta$ with $\partial/\partial\tilde{\theta}$ and their orthogonal complements with each other via TM, $\tilde{J}_+$ is essentially the same as J_+, but stretched in the directions of $\partial/\partial\theta$ and $J_+\partial/\partial\theta$ by g_0, while $\tilde{J}_-$ is J_- conjugated and stretched in those directions. In particular, J_+ and $\tilde{J}_+$ determine the same orientation while $\tilde{J}_-$ and J_- determine reverse orientations.

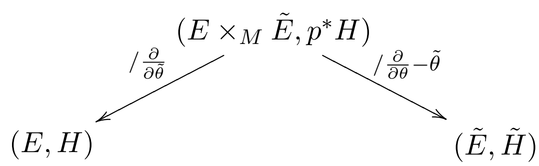
3. Reduction and T-duality

Now, let (E, H) and $(\tilde{E}, \tilde{H})$ be T-dual spaces and consider the correpondence space $E \times_M \tilde{E}$ with the 3-form p^*H:



There are two circle actions on this space with associated Lie algebra maps $\psi_i : \mathbb{R} \longrightarrow C^\infty(TM)$, $\psi_1(1) = \frac{\partial}{\partial\theta}$ and $\psi_2(1) = \frac{\partial}{\partial\tilde{\theta}}$. Since H is basic with respect to the action of $\frac{\partial}{\partial\tilde{\theta}}$, we can lift the action induced by ψ_2 and form the corresponding reduced algebroid over $E = M/S^1$, which is just $(TE + T^*E, [,]_H, \langle,\rangle)$.

On the other hand, H has an equivariantly closed extension with respect to the action of $\frac{\partial}{\partial\theta}$, since $i_{\frac{\partial}{\partial\theta}} H = d\tilde{\theta}$, so we can lift the action of $\frac{\partial}{\partial\theta}$ as $\rho_1(1) = \frac{\partial}{\partial\theta} - \tilde{\theta}$. As in Example 3.12, the connection θ allows us to choose a natural splitting for the reduced algebroid over $\tilde{E}$, which according to (3.11) has curvarture $\tilde{H} = H - d(\tilde{\theta} \wedge \theta) = (d\theta) \wedge \tilde{\theta} + h$, hence we have the following



Observe that for the first reduction we had $K_1 = \{\partial/\partial\tilde{\theta}\}$ and for the second reduction we had $K_2 = \{\partial/\partial\theta + \tilde{\theta}\}$ and the natural pairing gives a nondegenerate pairing between these two spaces.

Theorem 4.13 (Cavalcanti and Gualtieri [20], Hu [39]). *If two principal torus bundles over a common base (E, H) and $(\tilde{E}, \tilde{H})$ are T-dual to each other then they can be obtained as reduced spaces from a common space $(M, \mathcal{H})$ by two torus actions.*

If K_1 and K_2 are the vector bundles generated by the lifts of each of these actions to the Courant algebroid $(TM + T^*M, [,]_{\mathcal{H}}, \langle,\rangle)$, then K_1 and K_2 are isotropic and the natural pairing is nondegenerate in $K_1 \times K_2 \longrightarrow \mathbb{R}$.

Finally, we observe that reducing $(TM + T^*M, [,]_{\mathcal{H}}, \langle , \rangle)$ by the full T^{2n} action renders a Courant algebroid over the common base M. The rank of this Courant algebroid is the same as the rank of the reduced algebroids over either E or $\tilde{E}$ and it can be geometrically interpreted in two different ways: invariant sections of $TE + T^*E$ or invariant sections $T\tilde{E} + T^*\tilde{E}$. Of course the algebroid itself does not depend on the particular interpretation, hence $(TE + T^*E)_{T^n}$ and $(T\tilde{E} + T^*\tilde{E})_{T^n}$ are isomorphic as Courant algebroids over M, which is precisely the result of Theorem 4.6

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