Pseudodifferential Operators

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1 Introduction

One of the leading ideas in the theory of pseudodifferential operators is to reduce the study of properties of a linear differential operator

$$P = \sum_{|\alpha|\leq m} c_\alpha(x)\partial_x^\alpha,$$

which is a polynomial in the derivatives $\partial_x = (\partial_{x_1}, \ldots, \partial_{x_n})$ with constants c_α depending on x, to its *symbol*

$$p(x,\xi) = \sum_{|\alpha|\leq m} c_\alpha(x)(i\xi)^\alpha,$$

which is a polynomial in the *phase variable* $\xi \in \mathbb{R}^n$ with constants depending on the *space variable* x. Then the question arises how properties of the differential operator $p(x, D_x)$ – as for example invertibility – is related to properties of the symbol $p(x,\xi)$.

The main tool in the theory is the *Fourier transformation*

$$\mathcal{F}[f](\xi) := \hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx, \qquad \xi \in \mathbb{R}^n, \tag{1.1}$$

which is defined for suitable functions $f\colon \mathbb{R}^n \to \mathbb{C}$. Here (1.1) can be interpreted as scalar product of the function $f(x)$ and the function $x \mapsto e^{ix\cdot\xi}$:

$$\mathcal{F}[f](\xi) = \int_{\mathbb{R}^n} f(x)\overline{e^{ix\cdot\xi}}dx = (f, e^{i.\cdot\xi})$$

for fixed $\xi \in \mathbb{R}^n$, where $(.,.)$ denotes the scalar product on $L^2(\mathbb{R}^n)$. Hence $\hat{f}(\xi)$ can be interpreted as the contribution of the complex (multidimensional) oscillation $x \mapsto e^{ix\cdot\xi}$ to the function, where the *phase variable* $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ describes the frequency of the oscillation.

Knowing the Fourier transform $g(\xi) = \hat{f}(\xi)$, the function f can be reconstructed with the aid of the *inverse Fourier transformation*

$$\mathcal{F}^{-1}[g](x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} g(\xi) d\xi;$$

indeed it holds that

$$\mathcal{F}^{-1}[\mathcal{F}[f]] \equiv \mathcal{F}^{-1}[\hat{f}] = f \qquad\qquad \textbf{(Inversion formula)}.$$

Here $\mathcal{F}^{-1}[g]$ can be interpreted as infinitesimal linear combination of $x \mapsto e^{ix\cdot\xi}$ with coefficients $g(\xi)$ (and a correcting factor $1/(2\pi)^n$).

Using the inversion formula,

$$\begin{aligned} Pf &= \sum_{|\alpha|\le m} c_\alpha(x) \partial_x^\alpha \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi) d\xi \\ &= \sum_{|\alpha|\le m} c_\alpha(x) \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (i\xi)^\alpha e^{ix\cdot\xi} \hat{f}(\xi) d\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi) \hat{f}(\xi) d\xi, \end{aligned}$$

where we have used $\partial_{x_j} e^{ix\cdot\xi} = i\xi_j e^{ix\cdot\xi}$ and therefore $\partial_x^\alpha e^{ix\cdot\xi} = (i\xi)^\alpha e^{ix\cdot\xi}$. This motivates the definition of the symbol $p(x,\xi)$ of P.

In the following let $p(x,\xi) = p(\xi)$ do not depend on $x \in \mathbb{R}^n$. Then

$$Pf = \mathcal{F}^{-1}[p(\xi)\hat{f}] = \mathcal{F}^{-1}[p(\xi)\mathcal{F}[f]].$$

Hence application of P to f acts as multiplication of $\hat{f}(\xi)$ by the symbol $p(\xi)$. This suggests that inversion of P corresponds to multiplication $1/p(\xi)$ on the side of the Fourier transformed functions. Therefore we define

$$Qf := \mathcal{F}^{-1}[p(\xi)^{-1}\hat{f}(\xi)] = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{1}{p(\xi)} \hat{f}(\xi) d\xi$$

assuming that $p(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$. Then $\mathcal{F}[Qf] = p(\xi)^{-1}\hat{f}(\xi)$ since $\mathcal{F}\mathcal{F}^{-1} = I$ (inversion formula) and therefore

$$PQf = \mathcal{F}^{-1}[p(\xi)F[Qf]] = \mathcal{F}^{-1}[p(\xi)p(\xi)^{-1}\hat{f}] = \mathcal{F}^{-1}[\hat{f}(\xi)] = f,$$

i.e., Q is indeed the inverse of P.

Of course Q is no longer a differential operator; it belongs to the class of *pseudo-differential operators*, which are defined as

$$q(x, D_x)f := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(x,\xi) \hat{f}(\xi) d\xi$$

where $q(x, \xi)$ is a suitable function not necessarily a polynomial.

In the case that the coefficients of P depend on x the inverse of P is not inverted that easy. If we define analogously to the constant coefficient case

$$Qf := \frac{1}{2\pi} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{1}{p(x,\xi)} \hat{f}(\xi) d\xi,$$

then

$$PQf = \frac{1}{2\pi} \int_{\mathbb{R}^n} P \left(e^{ix\cdot\xi} \frac{1}{p(x,\xi)} \right) \hat{f}(\xi) d\xi.$$

Because of the product rule,

$$P \left(e^{ix\cdot\xi} \frac{1}{p(x,\xi)} \right) = (Pe^{ix\cdot\xi}) \frac{1}{p(x,\xi)} + r(x,\xi) = p(x,\xi) \frac{1}{p(x,\xi)} + r(x,\xi) = 1 + r(x,\xi)$$

where the remainder term $r(x, \xi)$ consists of terms where $\frac{1}{p(x,\xi)}$ is differentiated with respect to x at least once. Hence

$$PQf = I + r(x, D_x),$$

where $r(x, D_x) \neq 0$ if $p(x, \xi)$ are not constant in x. But in some sense $r(x, D_x)$ is of *lower order* (order less than 0) and does not play an important rule for many purposes.

So far all consideration have been made formally. To make them precise and to prove everything rigorously will be a main part of the lecture.

The structure of the lecture is as follows: First we recall some basic facts from calculus in $\mathbb{R}^n$ and Lebesgue integration in Section 4.10. Then, in Section 3, we study some basic properties of the Fourier transformation. In particular, we prove the inversion formula for functions in $L^2(\mathbb{R}^n)$. The main part of the lecture consists of Section 4 in which basic properties of pseudodifferential operators are studied. More precisely, we study compositions, adjoints, and boundedness of pseudodifferential operators on certain function spaces including $L^2(\mathbb{R}^n)$ and L^2-*Sobolev spaces.* The results will be applied to elliptic differential and pseudodifferential equations.

2 Summary of Some Basic Results

2.1 Functions in $\mathbb{R}^n$

In this section we briefly summerize some known fact from the basic courses on calculus in $\mathbb{R}^n$.

In the following $\mathbb{N}$ will denote the set of all natural numbers zero not included and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Moreover, $\mathbb{R}$ and $\mathbb{C}$ denote the set of real and complex numbers, respectively. Constants appearing in inequalities will usually be denoted by C sometimes marked with an index as e.g. C_α to denote that C depends on α. In sequences of inequalities all constants will simply be denoted by C although they may change from line to line.

We will make use of the *multi-indices*, which will keep the notation (relatively) short. A multi-index is a vector $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$. Given $\alpha \in \mathbb{N}_0^n$ we define the length of α as $|\alpha| = \alpha_1 + \dots + \alpha_n$ and its factorial as $\alpha! = \alpha_1! \cdot \ldots \cdot \alpha_n!$.

The multi-indices are used to write polynomials: for $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$ we define $x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$. Then x^α is a polynomial of degree $|\alpha|$ and an arbitrary polynomial $p \colon \mathbb{R}^n \to \mathbb{C}$ of order $m \in \mathbb{N}_0$ can be written as

$$p(x) = \sum_{|\alpha| \le m} c_\alpha x^\alpha$$

with coefficients $c_\alpha \in \mathbb{C}$. Here $\sum_{|\alpha| \le m}$ denotes summation with respect to all multi-indices $\alpha \in \mathbb{N}_0^n$ with length $|\alpha| \le m$. Moreover, if $\alpha, \beta \in \mathbb{N}_0^n$, we write $\alpha \le \beta$ if and only if $\alpha_j \le \beta_j$ for all $j = 1, \dots, n$.

The so-called binomial coefficients are defined by

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha - \beta)!}$$

if $\beta \le \alpha$ and $\binom{\alpha}{\beta} = 0$ otherwise. Then it is easy to check that

$$\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdot \ldots \cdot \binom{\alpha_n}{\beta_n}.$$

Using this notation and the fundamental relation

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

for $n, k \in \mathbb{N}_0$, one can prove by induction that

$$(x + y)^\alpha = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} x^\beta y^{\alpha - \beta}$$

for all $x, y \in \mathbb{R}^n$, which generalizes the well-known binomial formula in one-dimension.

Recall that the space $C^k(\mathbb{R}^n)$ consists of all k-times differentiable functions $f\colon \mathbb{R}^n \to \mathbb{C}$ or $f\colon \mathbb{R}^n \to \mathbb{R}$ with continuous derivatives up to order k. If $\alpha \in \mathbb{N}_0^n$ and $u \in C^{|\alpha|}(\mathbb{R}^n)$, then we define

$$\partial^\alpha u(x) = \partial_x^\alpha u(x) = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}} u(x).$$

If u depends on several variables, say x and y, $\partial_x^\alpha u$ and $\partial_y^\alpha u$ denote the derivative defined above with respect to x and y, respectively.

Using the product rule of differentiation, one can prove the following **Leibniz's formula**:

$$\partial^\alpha(uv) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (\partial^\beta u)(\partial^{\alpha-\beta} v) \tag{2.1}$$

for all $\alpha \in \mathbb{N}_0^n$ and $u, v \in C^{|\alpha|}(\mathbb{R}^n)$.

Finally, we recall Taylor's formula:

THEOREM 2.1 (Taylor's formula) *Let $u \in C^k(\mathbb{R}^n)$, $k \in \mathbb{N}$. Then for any x and $y \in \mathbb{R}^n$ one has*

$$u(x+y) = \sum_{|\alpha|<k} \frac{y^\alpha}{\alpha!} \partial^\alpha u(x) + \sum_{|\alpha|=k} k \frac{y^\alpha}{\alpha!} \int_0^1 (1-t)^{k-1} \partial^\alpha u(x+ty)dt$$

Proof: See basic course or [Ray91, Chapter 1, Theorem 1.1]. ■

Finally, we denote by $C_b^k(\mathbb{R}^n)$, $k \in \mathbb{N}_0$, the space of all $f \in C^k(\mathbb{R}^n)$ with *bounded* derivatives up to order k, i.e., $\partial^\alpha f$ is bounded for all $|\alpha| \leq k$. Moreover, $C_b^\infty(\mathbb{R}^n) := \bigcap_{k=0}^\infty C_b^k(\mathbb{R}^n)$. It is well-known from basic courses in functional analysis that $C_b^k(\mathbb{R}^n)$ equipped with the norm

$$\|f\|_{C_b^k} = \sup_{|\alpha|\leq k} \sup_{x \in \mathbb{R}^n} |\partial^\alpha f(x)|$$

is a Banach space.

2.2 Lebesgue Integral and L^p-spaces

Recall that a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is *Lebesgue-integrable* or just integrable if f is measurable and

$$\int_{\mathbb{R}^n} |f(x)|dx < \infty. \tag{2.2}$$

Note that $|f| \geq 0$ is measurable if f is measurable and for a non-negative measurable function $g\colon \mathbb{R}^n \to [0,\infty]$ there is always a well-defined$\int_{\mathbb{R}^n} g(x)dx$, which is possibly infinite. Moreover, if (2.2) holds, $f^+ := \max\{f, 0\}$ and $f^- : \max\{-f, 0\}$ are non-negative functions with finite integral and

$$\int_{\mathbb{R}^n} f(x)dx := \int_{\mathbb{R}^n} f^+(x)dx - \int_{\mathbb{R}^n} f^-(x)dx$$

is well-defined. More generally, a function $f\colon \mathbb{R}^n \to \mathbb{C}$ is Lebesgue-integrable if and only if $\operatorname{Re} f$ and $\operatorname{Im} f$ are Lebesgue-integrable. Then

$$\int_{\mathbb{R}^n} f(x)dx := \int_{\mathbb{R}^n} \operatorname{Re} f(x)dx + i \int_{\mathbb{R}^n} \operatorname{Im} f(x)dx.$$

The space of all Lebesgue-integrable functions $f\colon \mathbb{R}^n \to \mathbb{C}$ is denoted by $L^1(\mathbb{R}^n)$. It is a linear vector space, which can be normed by

$$\|f\|_1 := \int_{\mathbb{R}^n} |f(x)|dx.$$

$L^1(\mathbb{R}^n)$ is complete with respect to the norm $\|.\|_1$ if functions which differ on a set of measure zero are identified.

In the following almost all functions will be at least continuous, in most cases even infinitely differentiable. If f and g are continuous and $f = g$ *almost everywhere*, i.e., they differ on a set of measure zero, then $f(x) = g(x)$ for *every* $x \in \mathbb{R}^n$. Therefore we do not have to pay attention on the possible null-sets on which the functions possibly differ if we only work with continuous functions.

We need the following theorems from integration theory:

THEOREM 2.2 (Lebesgue's Theorem on Dominated Convergence) *Let $f_k \in L^1(\mathbb{R}^n)$ be sequence of functions such that*

$$\begin{aligned} \lim_{k\to\infty} f_k(x) &= f(x) \qquad a.e., \\ |f_k(x)| &\leq g(x) \qquad a.e., \end{aligned}$$

where $g \in L^1(\mathbb{R}^n)$. Then $f \in L^1(\mathbb{R}^n)$ and

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} f_k(x)dx = \int_{\mathbb{R}^n} f(x)dx.$$

As an application of Lebesgue's Theorem on Dominated Convergence one can prove the following two theorems on parameter-dependent integrals, which will often be used in the following.

THEOREM 2.3 *Let $U \subset \mathbb{R}^m$ be an open set and $a \in U$. Moreover, let $f\colon \mathbb{R}^n \times U \to \mathbb{C}$ be functions such that:*

1. *For all $x \in \mathbb{R}^n$ the function $t \mapsto f(x,t)$ is continuous in a.*

2. *For all $t \in U$ the function $x \mapsto f(x,t)$ is integrable on $\mathbb{R}^n$.*

3. *There is an $F \in L^1(\mathbb{R}^n)$ such that $|f(x,t)| \leq F(x)$ for almost all $x \in \mathbb{R}^n$ and all $t \in U$.*

Then the function

$$g(t) := \int_{\mathbb{R}^n} f(x,t)dx, \qquad t \in U,$$

is continuous in a.

THEOREM 2.4 *Let $I \subset \mathbb{R}$ be an open interval and $f \colon \mathbb{R}^n \times I \to \mathbb{C}$ be such that*

1. *For every $x \in \mathbb{R}^n$ the function $t \mapsto f(x,t)$ is differentiable on I.*
2. *For every $t \in I$ the function $x \mapsto f(x,t)$ is integrable on $\mathbb{R}^n$.*
3. *There is an $F \in L^1(\mathbb{R}^n)$ such that $|\partial_t f(x,t)| \le F(x)$.*

Then

$$g(t) := \int_{\mathbb{R}^n} f(x,t)dx$$

is a differentiable function on I and for all $t \in I$

$$g'(t) = \int_{\mathbb{R}^n} \partial_t f(x,t)dx.$$

The proofs of these theorems are left to the reader as an exercise. Alternatively, see [For84, §11, Satz 1/Satz 2].

THEOREM 2.5 (Fubini's Theorem)
Let $f \in L^1(\mathbb{R}^k \times \mathbb{R}^l)$, $k,l \in \mathbb{N}$. Then $f(x,.) \in L^1(\mathbb{R}^l)$ for almost all $x \in \mathbb{R}^k$, $x \mapsto \int_{\mathbb{R}^k} f(x,y)dy \in L^1(\mathbb{R}^k)$, and

$$\int_{\mathbb{R}^k \times \mathbb{R}^l} f(x,y)d(x,y) = \int_{\mathbb{R}^k} \left(\int_{\mathbb{R}^l} f(x,y)dy \right) dx. \tag{2.3}$$

Conversely, if $f(x,.) \in L^1(\mathbb{R}^l)$ for almost all $x \in \mathbb{R}^k$ and $x \mapsto \int_{\mathbb{R}^k} f(x,y)dy \in L^1(\mathbb{R}^k)$, then $f \in L^1(\mathbb{R}^k \times \mathbb{R}^l)$ and (2.3) holds.

The following simple lemma will be essential

Lemma 2.6 *Let $s > n$. Then $\langle x \rangle^{-s} \in L^1(\mathbb{R}^n)$ and $(1+|x|)^{-s} \in L^1(\mathbb{R}^n)$.*

Proof: The lemma can easily be proved e.g. by using polar coordinates. An alternative approach can be found in [Ray91, Lemma 1.3]. ■
We will also use the change-of-variable theorem; but only in the following special case:

$$\int_{\mathbb{R}^n} f(Ax+b)|\det A|dx = \int_{\mathbb{R}^n} f(y)dy$$

for all $f \in L^1(\mathbb{R}^n)$, $b \in \mathbb{R}^n$, and $A \in \mathbb{R}^{n \times n}$ with $\det A \neq 0$.

We will also use with spaces $L^p(\mathbb{R}^n)$, $1 \le p < \infty$, which consist of all measurable functions $f\colon \mathbb{R}^n \to \mathbb{C}$ such that

$$\|f\|_p := \left(\int_{\mathbb{R}^n} |f(x)|^p\right)^{\frac{1}{p}} < \infty.$$

All these spaces are Banach spaces (again identifying functions which coincide almost everywhere). We will mainly work with $L^2(\mathbb{R}^n)$ which is also a Hilbert space; its norm a given by the scalar product:

$$\begin{aligned} \|f\|_2 &= \sqrt{(f,f)}, \\ (f,g) &= \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx. \end{aligned}$$

The L^2-scalar product will play a fundamental role.

An important subspace of $L^p(\mathbb{R}^n)$ is $C_0^\infty(\mathbb{R}^n)$, which is the set of all smooth $f\colon \mathbb{R}^n \to \mathbb{C}$ with compact support

$$\operatorname{supp} f := \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}.$$

Lemma 2.7 *Let $1 \le p < \infty$. Then $C_0^\infty(\mathbb{R}^n)$ is* dense *in $L^p(\mathbb{R}^n)$, i.e., for every $f \in L^p(\mathbb{R}^n)$ there is a sequence $f_k \in C_0^\infty(\mathbb{R}^n)$ such that* $\lim_{k\to\infty} \|f - f_k\|_p = 0$. *For short: $\overline{C_0^\infty(\mathbb{R}^n)}^{\|\cdot\|_p} = L^p(\mathbb{R}^n)$.*

Proof: See [Alt85, Satz 2.10] or [For84, §10 Satz 3]. ∎

3 Fourier Transformation

3.1 Definition and Basic Properties

Let $f \in L^1(\mathbb{R}^n)$. Then we define the *Fourier transform* of f by

$$\hat{f}(\xi) := \mathcal{F}[f](\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx \tag{3.1}$$

for $\xi \in \mathbb{R}^n$. Since $|e^{-ix\cdot\xi} f(x)| = |f(x)|$ for all $\xi \in \mathbb{R}^n$, we have $e^{-ix\cdot\xi} f(x) \in L^1(\mathbb{R}^n)$ with respect to x and (3.1) is well-defined.

Moreover, we have the following elementary properties:

Lemma 3.1 *1. $\mathcal{F}\colon L^1(\mathbb{R}^n) \to C^0_b(\mathbb{R}^n)$ is a linear bounded operator.*

2. *If $f\colon \mathbb{R}^n \to \mathbb{C}$ is a continuously differentiable function such that $f \in L^1(\mathbb{R}^n)$ and $\partial_j f \in L^1(\mathbb{R}^n)$, then*
$$\mathcal{F}[\partial_j f] = i\xi_j \mathcal{F}[f] = i\xi_j \hat{f}. \tag{3.2}$$

3. *If $f \in L^1(\mathbb{R}^n)$ such that $x_j f \in L^1(\mathbb{R}^n)$, then $\hat{f}(\xi)$ is continuously partial differentiable with respect to ξ_j and*
$$\partial_{\xi_j} \hat{f}(\xi) = \mathcal{F}[-ix_j f(x)].$$

4. *Let $f \in L^1(\mathbb{R}^n)$ and $(\tau_y f)(x) := f(x+y)$, $y \in \mathbb{R}^n$, denote the* translation *of f by y. Then $\mathcal{F}[\tau_y f](\xi) = e^{iy\cdot\xi}\hat{f}(\xi)$ for all $\xi \in \mathbb{R}^n$.*

5. *If $f, g \in L^1(\mathbb{R}^n)$, then*
$$\hat{f}(\xi)\hat{g}(\xi) = \mathcal{F}[f * g],$$
where
$$f * g(x) := \int_{\mathbb{R}^n} f(x-y)g(y)dy$$
denotes the convolution *of f and g.*

6. *Let $f \in L^1(\mathbb{R}^n)$ and let $(\rho_\varepsilon f)(x) := f(\varepsilon x)$, $\varepsilon > 0$, denote the* dilation *of f by ε. Then*
$$\mathcal{F}[\rho_\varepsilon f](\xi) = \varepsilon^{-n}\hat{f}(\xi/\varepsilon) = \varepsilon^{-n}(\rho_{\varepsilon^{-1}}\hat{f})(\xi).$$

Proof:

1. First of all
$$\|\hat{f}\|_\infty = \sup_{\xi\in\mathbb{R}^n} |\hat{f}(\xi)| \le \sup_{\xi\in\mathbb{R}^n} \int_{\mathbb{R}^n} \left|e^{-ix\cdot\xi} f(x)\right| dx = \|f\|_1.$$
Hence $\hat{f}$ is bounded. Let $g(x,\xi) = e^{-ix\cdot\xi} f(x)$. Then $\xi \mapsto g(x,\xi)$ is continuous for every fixed $x \in \mathbb{R}^n$, $x \mapsto g(x,\xi)$ is measurable for every $\xi \in \mathbb{R}^n$, and $|g(x,\xi)| \le |f(x)| \in L^1(\mathbb{R}^n)$. Therefore we can apply Theorem 2.3 which implies the continuity of f.

2. Use integration by parts. See Exercise Sheet 2.

3. Use Theorem 2.4. See Exercise Sheet 2.

4. Use the change-of-variables theorem. See Exercise Sheet 2.

5. First of all, $f(x-y)g(y) \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$ with respect to (x, y) since

$$\|f * g\|_1 \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)g(y)| dy dx = \int_{\mathbb{R}^n \times \mathbb{R}^n} |f(z)g(y)| d(y,z) = \|f\|_1 \|g\|_1.$$

by Fubini's theorem and the change-of-variable theorem. Hence

$$\begin{aligned} \mathcal{F}[f * g](\xi) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x-y)g(y) dy dx \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{-ix\cdot\xi} (\tau_{-y} f)(x) dx \right) g(y) dy \\ &= \int_{\mathbb{R}^n} e^{-iy\cdot\xi} \hat{f}(\xi) g(y) dy = \hat{f}(\xi)\hat{g}(\xi), \end{aligned}$$

where we have used Fubini's theorem and the fourth statement of this lemma.

6. Use the change-of-variables theorem. See Exercise Sheet 2.

■

The relation (3.2) is the fundamental property of the Fourier transformation from the point of view of differential equations. Because of the factor i in (3.2), we define

$$D_{x_j} = \frac{1}{i}\partial_{x_j}, \qquad D_x = (D_{x_1}, \ldots, D_{x_n}).$$

Then (3.2) is equivalent to

$$\mathcal{F}[D_{x_j} f] = \xi_j \mathcal{F}[f] = \xi_j \hat{f}.$$

Moreover, if we express a linear differential operator P with constant coefficients as

$$Pf = \sum_{|\alpha| \leq m} c_\alpha D_x f$$

with some constants $c_\alpha \in \mathbb{C}$, then

$$\mathcal{F}[Pf] = \sum_{|\alpha| \leq m} c_\alpha \xi^\alpha \hat{f}.$$

Hence application of a linear differential operator P to f corresponds to multiplication of $\hat{f}$ with the polynomial

$$p(\xi) := \sum_{|\alpha| \leq m} c_\alpha \xi^\alpha.$$

The function $p(\xi)$ is called the *symbol* of P. Moreover, we write $P = p(D_x)$.

Remark 3.2 If $f \in L^1(\mathbb{R}^n)$ is continuously differentiable and $\partial_j f \in L^1(\mathbb{R}^n)$ for all $j = 1, \dots, n$, then $\hat{f}(\xi)$ and $\xi_j \hat{f}(\xi)$ are bounded functions. Hence

$$(1 + |\xi|)|\hat{f}(\xi)| \le C \quad \Leftrightarrow \quad |\hat{f}(\xi)| \le \frac{C}{1 + |\xi|},$$

which means that $\hat{f}(\xi)$ decays as $|\xi|^{-1}$ as $|\xi| \to \infty$.

More generally, if $f \in C^k(\mathbb{R}^n)$ such that $\partial_x^\alpha f \in L^1(\mathbb{R}^n)$ for all $|\alpha| \le k$, then one can show in the same way that

$$|\hat{f}(\xi)| \le \frac{C}{(1 + |\xi|)^k}.$$

For short: *Differentiability of f implies a polynomial decay of $\hat{f}$ as $|\xi| \to \infty$.*

On the other hand, if $(1 + |x|)^k f(x) \in L^1(\mathbb{R}^n)$ for some $k \in \mathbb{N}$, we can apply Lemma 3.1.3 successively to conclude that $\hat{f} \in C^k(\mathbb{R}^n)$. Hence faster decay of $f(x)$ as $|x| \to \infty$ yields to a higher differentiability of $\hat{f}$.

3.2 Rapidly decreasing functions – $\mathcal{S}(\mathbb{R}^n)$

Let $f \in C_0^\infty(\mathbb{R}^n)$. Then $\partial_x^\alpha f \in L^1(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}_0$. As seen in Remark 3.2, $\hat{f}(\xi)$ decays faster than any power $(1 + |x|)^{-k}$, $k \in \mathbb{N}$. Moreover, $(1 + |x|)^k f \in L^1(\mathbb{R}^n)$ for all $k \in \mathbb{N}$. Hence $\hat{f} \in C_b^k(\mathbb{R}^n)$ for all $k \in \mathbb{N}$, i.e., $\hat{f} \in C_b^\infty(\mathbb{R}^n) = \bigcap_{k=1}^\infty C_b^k(\mathbb{R}^n)$. But in general $\hat{f}(\xi)$ does not have compact support, cf. Exercise 1. If $f \in C_0^\infty(\mathbb{R}^n)$, then $\hat{f}$ belongs to the following function space.

Definition 3.3 The space $\mathcal{S}(\mathbb{R}^n)$ of all *rapidly decreasing smooth functions* is the set of all smooth $f\colon \mathbb{R}^n \to \mathbb{C}$ such that for all $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}_0$ there is a constant $C_{\alpha,N}$ such that

$$|\partial_x^\alpha f(x)| \le C_{\alpha,N}(1 + |x|)^{-N} \tag{3.3}$$

uniformly in $x \in \mathbb{R}^n$. If $f \in \mathcal{S}(\mathbb{R}^n)$ and $m \in \mathbb{N}$, we define the semi-norm:

$$|f|_{m,\mathcal{S}} := \sup_{|\alpha|+|\beta| \le m} \sup_{x \in \mathbb{R}^n} |x^\alpha \partial_x^\beta f(x)|$$

Obviously, $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$. The inclusion is strict since $f(x) = e^{-|x|^2} \in \mathcal{S}(\mathbb{R}^n)$. Moreover, $\mathcal{S}(\mathbb{R}^n) \subset C_b^\infty(\mathbb{R}^n)$.

Remark 3.4 At first sight, it may be more natural to use

$$|f|'_{m,\mathcal{S}} := \sup_{k+|\alpha|} \sup_{x \in \mathbb{R}^n} (1 + |x|)^k |\partial^\alpha f(x)|$$

as semi-norms on $\mathcal{S}(\mathbb{R}^n)$ since they are more closely related to the inequality (3.3). But the definition of $|.|_{m,\mathcal{S}}$ is more convenient when dealing with Fourier transformation, which will be demonstrated in the proof of the next lemma. Moreover, it does not matter if we use the semi-norms $|.|_{m,\mathcal{S}}$ or $|.|'_{m,\mathcal{S}}$ – the semi-norms are equivalent in the following sense: For every $m \in \mathbb{N}_0$ there is an $k(m) \in \mathbb{N}_0$ such that

$$|f|'_{m,\mathcal{S}} \leq C_m |f|_{k(m),\mathcal{S}} \quad \text{and} \quad |f|'_{m,\mathcal{S}} \leq C'_m |f|_{k(m),\mathcal{S}}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. (Actually, in this special case we can simply choose $k(m) = m$.)

Finally, we note that we can replace $(1 + |x|)$ in (3.3) by $\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}$ since $\langle x \rangle \leq (1 + |x|) \leq \sqrt{2}\langle x \rangle$.

For our purposes, the fundamental property of $\mathcal{S}(\mathbb{R}^n)$ is the following:

Lemma 3.5 *$\mathcal{F} \colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is linear mapping. More precisely, $|\hat{f}|_{m,\mathcal{S}} \leq C|f|_{m+n+1,\mathcal{S}}$ for all $m \in \mathbb{N}_0$ where C depends only on n and m.*

Proof: First of all, if $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned} \|f\|_1 &= \int_{\mathbb{R}^n} (1 + |x|)^{-n-1} \left((1 + |x|)^{n+1} |f(x)|\right) dx \\ &\leq C \int_{\mathbb{R}^n} (1 + |x|)^{-n-1} dx |f|_{n+1,\mathcal{S}} = C|f|_{n+1,\mathcal{S}}, \end{aligned}$$

where C depends only on the dimension n. Thus

$$|\hat{f}|_{0,\mathcal{S}} = \|\hat{f}\|_\infty \leq \|f\|_1 \leq C|f|_{n+1,\mathcal{S}} \tag{3.4}$$

by Lemma 3.1.1 and the previous estimate. Because of Lemma 3.1.2/3,

$$\xi^\alpha D_\xi^\beta \hat{f}(\xi) = \mathcal{F}\left[D_x^\alpha \left(x^\beta f(x)\right)\right].$$

Hence

$$\|\xi^\alpha D_\xi^\beta \hat{f}\|_\infty \leq C|D_x^\alpha \left(x^\beta f(x)\right)|_{n+1,\mathcal{S}}$$

by (3.4). Using Leibniz's formula (2.1),

$$D_x^\alpha \left(x^\beta f(x)\right) = \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} (D_x^\gamma x^\beta)(D_x^{\alpha-\gamma} f(x)).$$

Since $D_x^\gamma x^\beta$ is polynomial of degree less than $|\beta|$,

$$|D_x^\alpha \left(x^\beta f(x)\right)|_{n+1,\mathcal{S}} \leq C_{\alpha,\beta} |f|_{|\alpha|+|\beta|+n+1}.$$

Collecting all estimates and taking the supremum over $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| + |\beta| \leq m$, we finally conclude

$$|\hat{f}|_{m,\mathcal{S}} \leq C_m |f|_{m+n+1,\mathcal{S}}$$

with arbitrary $m \in \mathbb{N}_0$ where C_m depends only on n and m. Hence $\mathcal{F}[f] \in \mathcal{S}(\mathbb{R}^n)$ for all $f \in \mathcal{S}(\mathbb{R}^n)$. ∎

Example 3.6 Let $f(x) = e^{-|x|^2/2}$, $x \in \mathbb{R}^n$. Then $\hat{f}(\xi) = (2\pi)^{n/2} e^{-|\xi|^2/2}$.

Proof: Since $f(x) = e^{-x_1^2/2} \cdots e^{-x_n^2/2}$ and $e^{-ix\cdot\xi} = e^{-ix_1\xi_1} \cdots e^{-ix_n\xi_n}$,

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix_1\xi_1} e^{-x_1^2/2} dx_1 \cdots \int_{\mathbb{R}} e^{-ix_1\xi_1} e^{-x_n^2/2} dx_n = \hat{g}(\xi_1) \cdots \hat{g}(\xi_n)$$

where $g(x) = e^{-x^2/2}$, $x \in \mathbb{R}$. Hence it is sufficient to consider the case $n = 1$.

Because of Lemma 3.1.3, $\hat{g}(\xi)$ is continuously differentiable and $\hat{g}'(\xi) = \mathcal{F}[-ixg(x)]$. Moreover, $-xg(x) = \frac{d}{dx} e^{-x^2/2} = g'(x)$. Therefore, using Lemma 3.1.2,

$$\hat{g}'(\xi) = i\mathcal{F}[g'(x)] = -\xi g(\xi), \qquad \xi \in \mathbb{R},$$

and $\hat{g}(0) = \int_{\mathbb{R}} e^{-x^2/2} dx = \sqrt{2\pi}$. Hence $\hat{g}$ is uniquely determined by the letter initial value problem, which has the unique solution $\hat{g}(\xi) = \sqrt{2\pi} e^{-\xi^2/2}$. ∎

An alternative proof using the Cauchy integral formula can be found in [Ray91, Example 1.7].

Exercise 1 *Let* $f \in C_0^\infty(\mathbb{R})$ *such that* $\operatorname{supp} f \subseteq B_R(0)$. *Prove that*

$$\hat{f}(\xi) = \int_{-R}^{R} e^{i\xi x} f(x) dx, \qquad \xi \in \mathbb{C},$$

is a holomorphic function in $\mathbb{C}$. *Moreover,* $|\hat{f}(\xi)| \leq Ce^{R|\operatorname{Im}\xi|}$.

In particular, $\hat{f}(\xi)$, $\xi \in \mathbb{R}$, *is real analytic and* $\operatorname{supp} \hat{f}$ *is not compact unless* $\hat{f}(\xi) \equiv 0$.

3.3 Ex-course: Fréchet spaces

Throughout the lecture, we will work with many different spaces of smooth functions with certain properties. They all have in common that they cannot be normed to become a Banach space.

As an example we consider the space $C_b^\infty(\mathbb{R}^n)$ of all smooth and bounded functions $f: \mathbb{R}^n \to \mathbb{C}$ with bounded derivatives. Then

$$C_b^\infty(\mathbb{R}^n) = \bigcap_{k=0}^{\infty} C_b^k(\mathbb{R}^n),$$

where $C_b^k(\mathbb{R}^n)$ is the space of all k-times differentiable functions $f: \mathbb{R}^n \to \mathbb{C}$ with bounded derivatives up to order k. In contrast to $C_b^\infty(\mathbb{R}^n)$ the spaces $C_b^k(\mathbb{R}^n)$, $k \in \mathbb{N}_0$, are Banach spaces equipped with the norm

$$\|f\|_{C_b^k} = \sup_{|\alpha|\leq k} \sup_{x\in\mathbb{R}^n} |\partial^\alpha f(x)|.$$

Since in spaces smooth functions, all (infinitely many) derivatives have to be controlled, in general they cannot be normed by a single norm such that the space is complete. But these spaces can be "normed" by an infinite sequence of (semi-)norms, e.g. $f \in C_b^\infty(\mathbb{R}^n)$ if and only if $\|f\|_{C_b^k}$ is finite for *all* $k \in \mathbb{N}$.

Definition 3.7 Let V be a (complex or real) linear space. Then a mapping $\rho\colon V \to [0,\infty)$ is a *semi-norm* if

1. $\rho(rf) = |r|\rho(f)$ for all $f \in V$ and $r \in \mathbb{R}, \mathbb{C}$, resp.,
2. $\rho(f+g) \le \rho(f) + \rho(g)$ for all $f, g \in V$.

If ρ_m, $m \in \mathbb{N}$, is a sequence of semi-norms on a linear space V and $f_j \in V$, $j \in \mathbb{N}$, we say that $(f_j)_{j\in\mathbb{N}}$ is a *Cauchy sequence* if $\lim_{i,j\to\infty} \rho_m(f_i - f_j) = 0$ for all $m \in \mathbb{N}$. Moreover, we say that f_j converges to $f \in V$ if and only if $\lim_{j\to\infty} \rho_m(f_j - f) = 0$ for all $m \in \mathbb{N}$.

Definition 3.8 A linear space V is called a *Fréchet space* if there is a sequence of semi-norms $(\rho_m)_{m\in\mathbb{N}}$ satisfying the following conditions:

1. $(\rho_m)_{m\in\mathbb{N}}$ is an *increasing* sequence: $\rho_1(f) \le \rho_2(f) \le \ldots \le \rho_m(f) \le \ldots$ for all $f \in V$.
2. $(\rho_m)_{m\in\mathbb{N}}$ *separates points*: for any $f \neq 0$ there is an $m \in \mathbb{N}$ such that $\rho_m(f) \neq 0$.
3. V is *complete*: any Cauchy sequence $(f_j)_{j\in\mathbb{N}}$ converges to some $f \in V$.

Obviously, every Banach space is a Fréchet space.

Exercise 2 *Prove that $C_b^\infty(\mathbb{R}^n)$ equipped with the norms $\rho_m = \|.\|_{C_b^m}$ is a Fréchet space. – It can be used that $C_b^k(\mathbb{R}^n)$ is a Banach space.*

Lemma 3.9 *The space $\mathcal{S}(\mathbb{R}^n)$ together with the $|.|_{m,\mathcal{S}}$, $m \in \mathbb{N}$, is a Fréchet space.*

The proof is left as exercise for the interested reader. We just note that it mainly remains to check the completeness of $\mathcal{S}(\mathbb{R}^n)$, which can be done by using the completeness of $C_b^\infty(\mathbb{R}^n)$. But the completeness of $\mathcal{S}(\mathbb{R}^n)$ will not be used in the following. Therefore we do not give a proof.

Remark 3.10 (For readers interested in topology)
Every Fréchet space $(V, (\rho_m)_{m\in\mathbb{N}})$ is a complete metric space (V, d) where

$$d(f,g) := \sum_{m=1}^{\infty} 2^{-m} \frac{\rho_m(f-g)}{1+\rho_m(f-g)}. \tag{3.5}$$

Exercise 3 *1. Prove that d defined in (3.5) is a metric.*

2. Prove that $\lim_{j\to\infty} d(f_j, f)$ if and only if $\rho_m(f_j - f) = 0$ for all $m \in \mathbb{N}$.

Then convergence with respect to the semi-norms coincides with convergence in the metric sense. Since every metric space carries a natural topology which is defined by the neighborhood basis

$$U_l(f_0) = \left\{ f \in V : d(f - f_0) < \frac{1}{l} \right\}, \quad l \in \mathbb{N}, f_0 \in V,$$

every Fréchet space is a topological space. Equipped with this topology, the operations of multiplication by scalars and addition of vectors are continuous, i.e., V is a *linear topological space*.

Moreover, every Fréchet space is *locally convex*, i.e. there is a neighborhood basis at the origin that consists of convex sets. Finally, the Fréchet spaces can be characterized as the *locally convex* linear topological space that can be equipped with a metric and are complete. See e.g. [Tre67].

Definition 3.11 Let V and W be Fréchet spaces with semi-norms $(\rho_m)_{m\in\mathbb{N}}$ and $(\tau_m)_{m\in\mathbb{N}}$. Then a linear mapping $T\colon V \to W$ is *bounded* if for every $m \in \mathbb{N}$ there is a $k = k(m) \in \mathbb{N}$ and a C_m such that

$$\tau_m(Tf) \le C_m \rho_{k(m)}(f) \qquad \text{for all } f \in V.$$

THEOREM 3.12 *Let V and W be Fréchet spaces and $T\colon V \to W$ be a* linear *mapping. Then T is bounded if and only if T is continuous in the sense that* $\lim_{j\to\infty} f_j = f$ *implies* $\lim_{j\to\infty} Tf_j = Tf$.

In particular, if W is Banach space, $T\colon V \to W$ is continuous if and only if

$$\|Tf\|_W \le C\rho_m(f) \qquad \textit{for all } f \in V$$

with some $C > 0$ and $m \in \mathbb{N}$.

Proof: Let ρ_m and τ_m, $m \in \mathbb{N}$, denote the semi-norms of V and W, respectively.

Firstly, let T be bounded and f_j be a sequence such that $\lim_{j\to\infty} f_j = f$, i.e., $\lim_{j\to\infty} \rho_m(f_j - f) = 0$ for all $m \in \mathbb{N}$. Since T is bounded for every $m \in \mathbb{N}$ there exists a $k(m)$ such that $\tau_m(Tf) \le C_m\rho_{k(m)}(f)$, $f \in V$. Hence

$$\tau_m(Tf_j - Tf) = \tau_m(T(f_j - f)) \le C_m\rho_{k(m)}(f_j - f) \to 0 \quad \text{as } j \to \infty$$

for every $m \in \mathbb{N}$, which means $\lim_{j\to\infty} Tf_j = Tf$. Thus T is continuous.

Secondly, let T be continuous. Assume that T is not bounded. Then there is an $m \in \mathbb{N}$ and a sequence $f_j \in V$, $j \in \mathbb{N}$, such that for every $C > 0$ and $k \in \mathbb{N}$ there is a j_0 with

$$\tau_m(Tf_{j_0}) > C\rho_k(f_{j_0}).$$

In particular, we can choose a subsequence f_{j_k}, $k \in \mathbb{N}$, such that

$$\tau_m(Tf_{j_k}) > k\rho_k(f_{j_k}).$$

Then $\tilde{f}_k := f_{j_k}/\tau_m(Tf_{j_k})$, $k \in \mathbb{N}$, is a sequence such that

$$\rho_m(\tilde{f}_k) \leq \rho_k(\tilde{f}_k) < \frac{1}{k} \qquad \text{for all } m \leq k \in \mathbb{N}.$$

Thus $\lim_{k\to\infty} \rho_m(\tilde{f}_k) = 0$ for all $m \in \mathbb{N}$, i.e., $\lim_{k\to\infty} \tilde{f}_k = 0$. Hence $\lim_{k\to\infty} T\tilde{f}_k = 0$ because of the continuity of T. On the other hand $\tau_m(T\tilde{f}_k) = 1$ for all $k \in \mathbb{N}$ which contradicts $\lim_{k\to\infty} \tilde{f}_k = 0$. ■

Exercise 4 *A subset A of a Fréchet space V is called* bounded *if* $\sup_{f\in A} \rho_m(f) < \infty$ *for every $m \in \mathbb{N}$. – Prove that $T\colon V \to W$ is bounded if and only if $T(A)$ is bounded for every bounded $A \subset V$.*

3.4 Inverse Fourier transformation and Parseval's formula

In the following we will show that the Fourier transformation $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is invertible and that its inverse is given by

$$\mathcal{F}^{-1}[g](x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} g(\xi) d\xi,$$

which is well-defined for all $g \in L^1(\mathbb{R}^n)$. – Note that

$$\mathcal{F}^{-1}[g](x) = (2\pi)^n \mathcal{F}[g](-x). \tag{3.6}$$

If $f(x) = e^{-|x|^2}$ is the function discussed in Example 3.6, then

$$\mathcal{F}^{-1}[\hat{f}(\xi)] = (2\pi)^{-n/2} \mathcal{F}\left[e^{-|\xi|^2}\right](-x) = e^{-|x|^2} = f(x).$$

Hence $\mathcal{F}^{-1}[\hat{f}] = f$ for this special f.

Lemma 3.13 (Inversion Formula)
Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then $f(x) = \mathcal{F}^{-1}[\hat{f}](x)$ for all $x \in \mathbb{R}^n$. In particular, $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a linear isomorphism.

Proof: First of all,

$$\mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} f(y) dy \right) d\xi.$$

Since $e^{i(x-y)\cdot\xi} f(y) \notin L^1(\mathbb{R}^{2n})$ as function in (y,ξ), we cannot apply Fubini's theorem directly. Therefore we introduce a factor $\psi_\varepsilon(\xi) := e^{-\varepsilon^2|\xi|^2/2}$, $\varepsilon > 0$, which assures absolute integrability. Since $\lim_{\varepsilon\to 0} \psi_\varepsilon(\xi) = 1$ for all $x \in \mathbb{R}^n$, we have by Lebesgue's and Fubini's theorem

$$\begin{aligned} \mathcal{F}^{-1}[\hat{f}](x) &= \lim_{\varepsilon\to 0} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} \psi_\varepsilon(\xi) f(y) dy \right) d\xi \\ &= \lim_{\varepsilon\to 0} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} e^{-\varepsilon^2|\xi|^2} f(y) d(y,\xi). \end{aligned}$$

Using the substitution $\xi = \eta/\varepsilon$ and $y = x + \varepsilon z$,

$$\begin{aligned}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}e^{-\varepsilon^2|\xi|^2/2}f(y)d(y,\xi) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} e^{-iz\cdot\eta}e^{-|\eta|^2/2}d\eta\right)f(x+\varepsilon z)dz \\ &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-|z|^2/2}f(x+\varepsilon z)dz.\end{aligned}$$

Since f is continuous, we get

$$\lim_{\varepsilon\to 0}\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-|z|^2/2}f(x+\varepsilon z)dz = f(x)\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-|z|^2/2}dz = f(x)$$

by Lebesgue's theorem. Hence $\mathcal{F}^{-1}[\hat{f}](x) = f(x)$, which proves the lemma. ■

Remark 3.14 The technique of inserting a rapidly decreasing factor like $e^{-\varepsilon^2|\xi|^2/2}$, $\varepsilon > 0$, and passing to the limit afterwards will be fundamental in the following. It is the basis of the definition of the *oscillatority integrals* in Section 4.3 below.

Corollary 3.15 (Parseval's formula)
For every $f, g \in \mathcal{S}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} f(x)\overline{g(x)}dx = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi. \tag{3.7}$$

In particular,

$$\|f\|_2^2 = \frac{1}{(2\pi)^n}\|\hat{f}\|_2^2.$$

and $\mathcal{F}$ extends to an isomorphism $\mathcal{F}\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

Proof: Using Fubini's theorem, it is easy to check that

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\mathcal{F}[f](\xi)\overline{\hat{g}(\xi)}d\xi = \int_{\mathbb{R}^n} f(x)\overline{\mathcal{F}^{-1}[\hat{g}](x)}dx.$$

Because of Lemma 3.13, $g = \mathcal{F}^{-1}[\hat{g}]$, which implies (3.7). In particular, if $f = g$,

$$\frac{1}{(2\pi)^n}\|\mathcal{F}[f]\|_2^2 = \|f\|_2^2.$$

Since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ and obviously $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, $\mathcal{F}$ can be extended by continuity to a bounded operator $\mathcal{F}\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. By (3.6) the same is true for $\mathcal{F}^{-1}$. Moreover, $\mathcal{F}^{-1}[\mathcal{F}[f]] = f$ for all $f \in L^2(\mathbb{R}^n)$ since this is true for all $f \in \mathcal{S}(\mathbb{R}^n)$. ■

Remark 3.16 Because of the factor $(2\pi)^{-n}$ in the definition of the inverse Fourier transformation and in Parseval's formula (3.7), we introduce the common notation $đ\xi := \frac{d\xi}{(2\pi)^n}$. Then

$$\mathcal{F}^{-1}[g](x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} g(\xi)\, đ\xi$$

and

$$\int_{\mathbb{R}^n} f(x)\overline{g(x)}dx = \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, đ\xi.$$

In the following all integrals with respect to a *phase variable* $\xi, \eta, \ldots$ will be taken with respect to the scaled Lebesgue measure $đ\xi, đ\eta, \ldots$. All integrals with respect to a space variable $x, y, z, \ldots$ will be integrals using the usual Lebesgue measure $dx, dy, dz, \ldots$. Then all constants are part of the measure and we do not have to worry about them.

Lemma 3.17 (Fourier multipliers on $L^2(\mathbb{R}^n)$)
Let $m\colon \mathbb{R}^n \to \mathbb{C}$ be a measurable functions. Then

$$m(D_x)f := \mathcal{F}^{-1}[m(\xi)\hat{f}(\xi)]$$

is a well-defined bounded operator $m(D_x)\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ if and only if $m \in L^\infty(\mathbb{R}^n)$. Moreover, $\|m(D_x)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} = \|m\|_\infty$.

Proof: If $m \in L^\infty$, obviously $m(\xi)\hat{f}(\xi) \in L^2(\mathbb{R}^n)$ for every $f \in L^2(\mathbb{R}^n)$ because of Corollary 3.15. Moreover,

$$\|m(D_x)f\|_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^{\frac{n}{2}}}\|m(\xi)\hat{f}\|_2 \leq \frac{1}{(2\pi)^{\frac{n}{2}}}\|m\|_\infty\|\hat{f}\|_2 \leq= \|m\|_\infty\|f\|_2.$$

Hence $m(D_x)\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded linear operator and $\|m(D_x)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq \|m\|_\infty$.

For the converse implication it is sufficient to prove that a multiplication operator $M\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $(M\hat{f})(\xi) = m(\xi)\hat{f}(\xi)$ is bounded if and only if $m \in L^\infty(\mathbb{R}^n)$ and $\|M\|_{\mathcal{L}(L^2(\mathbb{R}^n))} = \|m\|_\infty$. We skip the details since we will not use the converse implication. See also [Ste70, Chapter II, Proposition 2]. ■

Examples 3.18 1. If $m(\xi) = \mathcal{F}[k]$, where $k \in L^1(\mathbb{R}^n)$ then $m \in C_b^0(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ due to Lemma 3.1. Moreover, $m(D_x)f = k * f$ and

$$\|m(D_x)f\|_2 \leq \|m\|_\infty\|f\|_2 \leq \|k\|_1\|f\|_2.$$

(Note that the latter inequality is also a simple consequence of the well-known estimate $\|k * f\|_p \leq \|k\|_1\|f\|_p$ for $1 \leq p \leq \infty$.)

2. Let $m_j(\xi) = \frac{i\xi_j}{|\xi|}$, $j = 1, \ldots, n$. Then $\|m_j\|_\infty = 1$ and $R_j := m_j(D_x)\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded operator. R_j is called *Riesz operator*. It can be shown that
$$R_j f = \lim_{\varepsilon \to 0} c_n \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)dy \quad = \text{"} k_j * f \text{"},$$
where $k_j(z) = z_j/|z|^{n+1}$, cf. [Èsk81, Chapter I, Example 3.4]. Note that the kernel $k_j \notin L^1(\mathbb{R}^n)$. These operators are typical examples of *singular integral operators*. If $n = 1$, $m_1(\xi) = \operatorname{sign} \xi$ and R_1 is also called *Hilbert transformation*.

3. The symbol of $-\Delta$ is $|\xi|^2$, i.e., $-\Delta f = \mathcal{F}^{-1}[|\xi|^2 \hat{f}(\xi)]$. Hence formally the inverse of $-\Delta$ is given by the operator $(-\Delta)^{-1} f = \mathcal{F}^{-1}[\hat{f}(\xi)/|\xi|^2]$ and therefore
$$\partial_j \partial_k (-\Delta)^{-1} f = \mathcal{F}^{-1}\left[\frac{i\xi_j}{|\xi|}\frac{i\xi_j}{|\xi|}\hat{f}(\xi)\right] = R_j R_k f,$$
where R_j, R_k are the Riesz operator defined above. Hence in some sense $\partial_j \partial_k (-\Delta)^{-1} f \in L^2(\mathbb{R}^n)$ if $f \in L^2(\mathbb{R}^n)$; but the derivatives do not have exist in the classical sense. (They have to be understood as distributional derivatives, which will be defined in the next section.)

3.5 Tempered Distributions and Fourier Transformation

Definition 3.19 The space *tempered distributions* is $\mathcal{S}'(\mathbb{R}^n) := (\mathcal{S}(\mathbb{R}^n))'$ – the space of linear and bounded functionals $f\colon \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$. A sequence $f_k \in \mathcal{S}'(\mathbb{R}^n)$ converges to $f \in \mathcal{S}'(\mathbb{R}^n)$ if and only if $\lim_{k\to\infty}\langle f_k, \varphi\rangle = \langle f, \varphi\rangle$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Here $\langle f, \varphi\rangle := f(\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ denotes the duality product.

Remarks 3.20 1. If $f\colon \mathbb{R}^n \to \mathbb{C}$ is a measurable function such that $\langle x\rangle^{-N} f(x) \in L^1(\mathbb{R}^n)$ for some $N \in \mathbb{N}_0$, then f is naturally identified with a tempered distribution $F_f \in \mathcal{S}(\mathbb{R}^n)'$ defined by
$$\langle F_f, \varphi\rangle := \int_{\mathbb{R}^n} f(x)\varphi(x)dx, \qquad \varphi \in \mathcal{S}(\mathbb{R}^n).$$
Moreover,
$$|\langle F_f, \varphi\rangle| \leq \|\langle x\rangle^{-N} f\|_1 |\varphi|'_{N,\mathcal{S}} \leq C\|\langle x\rangle^{-N} f\|_1 |\varphi|_{N,\mathcal{S}}, \qquad \varphi \in \mathcal{S}(\mathbb{R}^n),$$
cf. Remark 3.4. In the following we will identify the function f with its associated distribution F_f and will denote F_f simply by f.

2. If $f\colon \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is linear mapping, then by definition f is bounded if an only if there is an $m \in \mathbb{N}_0$ and a constant $C > 0$ such that
$$|\langle f, \varphi\rangle| \leq C|\varphi|_{m,\mathcal{S}}, \qquad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Example 3.21 The famous delta distribution $\delta \in \mathcal{S}'(\mathbb{R}^n)$ is defined by

$$\langle \delta, \varphi \rangle := \varphi(0), \qquad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

One of the most important properties of distributions is that they can be differentiated infinitely many times (in a generalized sense).

Definition 3.22 Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then the distributional derivative $\partial^\alpha f \in \mathcal{S}'(\mathbb{R}^n)$ of f is the distribution defined by

$$\langle \partial^\alpha f, \varphi \rangle := (-1)^{|\alpha|} \langle f, \partial^\alpha \varphi \rangle \qquad \text{for } \ \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Remarks 3.23

1. Recall that the adjoint $T' \colon Y' \to X'$ of a linear operator $T \colon X \to Y$ is defined by $\langle T'y', x \rangle := \langle y', Tx \rangle$ for $y' \in Y'$ and $x \in X$. Hence the derivative $\partial_x^\alpha \colon \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is the adjoint of $(-1)^{|\alpha|} \partial_x^\alpha \colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$.

2. If $f \colon \mathbb{R}^n \to \mathbb{C}$ is a k-times continuously differentiable function with $\langle x \rangle^{-N} f \in L^1(\mathbb{R}^n)$, then f can be naturally considered as a tempered distribution, cf. Remark3.20.1, Moreover, the distributional derivatives up to order k coincides with the usual derivative, i.e.

$$\langle \partial^\alpha f, \varphi(x) \rangle = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(x) \partial^\alpha \varphi(x) dx = \int_{\mathbb{R}^n} \partial^\alpha f(x) \varphi(x) dx$$

   for all $|\alpha| \leq k$ due to integration by parts.

3. Since every measurable function $f \colon \mathbb{R}^n \to \mathbb{C}$ such that $\langle x \rangle^{-N} f(x) \in L^1(\mathbb{R}^n)$ for some $N \in \mathbb{N}_0$ can be considered as a distribution, it can be differentiated infinitely many times in the sense of distributions although it may even not be continuous.

Examples 3.24

1. Let f be the *Heavyside function*, i.e. $f(x) = 1$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$. Then $f \in \mathcal{S}'(\mathbb{R})$ and the distributional derivative f' is

$$\langle f', \varphi \rangle = -\int_{\mathbb{R}} f(x) \varphi'(x) dx = -\int_0^\infty \varphi'(x) dx = \varphi(0)$$

   for all $\varphi \in \mathcal{S}(\mathbb{R})$. Hence $f' = \delta$ is the delta distribution.

2. By definition the distributional derivative of the delta distribution is

$$\langle \partial^\alpha \delta, \varphi \rangle = (-1)^{|\alpha|} \langle \delta, \partial^\alpha \varphi \rangle = (-1)^{|\alpha|} \partial^\alpha \varphi(0).$$

Definition 3.25 Let $f \in \mathcal{S}'(\mathbb{R}^n)$. The Fourier transform $\mathcal{F}[f]$ of f is defined as the distribution

$$\langle \mathcal{F}[f], \varphi \rangle = \langle f, \bar{\mathcal{F}}[\varphi] \rangle \qquad \text{for } \ \varphi \in \mathcal{S}(\mathbb{R}^n),$$

where

$$\bar{\mathcal{F}}[\varphi](\xi) := \int_{\mathbb{R}^n} e^{ix \cdot \xi} f(x) dx$$

is the *conjugate Fourier transformation.*

Proposition 3.26 The Fourier transformation is a continuous linear mapping $\mathcal{F}\colon \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. Moreover, $\mathcal{F}$ is a bijection on $\mathcal{S}'(\mathbb{R}^n)$.

Proof: Since $\bar{\mathcal{F}}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a continuous linear operator and

$$\langle \mathcal{F}[f], \varphi \rangle = \langle f, \bar{\mathcal{F}}[\varphi] \rangle = (f \circ \bar{\mathcal{F}})(\varphi),$$

$\mathcal{F}[f] = f \circ \bar{\mathcal{F}}\colon \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is a continuous linear operator. Moreover, if $f_k \to f$ in $\mathcal{S}'(\mathbb{R}^n)$ as $k \to \infty$, then

$$\langle \mathcal{F}[f_k], \varphi \rangle = \langle f_k, \bar{\mathcal{F}}[\varphi] \rangle \to \langle f, \bar{\mathcal{F}}[\varphi] \rangle = \langle \mathcal{F}[f], \varphi \rangle$$

as $k \to \infty$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Hence $\mathcal{F}\colon \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is continuous.

Finally, if we define $\mathcal{F}^{-1}\colon \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ by

$$\langle \mathcal{F}^{-1}[f], \varphi \rangle := \langle f, \bar{\mathcal{F}}^{-1}[\varphi] \rangle, \qquad \varphi \in \mathcal{S}(\mathbb{R}^n),$$

$\mathcal{F}^{-1}$ has the same properties as $\mathcal{F}$ and obviously $\mathcal{F}^{-1}[\mathcal{F}[f]] = f$ for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

■

Remark 3.27 As differentiation the Fourier transformation is defined by duality, i.e., $\mathcal{F}\colon \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is defined as th adjoint of $\bar{\mathcal{F}}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$.

Example 3.28 The Fourier transform of the delta distribution δ can be simply calculated:

$$\langle \mathcal{F}[\delta], \varphi \rangle = \langle \delta, \bar{\mathcal{F}}[\varphi] \rangle = \bar{\mathcal{F}}[\varphi](0) = \int_{\mathbb{R}^n} \varphi(x)dx = \langle 1, \varphi \rangle$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Hence $\mathcal{F}[\delta] = 1$.

4 Basic Calculus of Pseudodifferential Operators

4.1 Symbol Classes and Basic Properties

We now introduce the basic pseudodifferential symbol class $S^m_{1,0}$. There are many other more general or modified symbol classes, which are used in the literature and research for different purposes. But the following symbol class is the most simple and most common. It is a natural symbol class that contains the symbols of differential operators with smooth coefficients.

Definition 4.1 Let $m \in \mathbb{R}$, $n, N \in \mathbb{N}$. Then $S^m_{1,0}(\mathbb{R}^N \times \mathbb{R}^n)$ is the vector-space of all smooth functions $p\colon \mathbb{R}^N \times \mathbb{R}^n \to \mathbb{C}$ such that

$$|D_\xi^\alpha D_x^\beta p(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|} \tag{4.1}$$

holds for all $\alpha \in \mathbb{N}_0^n$, $\beta \in \mathbb{N}_0^N$, where $C_{\alpha,\beta}$ is independent of $x \in \mathbb{R}^N, \xi \in \mathbb{R}^n$. The function p is called *pseudodifferential symbol* and m is called the *order* of p. Moreover,

$$S^\infty_{1,0}(\mathbb{R}^N \times \mathbb{R}^n) := \bigcup_{m\in\mathbb{R}} S^m_{1,0}(\mathbb{R}^N \times \mathbb{R}^n) \quad \text{and} \quad S^{-\infty}_{1,0}(\mathbb{R}^N \times \mathbb{R}^n) := \bigcap_{m\in\mathbb{R}} S^m_{1,0}(\mathbb{R}^N \times \mathbb{R}^n).$$

For short we also write $S^m_{1,0}$ instead of $S^m_{1,0}(\mathbb{R}^N \times \mathbb{R}^n)$.

Remark 4.2 In the following we usually deal with the case $N = n$. But sometimes it is useful to have defined the symbol classes for general $N, n \in \mathbb{N}$.

If $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ is a symbol, then

$$p(x, D_x)f := \mathrm{OP}(p)f := \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\hat{f}(\xi)\,đ\xi \tag{4.2}$$

defines the associated *pseudodifferential operator*, where $f\colon \mathbb{R}^n \to \mathbb{C}$ is a suitable function. If $f \in \mathcal{S}(\mathbb{R}^n)$, then $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$ and therefore $p(x,\xi)\hat{\xi} \in \mathcal{S}(\mathbb{R}^n)$ for every fixed $x \in \mathbb{R}^n$. Therefore the integral in (4.2) exists and $p(x, D_x)f$ is well-defined. In the following we will prove that $p(x, D_x)\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a continuous mapping. But before we prove this fact, we discuss some examples and make some simple observations.

Examples 4.3 1. Let $p(x,\xi) = \sum_{|\alpha|\leq m} c_\alpha(x)\xi^\alpha$ be a polynomial in ξ of order $m \in \mathbb{N}_0$ with smooth coefficients $c_\alpha(x) \in C_b^\infty(\mathbb{R}^n)$. Then $p \in S^m_{1,0}$ and

$$p(x, D_x)f = \sum_{|\alpha|\leq m} c_\alpha(x) D_x^\alpha f$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$. Hence every linear differential operator with smooth and bounded coefficients is a pseudodifferential operator. In particular the Laplacian $\Delta = \partial_1^2 + \ldots + \partial_n^2$ is a pseudodifferential operator with symbol $-|\xi|^2 = -\xi_1^2 - \ldots - \xi_n^2$.

2. The function $\langle\xi\rangle := \sqrt{1+|\xi|^2}$ is a pseudodifferential symbol of order 1, see Exercise 4.1. Since $1+|\xi|^2$ is the symbol of $1-\Delta$, the associated pseudodifferential operator
$$\langle D_x\rangle f = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\sqrt{1+|\xi|^2}\hat{f}(\xi)d\xi$$
can be considered as the *square root of* $1-\Delta$. For short: $\langle D_x\rangle = \sqrt{1-\Delta}$.
More generally, $\langle\xi\rangle^m \in S^m_{1,0}$ for every $m\in\mathbb{R}$ and $\langle D_x\rangle^m = (1-\Delta)^{\frac{m}{2}}$.

Exercise 5 *Let* $\langle\xi\rangle = \sqrt{1+|\xi|^2}$*. Prove that* $\langle\xi\rangle^m \in S^m_{1,0}$ *for every* $m\in\mathbb{R}$*.*
Hint: *Consider the function* $f(a,x) := (a^2+|x|^2)^{\frac{m}{2}}$*, where* $a\in\mathbb{R}$*,* $x\in\mathbb{R}^n$*. Use that* f *is* homogeneous of degree m*, i.e.,* $f(ra,rx) = r^m f(a,x)$ *for all* $r>0$*,* $a\in\mathbb{R}$*,* $x\in\mathbb{R}^n$*.*

Exercise 6 *Let* $p\colon \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}$ *be a smooth function which is homogeneous in* ξ *of degree* $m\in\mathbb{R}$ *for* $|\xi|\geq 1$*, i.e.* $p(x,r\xi) = r^m p(x,\xi)$ *for all* $|\xi|\geq 1$ *and* $r\geq 1$*. Moreover, let* $\partial_\xi^\alpha\partial_x^\beta p \in C_b^\infty(\mathbb{R}^n)$ *w.r.t.* x *for all* $\alpha,\beta\in\mathbb{N}_0^n$*. Prove that* $p\in S^m_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$*.*

$$c\langle\xi\rangle \leq (1+|\xi|) \leq C\langle\xi\rangle, \qquad \xi\in\mathbb{R}^n,$$

We can define a sequence of semi-norms on $S^m_{1,0}$, which is related to the family of inequalities (4.1) in a natural way. Let
$$|p|_k^{(m)} := \max_{|\alpha|,|\beta|\leq k}\sup_{x,\xi\in\mathbb{R}^n} |D_\xi^\alpha D_x^\beta p(x,\xi)|(1+|\xi|)^{-m+|\alpha|} \tag{4.3}$$
for $k\in\mathbb{N}$. Here $\sup_{x,\xi\in\mathbb{R}^n}|D_\xi^\alpha D_x^\beta p(x,\xi)|(1+|\xi|)^{-m+|\alpha|}$ is the smallest constant $C_{\alpha,\beta}$ such that (4.1) holds for all $x,\xi\in\mathbb{R}^n$ and fixed $\alpha,\beta\in\mathbb{N}_0$.

It is not difficult to check that $S^m_{1,0}$ with these semi-norms is a Fréchet space.

Remark 4.4 In the literature the function $(1+|\xi|)$ in the estimates (4.1) is often replaced by $\langle\xi\rangle = \sqrt{1+|\xi|^2}$. This can be done without changing the symbol classes since
$$\sqrt{1+|\xi|^2} \leq (1+|\xi|) \leq \sqrt{2}\sqrt{1+|\xi|^2},$$
cf. proof of Lemma 4.7 below. Using $\langle\xi\rangle$ instead of $(1+|\xi|)$, the notation becomes a bit shorter.

Proposition 4.5 Let $p_j\in S^{m_j}_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$, $m_j\in\mathbb{R}$, $j=1,2$. Then $p(x,\xi) := p_1(x,\xi)p_2(x,\xi)\in S^{m_1+m_2}_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$ and $|p|_k^{(m_1+m_2)} \leq C_k|p_1|_k^{(m_1)}|p_2|_k^{(m_2)}$, where C_k depends only on k and n.

The lemma is a simple consequence of the Leibniz formula. The main result of this section is

THEOREM 4.6 *Let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$, be a pseudodifferential symbol. Then*

$$p(x, D_x)\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

is a continuous mapping. More precisely, $|p(x, D_x)f|_{k,\mathcal{S}} \leq C_k |p|^{(m)}_k |f|_{m+2(n+1)+k,\mathcal{S}}$ for all $k \in \mathbb{N}$ and $f \in \mathcal{S}(\mathbb{R}^n)$.

Proof: Since $f \in \mathcal{S}(\mathbb{R}^n)$, $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$ due to Lemma 3.5. First of all,

$$\begin{aligned}
\sup_{x\in\mathbb{R}^n} |p(x, D_x)f(x)| &\leq \sup_{x\in\mathbb{R}^n} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \langle\xi\rangle^{-n-1} |\langle\xi\rangle^{-m} p(x,\xi)| |\langle\xi\rangle^{n+m+1} \hat{f}(\xi) d\xi \\
&\leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \langle\xi\rangle^{-n-1} d\xi |p|^{(m)}_0 \|\langle\xi\rangle^{n+m+1}\hat{f}\|_\infty \\
&\leq C|p|^{(m)}_0 |\hat{f}|_{m+n+1,\mathcal{S}} \leq C|p|^{(m)}_0 |f|_{m+2n+2,\mathcal{S}} \qquad (4.4)
\end{aligned}$$

by Lemma 2.6 and Lemma 3.5, where C depends only on the dimension. Moreover, we note that (4.4) holds for arbitrary $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ and $m \in \mathbb{R}$ — not just for the one in the assumption of this theorem.

In order to estimate the derivatives, we calculate

$$\begin{aligned}
\partial_{x_j}(p(x, D_x)f(x)) &= \partial_{x_j} \int e^{ix\cdot\xi} p(x,\xi)\hat{f}(\xi) đ\xi \\
&= \int e^{ix\cdot\xi} p(x,\xi) i\xi_j \hat{f}(\xi) đ\xi + \int e^{ix\cdot\xi} \partial_{x_j} p(x,\xi)\hat{f}(\xi) đ\xi \\
&= p(x, D_x)(\partial_j f)(x) + (\partial_{x_j} p)(x, D_x) f,
\end{aligned}$$

where have applied Theorem 2.4. Hence using (4.4) first with f replaced by $\partial_j f$ and then p replaced by $\partial_{x_j} p$, we obtain

$$\begin{aligned}
\sup_{x\in\mathbb{R}^n} |\partial_j (p(x, D_x)f(x)| &\leq C\left(|p|^{(m)}_0 |\partial_j f|_{m+2n+2,\mathcal{S}} + |\partial_{x_j} p|^{(m)}_0 |f|_{m+2n+2,\mathcal{S}}\right) \\
&\leq C|p|^{(m)}_1 |f|_{m+2n+3,\mathcal{S}}. \qquad (4.5)
\end{aligned}$$

Similarly,

$$\begin{aligned}
ix_j p(x, D_x)f(x) &= \int (\partial_{\xi_j} e^{ix\cdot\xi}) p(x,\xi)\hat{f}(\xi) đ\xi \\
&= \int e^{ix\cdot\xi} p(x,\xi) \partial_j \hat{f}(\xi) đ\xi + \int e^{ix\cdot\xi} (\partial_{\xi_j} p)(x,\xi)\hat{f}(\xi) đ\xi \\
&= p(x, D_x)(ix_j f(x)) + (\partial_{\xi_j} p)(x, D_x) f
\end{aligned}$$

and therefore

$$\begin{aligned}
\sup_{x\in\mathbb{R}^n} |x_j p(x, D_x)f(x)| &\leq C\left(|p|^{(m)}_0 |x_j f|_{m+2n+2,\mathcal{S}} + |\partial_{\xi_j} p|^{(m-1)}_0 |f|_{m+2n+1,\mathcal{S}}\right) \\
&\leq C|p|^{(m)}_1 |f|_{m+2n+3,\mathcal{S}} \qquad (4.6)
\end{aligned}$$

by (4.4), where we note that $\partial_{\xi_j} p(x,\xi)$ is of order $m-1$ and $|\partial_{\xi_j} p|_0^{(m-1)} \leq |p|_1^{(m)}$ by definition of the semi-norms.

Using (4.5) and (4.6), one can easily prove by induction that

$$\sup_{x\in\mathbb{R}^n} |x^\alpha \partial_x^\beta p(x,D_x)f(x)| \quad \leq \quad C_{\alpha,\beta}|p|^{(m)}_{|\alpha|+|\beta|}|f|_{m+2(n+1)+|\alpha|+|\beta|,\mathcal{S}}$$

uniformly in $x \in \mathbb{R}^n$ for all $\alpha, \beta \in \mathbb{N}_0^n$. Hence $|p(x,D_x)f|_{k,\mathcal{S}} \leq C_k|p|_k^{(m)}|f|_{m+2(n+1)+k,\mathcal{S}}$, which proves the theorem. ■

Finally, we prove the following simple but important inequality:

Lemma 4.7 (Peetre's inequality)
Let $\langle\xi\rangle := (1+|\xi|^2)^{\frac{1}{2}}$, $\xi \in \mathbb{R}^n$. *Then for all* $s \in \mathbb{R}$

$$\langle\xi\rangle^s \leq 2^{|s|}\langle\xi-\eta\rangle\langle\eta\rangle, \qquad \xi,\eta \in \mathbb{R}^n.$$

Proof: First of all we have

$$\langle\xi\rangle^2 = (1+|\xi|^2) \leq (1+|\xi|)^2 \leq (1+|\xi|)^2 + (1-|\xi|)^2 = 2(1+|\xi|^2).$$

Hence

$$\langle\xi\rangle \leq (1+|\xi|) \leq \sqrt{2}\langle\xi\rangle. \tag{4.7}$$

First let $s \geq 0$. Then the triangle inequality implies

$$(1+|\xi|) \leq (1+|\xi-\eta|+|\eta|) \leq (1+|\xi-\eta|)(1+|\eta|)$$

and therefore

$$\langle\xi\rangle^s \leq (1+|\xi-\eta|)^s(1+|\eta|)^s \leq 2^s\langle\xi-\eta\rangle^s\langle\eta\rangle^s$$

by (4.7). If $s<0$, we can use the previous inequality with interchanged role of ξ and η and s replaced by $-s$ to conclude

$$\langle\eta\rangle^{-s} \leq 2^{-s}\langle\xi-\eta\rangle^{-s}\langle\xi\rangle^{-s}$$

which is equivalent to

$$\langle\xi\rangle^s \leq 2^{|s|}\langle\eta\rangle^{|s|}\langle\xi-\eta\rangle^{-s}.$$

■

4.2 Composition of Pseudodifferential Operators: Motivation

Because of Theorem 4.6, the composition of two pseudodifferential operators $p_1(x, D_x)$ and $p_2(x, D_x)$ is a well-defined bounded operator

$$p_1(x, D_x)p_2(x, D_x) \colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n).$$

The natural question arises if this operator is again a pseudodifferential operator, i.e., if there is a symbol $p \in S^{\infty}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$p(x, D_x) = p_1(x, D_x)p_2(x, D_x).$$

If this is the case, it is of interest how the symbol $p(x, \xi)$ is related to the symbols $p_1(x, \xi)$ and $p_2(x, \xi)$.

The behavior of pseudodifferential operators under composition is of particular interest for calculation inverses or at least approximate inverses of pseudodifferential operators, which are also called *parametrices.*

In order to motivate the following sections, we calculate the composition of $p_1(x, D_x)p_2(x, D_x)$ formally, ignoring all technical difficulties. First of all,

$$p_1(x, D_x)g = \iint e^{i(x-y)\cdot\eta} p_1(x, \eta) g(y) dy đ\eta$$

and

$$p_2(x, D_x)f|_{x=y} = \iint e^{i(y-z)\cdot\xi} p_2(y, \xi) f(z) dz đ\xi.$$

Hence we get for $g(y) = p_2(x, D_x)f|_{x=y}$

$$\begin{aligned} p_1(x, D_x)p_2(x, D_x)f &= \iint e^{i(x-y)\cdot\eta} p_1(x, \eta) \left(\iint e^{i(y-z)\cdot\xi} p_2(y, \xi) f(z) dz đ\xi \right) dy đ\eta \\ &= \iiiint e^{i(x-z)\cdot\xi} e^{-i(x-y)\cdot(\xi-\eta)} p(x, \eta) q(y, \xi) f(z) dy đ\eta dz đ\xi. \end{aligned}$$

Using the substitution $x' = x - y$ and $\xi' = \xi - \eta$, we obtain

$$\begin{aligned} &p_1(x, D_x)p_2(x, D_x)f \\ &= \iint e^{i(x-z)\cdot\xi} \left(\iint e^{-ix'\cdot\xi'} p_1(x, \xi + \xi') p_2(x + x', \xi) dx' đ\xi' \right) f(z) dz đ\xi \end{aligned}$$

Hence formally the symbol of $p_1(x, D_x)p_2(x, D_x)$ is

$$(p_1 \# p_2)(x, \xi) := \iint e^{-ix'\cdot\xi'} p_1(x, \xi + \xi') p_2(x + x', \xi) dx' đ\xi'. \tag{4.8}$$

But the main problem is that the latter integral in general does not exists in the classical sense. We will define it as so called *oscillatority integral*:

$$\begin{aligned} &\text{Os}-\iint e^{-ix'\cdot\xi'} p_1(x, \xi + \xi') p_2(x + x', \xi) dx' đ\xi' := \\ &\lim_{\varepsilon \to 0} \iint \chi(\varepsilon x', \varepsilon \xi') e^{-ix'\cdot\xi'} p_1(x, \xi + \xi') p_2(x + x', \xi) dx' đ\xi', \end{aligned}$$

where $\chi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ with $\chi(0,0) = 1$. In the following section we prove that the oscillatority integral are well-defined for suitable integrands. Moreover, we will observe some elementary properties, which justify our formal calculations above.

4.3 Oscillatority Integrals

Definition 4.8 The space of *amplitudes* $\mathcal{A}^m_\tau(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$, $\tau \geq 0$, is the set of smooth functions $a\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ such that

$$|\partial^\alpha_\eta \partial^\beta_y a(y,\eta)| \leq C_{\alpha,\beta}(1+|\eta|)^m(1+|y|)^\tau$$

uniformly in $y, \eta \in \mathbb{R}^n$ for all $\alpha, \beta \in \mathbb{N}^n_0$. Moreover, let

$$|a|_{\mathcal{A}^m_\tau,k} := \max_{|\alpha|+|\beta|\leq k} \sup_{x,\xi\in\mathbb{R}^n} (1+|\eta|)^{-m}(1+|y|)^{-\tau}|\partial^\alpha_\eta \partial^\beta_y a(y,\eta)|,$$

$k \in \mathbb{N}$, be the associated sequence of monotone increasing semi-norms.

It is not difficult to check that $\mathcal{A}^m_\tau(\mathbb{R}^n \times \mathbb{R}^n)$ is a Fréchet space. Moreover, $S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) \subset \mathcal{A}^m_0(\mathbb{R}^n \times \mathbb{R}^n)$ with continuous embedding.

THEOREM 4.9 *Let $a \in \mathcal{A}^m_\tau(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$, $\tau \geq 0$, and let $\chi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ with $\chi(0,0) = 1$. Then*

$$\text{Os–}\iint e^{-iy\cdot\eta} a(y,\eta)\,dy\,đ\eta := \lim_{\varepsilon\to 0} \iint \chi(\varepsilon y, \varepsilon\eta) e^{-iy\cdot\eta} a(y,\eta)\,dy\,đ\eta$$

exists and

$$\text{Os–}\iint e^{-iy\cdot\eta} a(y,\eta)\,dy\,đ\eta = \iint e^{-iy\cdot\eta} \langle y\rangle^{-2l'}\langle D_\eta\rangle^{2l'} \left[\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l} a(y,\eta)\right] dy\,đ\eta, \tag{4.9}$$

where $l, l' \in \mathbb{N}_0$ are chosen such that $2l > n+m$ and $2l' > n+\tau$. In particular, the definition does not depend on the choice of χ and

$$\left|\text{Os–}\iint e^{-iy\cdot\eta} a(y,\eta)\,dy\,đ\eta\right| \leq C_m \|a\|_{\mathcal{A}^m_\tau,2(l+l')}. \tag{4.10}$$

Proof: Using $D^\alpha_y e^{-iy\cdot\eta} = (-\eta)^\alpha e^{-iy\cdot\eta}$ and $D^\beta_\eta e^{-iy\cdot\eta} = (-y)^\beta e^{-iy\cdot\eta}$ for $\alpha, \beta \in \mathbb{N}^n_0$, we have

$$\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l} e^{-iy\cdot\eta} \quad \text{and} \quad \langle y\rangle^{-2l'}\langle D_\eta\rangle^{2l'} e^{-iy\cdot\eta} \quad . \tag{4.11}$$

Since $\chi(\varepsilon y, \varepsilon\eta) \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ for fixed $\varepsilon > 0$, we can integrate by parts and obtain

$$\begin{aligned}
I_\varepsilon &:= \iint \chi(\varepsilon y, \varepsilon\eta) e^{iy\cdot\eta} a(y,\eta)\,dy\,đ\eta \\
&= \iint e^{iy\cdot\eta}\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}\left(\chi(\varepsilon y,\varepsilon\eta)a(y,\eta)\right) dy\,đ\eta \\
&= \iint e^{iy\cdot\eta}\langle y\rangle^{-2l'}\langle D_\eta\rangle^{2l'}\left[\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}\left(\chi(\varepsilon y,\varepsilon\eta)a(y,\eta)\right)\right] dy\,đ\eta
\end{aligned}$$

On the other hand, $\{\chi(\varepsilon y, \varepsilon\eta)\}_{0<\varepsilon<1}$ is bounded in $\mathcal{A}^0_0 = C^\infty_b(\mathbb{R}^n \times \mathbb{R}^n)$ and $\lim_{\varepsilon\to 0}\chi(\varepsilon y, \varepsilon\eta) = 1$ in $C^\infty_b(\mathbb{R}^n \times \mathbb{R}^n)$, cf. Exercise **??**. Hence there are constants $C_{\alpha,\beta}$ independent of both $0 < \varepsilon < 1$ and $a \in \mathcal{A}^m_\tau$ such that

$$\left|\partial_y^\alpha \partial_\eta^\beta \left(\chi(\varepsilon y, \varepsilon\eta) a(y,\varepsilon)\right)\right| \leq C_{\alpha,\beta}|a|_{|\alpha|+|\beta|}\langle\eta\rangle^m \langle y\rangle^\tau. \tag{4.12}$$

Moreover, since $\langle\xi\rangle^s \in S^s_{1,0}$,

$$|\partial^\alpha_\eta \langle\eta\rangle^s| \leq C_{s,\alpha}\langle\eta\rangle^{s-|\alpha|}. \tag{4.13}$$

Combining (4.12) and (4.13), there are constants $C_{l,\alpha}$ independent of $0 < \varepsilon < 1$ such that

$$\left|\partial^\alpha_\eta \left[\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}\left(\chi(\varepsilon y, \varepsilon\eta)a(y,\varepsilon)\right)\right]\right| \leq C_{l,\alpha}|a|_{2l+|\alpha|}\langle\eta\rangle^{m-2l}\langle y\rangle^\tau.$$

Consequently there are constants $C_{l,l'}$ independent of $0 < \varepsilon < 1$ and a such that

$$\left|\langle y\rangle^{-2l'}\langle D_\eta\rangle^{2l'}\left[\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}\left(\chi(\varepsilon y, \varepsilon\eta)a(y,\varepsilon)\right)\right]\right| \leq C_{l,l'}|a|_{2(l+l')}\langle\eta\rangle^{m-2l}\langle y\rangle^{\tau-2l'}. \tag{4.14}$$

If we now choose $2l > n+m$ and $2l' > n+\tau$, then $\langle\eta\rangle^{m-2l}\langle y\rangle^{\tau-2l'} \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$, cf. Lemma 2.6. Hence the Lebesgue dominated convergence theorem, $\lim_{\varepsilon\to 0}\chi(\varepsilon y, \varepsilon\eta) = 1$, and $\lim_{\varepsilon\to 0}\partial^\alpha_y\partial^\beta_\eta\chi(\varepsilon y, \varepsilon\eta) = 0$ for $\alpha+\beta > 0$ imply

$$\lim_{\varepsilon\to 0} I_\varepsilon = \iint e^{iy\cdot\eta}\langle y\rangle^{-2l'}\langle D_\eta\rangle^{2l'}\left[\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}a(y,\eta)\right] dy\, đ\eta.$$

Thus the limit in the definition of the oscillatority integral exists and (4.9) holds. Moreover, the representation (4.9) shows that the definition does not depend on the choice of χ. Finally, passing to the limit $\varepsilon \to 0$ in (4.14), (4.10) follows from (4.9) and Lemma 2.6. ■

Corollary 4.10 *Let $a_j \in \mathcal{A}^m_\tau$ be a bounded sequence such that $\lim_{j\to\infty}\partial^\alpha_y\partial^\beta_\eta a_j(y,\eta) = \partial^\alpha_y\partial^\beta_\eta a(y,\eta)$ for every $y,\eta \in \mathbb{R}^n$, $\alpha,\beta \in \mathbb{N}^n_0$ and $a \in \mathcal{A}^m_\tau$. Then*

$$\lim_{j\to\infty}\text{Os-}\iint e^{iy\cdot\eta}a_j(y,\eta)dy\, đ\eta = \text{Os-}\iint e^{iy\cdot\eta}a(y,\eta)dy\, đ\eta.$$

Proof: The assumptions imply that

$$\lim_{j\to\infty}\langle y\rangle^{-2l'}\langle D_\eta\rangle^{2l'}\left[\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}a_j(y,\eta)\right] = \langle y\rangle^{-2l'}\langle D_\eta\rangle^{2l'}\left[\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}a(y,\eta)\right]$$

for every $y,\eta \in \mathbb{R}^n$ (pointwise). Moreover, (4.14) implies

$$\left|\langle y\rangle^{-2l'}\langle D_\eta\rangle^{2l'}\left[\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}\left(\chi(\varepsilon y, \varepsilon\eta)a_j(y,\varepsilon)\right)\right]\right| \leq C_{l,l'}|a_j|_{2(l+l')}\langle\eta\rangle^{m-2l}\langle y\rangle^{\tau-2l'}.$$

Since the sequence a_j is bounded in $\mathcal{A}^m_\tau$, $|a_j|_{2(l+l')} \leq C$ uniformly in $j \in \mathbb{N}$. Hence the representation (4.9) and the Lebesgue dominated convergence theorem imply the statement of the theorem. ■

Example 4.11 Let $u \in C_b^\infty(\mathbb{R}^n)$. Then $a(y,\eta) = e^{ix\cdot\eta}u(y) \in \mathcal{A}_0^0$ and

$$\text{Os}-\iint e^{i(x-y)\cdot\eta}u(y)dy\,đ\eta$$

is well-defined. We can calculate the oscillatority integral explicitely: Let $\chi(y,\eta) = \psi(y)\psi(\eta)$, where $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\psi(0) = 1$. Then

$$\begin{aligned}
\text{Os}-\iint e^{i(x-y)\cdot\eta}u(y)dy\,đ\eta &= \lim_{\varepsilon\to 0}\int \psi(\varepsilon y)u(y)\left(\int e^{i(x-y)\cdot\eta}\psi(\varepsilon\eta)\,đ\eta\right)dy \\
&= \lim_{\varepsilon\to 0}\int \psi(\varepsilon y)u(y)\varepsilon^{-n}\mathcal{F}^{-1}[\psi]\left(\frac{x-y}{\varepsilon}\right)dy \\
&= \lim_{\varepsilon\to 0}\int \psi(\varepsilon(x-\varepsilon y')u(x-\varepsilon y')\mathcal{F}^{-1}[\psi](y')dy' \\
&= \int u(x)\mathcal{F}^{-1}[\psi](y')dy' = u(x)\mathcal{F}[\mathcal{F}^{-1}[\psi]](0) \\
&= u(x)\psi(0) = u(x)
\end{aligned}$$

since $\lim_{\varepsilon\to 0}\psi(\varepsilon(x-\varepsilon y')u(x-\varepsilon y') = \psi(0)u(x) = u(x)$. Thus formally we have $\mathcal{F}^{-1}[\mathcal{F}[u]] = u$ for all $u \in C_b^\infty(\mathbb{R}^n)$.

Lemma 4.12 *Let $a \in A_\tau^m$, $m \in \mathbb{R}$, $\tau \geq 0$ and let $\alpha \in \mathbb{N}_0^n$. Then*

$$\begin{aligned}
\text{Os}-\iint e^{iy\cdot\eta}y^\alpha a(y,\eta)dy\,đ\eta &= \text{Os}-\iint e^{iy\cdot\eta}D_\eta^\alpha a(y,\eta)dy\,đ\eta, \\
\text{Os}-\iint e^{iy\cdot\eta}\eta^\alpha a(y,\eta)dy\,đ\eta &= \text{Os}-\iint e^{iy\cdot\eta}D_y^\alpha a(y,\eta)dy\,đ\eta.
\end{aligned}$$

Proof: First of all we note that $y^\alpha a(y,\eta) \in A_{\tau+|\alpha|}^m$, $\eta^\alpha a(y,\eta) \in A_\tau^{m+|\alpha|}$, and $D_\eta^\alpha a(y,\eta), D_y^\alpha a(y,\eta) \in A_\tau^m$. Therefore the oscillatority integrals are well-defined.

We only prove the first identity since the proof of the second is done in the same way. Moreover, it is sufficient to consider the case $|\alpha| = 1$. (Then the general case follows by induction.)

If $|\alpha| = 1$, then $y^\alpha = y_j$ for $1 \leq j \leq n$. Moreover, we choose χ in the definition of the oscillatority integral as $\chi(y,\eta) = e^{-|(y,\eta)|^2/2}$. Then

$$\begin{aligned}
\iint \chi(\varepsilon y,\varepsilon\eta)e^{iy\cdot\eta}y_j a(y,\eta)dy\,đ\eta &= \iint \chi(\varepsilon y,\varepsilon\eta)(D_{y_j}e^{iy\cdot\eta})a(y,\eta)dy\,đ\eta \\
&= \iint e^{iy\cdot\eta}D_{\eta_j}\left(\chi(\varepsilon y,\varepsilon\eta)a(y,\eta)\right)dy\,đ\eta
\end{aligned}$$

Using $D_{\eta_j}\chi(\varepsilon y,\varepsilon\eta) = -i\varepsilon\eta_j\chi(\varepsilon y,\varepsilon\eta)$, we obtain

$$D_{y_j}\left(\chi(\varepsilon y,\varepsilon\eta)a(y,\eta)\right) = \big(\chi(\varepsilon y,\varepsilon\eta)D_{y_j}a(y,\eta) + i\varepsilon\chi(\varepsilon y,\varepsilon\eta)\eta_j a(y,\eta)\big).$$

Therefore

$$\iint \chi(\varepsilon y, \varepsilon\eta) e^{iy\cdot\eta} y_j a(y,\eta) dy\, đ\eta =$$

$$\iint \chi(\varepsilon y, \varepsilon\eta) e^{iy\cdot\eta} D_{\eta_j} a(y,\eta) dy\, đ\eta - i\varepsilon \iint \chi(\varepsilon y, \varepsilon\eta) e^{iy\cdot\eta} \eta_j a(y,\eta) dy\, đ\eta.$$

Passing to the limit $\varepsilon \to 0$ yields the first equality. ■

THEOREM 4.13 (Fubini's theorem for oscillatory integrals)
Let $a \in A^m_\tau(\mathbb{R}^{n+k} \times \mathbb{R}^{n+k})$, $n, k \in \mathbb{N}$. Then

$$b(y,\eta) := \mathrm{Os}\text{–}\iint e^{-iy'\cdot\eta'} a(y,y',\eta,\eta') dy'\, đ\eta' \in A^m_\tau(\mathbb{R}^n \times \mathbb{R}^n)$$

and

$$\partial_y^\alpha \partial_\eta^\beta b(y,\eta) = \mathrm{Os}\text{–}\iint e^{-iy'\cdot\eta'} \partial_y^\alpha \partial_\eta^\beta a(y,y',\eta,\eta') dy'\, đ\eta' \tag{4.15}$$

Moreover,

$$\begin{aligned} &\mathrm{Os}\text{–}\iiiint e^{-iy\cdot\eta - iy'\cdot\eta'} a(y,y',\eta,\eta') dy dy'\, đ\eta\, đ\eta' \\ &= \mathrm{Os}\text{–}\iint e^{-iy\cdot\eta} \left(\mathrm{Os}\text{–}\iint e^{-iy'\cdot\eta'} a(y,y',\eta,\eta') dy'\, đ\eta' \right) dy\, đ\eta. \end{aligned}$$

Proof: Because of Peetre's inequality,

$$\langle(\eta,\eta')\rangle^m \langle(y,y')\rangle^\tau \le 2^{|m|+\tau} \langle\eta\rangle^m \langle y\rangle^\tau \langle\eta'\rangle^{|m|} \langle y'\rangle^\tau.$$

Hence $\partial_y^\alpha \partial_\eta^\beta a(y,.,\eta,.) \in \mathcal{A}^{|m|}_\tau(\mathbb{R}^{2k})$ with respect to (y',η') and

$$|\partial_y^\alpha \partial_\eta^\beta a(y,.,\eta,.)|_{\mathcal{A}^{|m|}_\tau(\mathbb{R}^{2n}),j} \le C_{k,m} |a|_{\mathcal{A}^m_\tau(\mathbb{R}^{2(k+n)}),j+|\alpha|+|\beta|} \langle\eta\rangle^m \langle y\rangle^\tau.$$

Therefore (4.14) implies

$$\left| \mathrm{Os}\text{–}\iint e^{-iy'\cdot\eta'} \partial_y^\alpha \partial_\eta^\beta a(y,y',\eta,\eta') dy'\, đ\eta' \right| \le C_m |a|_{\mathcal{A}^m_\tau(\mathbb{R}^{4n}),2(l+l')+|\alpha|+|\beta|} \langle\eta\rangle^m \langle y\rangle^\tau,$$

where $2l > |m| + k$ and $2l' > k + \tau$. Moreover, we choose l, l' so large that $2l > m + n$ and $2l' > n + \tau$. Because of the representation (4.9) and Theorem 2.4, we can take $\partial_y^\alpha \partial_\eta^\beta$ out of the oscillatority integral, which proves (4.15). Thus

$$\left| \partial_y^\alpha \partial_\eta^\beta b(y,\eta) \right| \le C_m |a|_{\mathcal{A}^m_\tau(\mathbb{R}^{4n}),2(l+l')+|\alpha|+|\beta|} \langle\eta\rangle^m \langle y\rangle^\tau.$$

Hence $b(y,\eta) \in \mathcal{A}^m_\tau(\mathbb{R}^n \times \mathbb{R}^n)$. Thus $\langle y\rangle^{-2l'} \langle D_\eta\rangle^{2l'} \left[\langle\eta\rangle^{-2l} \langle D_y\rangle^{2l} b(y,\eta) \right] \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$ because of Theorem 4.9. Moreover,

$$b(y,\eta) = \iint e^{-iy'\cdot\eta'} \langle y'\rangle^{-2l'} \langle D_{\eta'}\rangle^{2l'} \left[\langle\eta'\rangle^{-2l} \langle D_{y'}\rangle^{2l} a(y,y',\eta,\eta') \right] dy'\, đ\eta'$$

and therefore

$$\begin{aligned}
&\langle y\rangle^{-2l'}\langle D_\eta\rangle^{2l'}\left[\langle\eta\rangle^{-2l}\langle D_y\rangle^{2l}b(y,\eta)\right]\\
&\quad= \iint e^{-iy'\cdot\eta'}(\langle y\rangle\langle y'\rangle)^{-2l'}(\langle D_\eta\rangle\langle D_{\eta'}\rangle)^{2l'}\left[(\langle\eta\rangle\langle\eta'\rangle)^{-2l}(\langle D_y\rangle\langle D_{y'}\rangle)^{2l}a\right]dy'\,đ\eta',
\end{aligned}$$

where

$$(\langle y\rangle\langle y'\rangle)^{-2l'}(\langle D_\eta\rangle\langle D_{\eta'}\rangle)^{2l'}\left[(\langle\eta\rangle\langle\eta'\rangle)^{-2l}(\langle D_y\rangle\langle D_{y'}\rangle)^{2l}a(y,y',\eta,\eta')\right]\in L^1(\mathbb{R}^{4n}).$$

Hence we can apply Fubini's theorem and get

$$\begin{aligned}
&\mathrm{Os}\text{–}\iint e^{-iy\cdot\eta}\left(\mathrm{Os}\text{–}\iint e^{-iy'\cdot\eta'}a(y,y',\eta,\eta')dy'\,đ\eta'\right)dy\,đ\eta\\
&= \int e^{-iy\cdot\eta-iy'\cdot\eta'}(\langle y\rangle\langle y'\rangle)^{-2l'}(\langle D_\eta\rangle\langle D_{\eta'}\rangle)^{2l'}\left[(\langle\eta\rangle\langle\eta'\rangle)^{-2l}(\langle D_y\rangle\langle D_{y'}\rangle)^{2l}a\right]\frac{d(y'y,\eta,\eta')}{(2\pi)^{n+k}}\\
&= \mathrm{Os}\text{–}\iiiint e^{-iy\cdot\eta-iy'\cdot\eta'}(\langle\eta\rangle\langle\eta'\rangle)^{-2l}(\langle D_y\rangle\langle D_{y'}\rangle)^{2l}a(y,y',\eta,\eta')dy'\,đ\eta'\,dy\,đ\eta\\
&= \mathrm{Os}\text{–}\iiiint e^{-iy\cdot\eta-iy'\cdot\eta'}a(y,y',\eta,\eta')dy'\,đ\eta'\,dy\,đ\eta,
\end{aligned}$$

where we have used Lemma 4.12. ■

4.4 Double Symbols

The composition $p_1(x,D_x)p_2(x,D_x)$ calculated in Section 4.2 is an example of a pseudodifferential operator in more general form – a pseudodifferential operator with a *double symbol*:

$$p(x,D_x,x',D_{x'})u = \iiiint e^{i(x-x')\cdot\xi+i(x'-x'')\cdot\xi'}p(x,\xi,x',\xi')u(x'')dx''\,đ\xi'\,dx'\,đ\xi$$

for $u\in\mathcal{S}(\mathbb{R}^n)$, where the integral has to be understood as iterated integral. See [KG74, Chapter 2, §2] for a more detailed discussion. Here $p(x,\xi,x',\xi') = p_1(x,\xi)p_2(x',\xi')\in S^{m_1,m_2}_{1,0}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n)$ defined as follows:

Definition 4.14 Let $m,m'\in\mathbb{R}$. Then the space of double pseudodifferential symbols $S^{m,m'}_{1,0}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n)$ is the space of all smooth functions $p\colon\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}$ such that

$$|D^\alpha_\xi D^\beta_x D^{\alpha'}_{\xi'}D^{\beta'}_{x'}p(x,\xi,x',\xi')|\le C_{\alpha,\beta,\alpha',\beta'}(1+|\xi|)^{m-|\alpha|}(1+|\xi'|)^{m'-|\alpha'|}$$

uniformly in $x,\xi,x',\xi'\in\mathbb{R}^n$ for arbitrary $\alpha,\beta,\alpha',\beta'\in\mathbb{N}_0^n$.

Note that $S^{m,m'}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n) \subset A_0^{\max(m,m',m+m')}(\mathbb{R}^{2n} \times \mathbb{R}^{2n})$.

The statements on composition of two pseudodifferential operators will be a corollary to the following more general theorem:

THEOREM 4.15 *Let $m, m' \in \mathbb{R}^n$ and let $p \in S^{m,m'}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ be a double symbol. Then*

$$p_L(x,\xi) := \text{Os--}\iint e^{-iy\cdot\eta} p(x, \xi+\eta, x+y, \xi)\, dy\, đ\eta \in S^{m+m'}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n).$$

Moreover,

$$p_L(x,\xi) \sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} \partial_\xi^\alpha D_{x'}^\alpha p(x,\xi,x',\xi')|_{x'=x,\xi'=\xi}$$

Proof: First of all, let $a_{x,\xi}(y,\eta) := p(x,\xi+\eta,x+y,\eta)$. Using Peetre's inequality,

$$\begin{aligned} |\partial_\eta^\alpha \partial_y^\beta p(x,\xi+\eta,x+y,\xi)| &\leq C_{\alpha,\beta}(1+|\xi+\eta|)^m (1+|\xi|)^{m'} \\ &\leq C_{\alpha,\beta} 2^{|m|}(1+|\eta|)^{|m|}(1+|\xi|)^{m+m'}. \end{aligned}$$

Hence $a_{x,\xi}(y,\eta) \in \mathcal{A}_0^{|m|}$ with $|a_{x,\xi}|_{\mathcal{A}_0^{|m|},|m|+n+2} \leq C(1+|\xi|)^{m+m'}$. Therefore

$$|p_L(x,\xi)| = \left| \text{Os--}\iint e^{-iy\cdot\eta} p(x,\xi+\eta,x+y,\xi)\, dy\, đ\eta \right| \leq C(1+|\xi|)^{m+m'} \tag{4.16}$$

because (4.10).

Because of (4.15),

$$\partial_\xi^\alpha \partial_x^\beta p_L(x,\xi) = \text{Os--}\iint e^{-iy\cdot\eta} \partial_\xi^\alpha \partial_x^\beta \left[p(x,\xi+\eta,x+y,\xi)\right] dy\, đ\eta. \tag{4.17}$$

Combining (4.16) and (4.17) yields

$$|\partial_\xi^\alpha \partial_x^\beta p_L(x,\xi)| = \left| \text{Os--}\iint e^{-iy\cdot\eta} \partial_\xi^\alpha \partial_x^\beta \left[p(x,\xi+\eta,x+y,\xi)\right] dy\, đ\eta \right| \leq C(1+|\xi|)^{m+m'-|\alpha|} \tag{4.18}$$

In order to prove the asymptotic expansion, we use the Taylor series expansion:

$$\begin{aligned} p(x,\xi+\eta,x+y,\xi) &= \sum_{|\alpha|\leq N} \frac{\eta^\alpha}{\alpha!} p_\alpha(x,\xi,x+y,\xi) \\ &\quad +(N+1) \sum_{|\alpha|=N+1} \frac{\eta^\alpha}{\alpha!} \int_0^1 (1-\theta)^N p_\alpha(x,\xi+\theta\eta,x+y,\xi)\, d\theta, \end{aligned}$$

where $p_\alpha(x,\xi,y,\eta) = \partial_\xi^\alpha p(x,\xi,y,\eta)$. Hence

$$\begin{aligned} p_L(x,\xi) &= \sum_{|\alpha|\leq N} \frac{1}{\alpha!} \text{Os--}\iint e^{-iy\cdot\eta} \eta^\alpha p_\alpha(x,\xi,x+y,\xi)\, dy\, đ\eta \\ &\quad +(N+1) \sum_{|\alpha|=N+1} \frac{1}{\alpha!} \int_0^1 r_{\alpha,\theta}(x,\xi)\, d\theta, \end{aligned}$$

where

$$r_{\alpha,\theta}(x,\xi) = \text{Os--}\iint e^{-iy\cdot\eta}(\partial_\xi^\alpha D_y^\alpha p)(x,\xi+\theta\eta,x+y,\xi)dy đ\eta.$$

Because of Lemma 4.12

$$\begin{aligned}\text{Os--}\iint e^{-iy\cdot\eta}\eta^\alpha p_\alpha(x,\xi,x+y,\xi)dyđ\eta &= \text{Os--}\iint e^{-iy\cdot\eta}D_y^\alpha p_\alpha(x,\xi,x+y,\xi)dyđ\eta \\ &= \partial_\xi^\alpha D_y^\alpha p(x,\xi,y,\eta)|_{y=x,\eta=\xi}.\end{aligned}$$

Therefore it remains to prove that $\{r_{\alpha,\theta}(x,\xi)\}_{0\leq\theta\leq 1}$ is uniformly bounded in $S_{1,0}^{m+m'-|\alpha|}$. As in the beginning of the proof,

$$\begin{aligned}\left|\partial_\eta^\beta\partial_y^\gamma\left((\partial_\xi^\alpha D_y^\alpha p)(x,\xi+\theta\eta,x+y,\xi)\right)\right| &\leq C_{\alpha,\beta,\gamma}2^{|m|}(1+|\theta\eta|)^{|m|}(1+|\xi|)^{m+m'-|\alpha|} \\ &\leq C_{\alpha,\beta,\gamma}2^{|m|}(1+|\eta|)^{|m|}(1+|\xi|)^{m+m'-|\alpha|},\end{aligned}$$

where $C_{\alpha,\beta,\gamma}$ does not depend on $\theta\in[0,1]$. Hence $\{p(x,\xi+\theta.,x+.,\xi)\}_{0\leq\theta\leq 1}$ are uniformly bounded in $\mathcal{A}_0^{|m|}$ as amplitudes in (y,η). Therefore

$$|r_{\alpha,\theta}(x,\xi)| = \left|\text{Os--}\iint e^{-iy\cdot\eta}(\partial_\xi^\alpha D_y^\alpha p)(x,\xi+\eta,x+y,\xi)dyđ\eta\right| \leq C_\alpha(1+|\xi|)^{m+m'-|\alpha|} \tag{4.19}$$

because (4.10). Finally, the derivatives $\partial_\xi^\beta\partial_x^\gamma r_{\alpha,\theta}(x,\xi)$ are estimated in the same way as before. ■

4.5 Composition

With the aid of the oscillatority integrals, we can make the formal calculations in Section 4.2 rigorous.

First of all, if $p\in S_{1,0}^m(\mathbb{R}^n\times\mathbb{R}^n)$, then

$$\begin{aligned}p(x,D_x)u &= \int\left(\int e^{i(x-y)\cdot\xi}p(x,\xi)u(y)dy\right)đ\xi \\ &= \text{Os--}\iint e^{-ix'\cdot\xi}p(x,\xi)u(x+x')dx'đ\xi \end{aligned}\tag{4.20}$$

for all $u\in\mathcal{S}(\mathbb{R}^n)$. The proof is left to the reader as an exercise.

Exercise 7 *Prove (4.20).*

Using this representation and Theorem 4.13, we easily get

$$
\begin{aligned}
& p_1(x, D_x)p_2(x, D_x)u \\
&\quad = \text{Os--}\iint e^{-ix'\cdot\xi} p_1(x,\xi)\left(\text{Os--}\iint e^{-ix''\cdot\xi'} p_2(x+x',\xi')u(x+x'+x'')dx''đ\xi'\right)dx'đ\xi \\
&\quad = \text{Os--}\iiiint e^{-ix'\cdot\xi - ix''\cdot\xi'} p_1(x,\xi)p_2(x+x',\xi')u(x+x'+x'')dx''đ\xi'dx'đ\xi \\
&\quad = \text{Os--}\iiiint e^{-ix'\cdot\eta - iy\cdot\xi'} p_1(x,\xi'+\eta)p_2(x+x',\xi')u(x+y)dx'đ\xi'dyđ\eta \\
&\quad = \text{Os--}\iint e^{-iy\cdot\xi'}\left(\text{Os--}\iint e^{-ix'\cdot\eta} p_1(x,\xi'+\eta)p_2(x+x',\xi')dx'đ\eta\right)u(x+y)dyđ\xi' \\
&\quad = \text{Os--}\iint e^{-iy\cdot\xi'} p_1\#p_2(x,\xi')u(x+y)dyđ\xi
\end{aligned}
$$

where we have used $\eta = \xi - \xi'$ and $y = x' + x''$ and $p_1\#p_2$ is defined as in (4.8).

THEOREM 4.16 *Let $p_j \in S^{m_j}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $j = 1, 2$, be two pseudodifferential symbols. Then there is a $p_1\#p_2 \in S^{m_1+m_2}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ such that*

$$p_1(x, D_x)p_2(x, D_x) = (p_1\#p_2)(x, D_x).$$

Moreover, $p_1\#p_2$ has the following asymptotic expansion:

$$p_1\#p_2(x,\xi) \sim \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!}\partial_\xi^\alpha p_1(x,\xi) D_x^\alpha p_2(x,\xi)$$

in the sense that

$$p_1\#p_2(x,\xi) - \sum_{|\alpha|<N} \frac{1}{\alpha!}\partial_\xi^\alpha p_1(x,\xi) D_x^\alpha p_2(x,\xi) \in S^{m_1+m_2-N}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$$

for all $N \in \mathbb{N}$.

Proof: Let $p(x,\xi,x',\xi') = p_1(x,\xi)p_2(x',\xi')$. Then $p \in S^{m_1,m_2}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ and $p_1\#p_2(x,\xi) = p_L(x,\xi)$ and the theorem is a consequence of Theorem 4.15. ∎

Hence the composition of two pseudodifferential operators is again a pseudodifferential operator. In particular, the asymptotic expansions implies:

$$p_1\#p_2(x,\xi) = p_1(x,\xi)p_2(x,\xi) + r(x,\xi),$$

where $r(x,\xi) \in S^{m_1+m_2-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Hence the symbol of the composition coincides with the product of the symbols modulo terms of lower order. This is the essential fact needed for the construction of a parametrix to an elliptic pseudodifferential operator.

Finally, we note that, if $p_2(x,\xi) = p_2(\xi)$ is independent of x, we simply have $p_1(x,D_x)p_2(D_x) = \mathrm{OP}(p_1(x,\xi)p_2(\xi))$ since $\mathcal{F}[p_2(D_x)u] = p_2(\xi)\hat{u}(\xi)$. Moreover, if $p_1(x,D_x)$ is a differential operator of order $m \in \mathbb{N}_0$ with coefficients in $C_b^\infty(\mathbb{R}^n)$, then

$$p_1(x,D_x)p_2(x,D_x) = (p_1\#p_2)(x,D_x)$$

where

$$(p_1\#p_2)(x,\xi) = \sum_{|\alpha|\leq m} \frac{1}{\alpha!}\partial_\xi^\alpha p_1(x,\xi)D_x^\alpha p_2(x,\xi).$$

Hence the asymptotic expansion of $p_1\#p_2(x,\xi)$ consists only of finitely many terms. This is an easy consequence of Leibniz's formula. (See exercise sheet 5.) Alternatively, it can be observed by modifying the proof of Theorem 4.15. (In this case there is no remainder term $r_{\alpha,\theta}$ if $N > m$.)

4.6 Application: Elliptic Pseudodifferential Operators and Parametrices

Definition 4.17 A symbol $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$, is called *elliptic* if there is an $R > 0$ such that

$$|p(x,\xi)| \geq C|\xi|^m \qquad \text{for all } |\xi| \geq R, x \in \mathbb{R}^n \tag{4.21}$$

Examples 4.18 1. Let $p(\xi) = |\xi|^2$ be the symbol of $-\Delta$. Then p is an elliptic symbol of order 2. Moreover, $q(\xi) = \langle\xi\rangle^m$, $m \in \mathbb{R}$, is an elliptic symbol of order m.

2. Let $A(x) \in C_b^\infty(\mathbb{R}^n)^{n\times b}$ be a matrix that is *uniformly positive definite*, i.e., there is a $c > 0$ such that

$$\xi^T A(x)\xi \geq c|\xi|^2, \qquad \text{for all } x,\xi \in \mathbb{R}^n.$$

Then $p(x,\xi) = \xi^T A(x)\xi$ is an elliptic symbol of order 2.

Lemma 4.19 *Let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$, be an elliptic symbol and $R > 0$ as in (4.21). Then*

$$q(x,\xi) := \psi(\xi)p^{-1}(x,\xi) \in S^{-m}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$$

where $\psi \in C_b^\infty(\mathbb{R}^n)$ such that $\psi(\xi) = 1$ for $|\xi| \geq R+1$ and $\psi(\xi) = 0$ for $|\xi| \leq R$.

Proof: Since $q(x,\xi) = \psi(\xi) = 0$ for $|\xi| \leq R$, $q(x,\xi)$ is obviously smooth in (x,ξ) and it suffices to consider $|\xi| \geq R$. Because of the chain rule,

$$\partial_{\xi_j}p^{-1}(x,\xi) = -p^{-2}(x,\xi)\partial_{\xi_j}p(x,\xi)$$

and the same identity with ∂_{ξ_j} replaced by ∂_{x_j}. Using (4.21),

$$|\partial_{\xi_j}p^{-1}(x,\xi)| \leq C|\xi|^{-2m}\langle\xi\rangle^{m-1} \leq C\langle\xi\rangle^{-m-1}$$

and

$$|\partial_{x_j} p^{-1}(x,\xi)| \leq C|\xi|^{-2m}\langle\xi\rangle^m \leq C\langle\xi\rangle^{-m}$$

since $|\xi| \geq R > 0$. In the same way one can easily prove by induction that

$$|\partial_\xi^\alpha \partial_x^\beta p^{-1}(x,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{-m-|\alpha|}$$

for all $|\xi| \geq R$. Since $\operatorname{supp}\partial_\xi^\alpha\psi \subset \overline{B_{R+1}(0)}$, $\psi \in S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Hence

$$|\partial_\xi^\alpha q(x,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{-m-|\alpha|}$$

for all $\xi \in \mathbb{R}^n$. ■

Corollary 4.20 *Let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ be an elliptic symbol. Then there is an $q \in S^{-m}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ such that $p(x,D_x)q(x,D_x) = I + r(x,D_x)$ with $r \in S^{-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$.*

Proof: Let q be defined as in Lemma 4.19. Then, because of Theorem 4.16, $p(x,D_x)q(x,D_x) = (pq)(x,D_x) + r'(x,D_x)$, where $r' \in S^{-1}(\mathbb{R}^n \times \mathbb{R}^n)$. Moreover, $p(x,\xi)q(x,\xi) = 1$ for all $|\xi| \geq R$. Hence $p(x,\xi)q(x,\xi) - 1 \in S^{-\infty}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ and

$$p(x,D_x)q(x,D_x) = I + r(x,D_x)$$

with $r(x,\xi) = p(x,\xi)q(x,\xi) - 1 + r'(x,\xi) \in S^{-1}_{1,0}(x,\xi)$. ■

THEOREM 4.21 *Let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$, $k \in \mathbb{N}$. Then the following conditions are equivalent:*

1. *p is elliptic.*
2. *There is a $q \in S^{-m}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ such that $p(x,D_x)q(x,D_x) = I + r(x,D_x)$, where $r \in S^{-1}_{1,0}$.*
3. *For every $N \in \mathbb{N}$ there is a $q_N \in S^{-m}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ such that $p(x,D_x)q_N(x,D_x) = I + r_N(x,D_x)$, where $r_N \in S^{-N}_{1,0}$.*

Proof: 1. implies 2.: This is a consequence of Corollary 4.20.

2. implies 3.: Let $r(x,\xi) := 1 - p\#q(x,\xi)$. Then $r \in S^{-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ because of the assumptions and

$$p(x,D_x)q(x,D_x) = I - r(x,D_x).$$

Now let $q_N \in S^{-m}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ be such that

$$q_N(x,D_x) = q(x,D_x)\sum_{k=0}^{N-1}(-r(x,D_x))^k.$$

Then

$$p(x,D_x)q_N(x,D_x) = (I - r(x,D_x))\sum_{k=0}^{N-1}(-r(x,D_x))^k = I + (-r(x,D_x))^N,$$

where $(-r(x,D_x))^N = r_N(x,D_x)$ with $r_N \in S^{-N}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$.

3. implies 1.: Since $p(x,D_x)q(x,D_x) = OP(p(x,\xi)q(x,\xi)) + r(x,D_x)$ and $p(x,D_x)q(x,D_x) = I + r'(x,D_x)$ with $r, r' \in S^{-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, we obtain

$$p(x,\xi)q(x,\xi) - 1 = r(x,\xi) + r'(x,\xi) \in S^{-1}_{1,0}.$$

In particular

$$|p(x,\xi)q(x,\xi) - 1| \leq C\langle\xi\rangle^{-1}$$

for all $x, \xi \in \mathbb{R}^n$. Hence there is an $R > 0$ such that $|p(x,\xi)q(x,\xi) - 1| \leq \frac{1}{2}$ for all $|\xi| \geq R$ and $x \in \mathbb{R}^n$. Thus $(p(x,\xi)q(x,\xi))^{-1} \leq 2$ for for all $|\xi| \geq R$ and $x \in \mathbb{R}^n$. Therefore we finally conclude that

$$|p^{-1}(x,\xi)| = |(p(x,\xi)q(x,\xi))^{-1}||q(x,\xi)| \leq 2C\langle\xi\rangle^{-m}$$

since $q \in S^{-m}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. ■

For completeness we note that for an elliptic symbol p there is also an $q_\infty \in S^{-m}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$p(x,D_x)q_\infty(x,D_x) = I + r_\infty(x,D_x) \tag{4.22}$$

with $r \in S^{-\infty}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. In order to prove this statement, the following lemma is needed:

Lemma 4.22 *Let $p_j \in S^{m_j}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ with $m_1 \geq \ldots \geq m_j \to -\infty$ as $j \to \infty$. Then there is an $p \in S^{m_1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ such that*

$$p(x,\xi) \sim \sum_{j=1}^{\infty} p_j(x,\xi) \;\; :\Leftrightarrow \;\; p(x,\xi) - \sum_{j=1}^{N-1} p_j(x,\xi) \in S^{m_N}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n).$$

Proof: See [KG74, Lemma 3.2] or [Ray91, Lemma 2.2]. ■

In order to prove (4.22), we can define

$$q_\infty(x,D_x) = q(x,D_x)q'(x,D_x),$$

where

$$q'(x,\xi) \sim \sum_{k=0}^{\infty} r^{\#k}(x,\xi),$$

r is the same as in the proof of Theorem 4.21, and

$$r^{\#k}(x,\xi) = \underbrace{r\#\ldots\#r(x,\xi)}_{k\text{-times}} \in S^{-k}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n).$$

4.7 Boundedness on $C_b^\infty(\mathbb{R}^n)$ and Uniqueness of the Symbol

As seen above,

$$p(x, D_x)u = \text{Os-}\iint e^{-ix'\cdot\xi} p(x,\xi)u(x+x')dx'đ\xi \tag{4.23}$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$. But the latter oscillatory integral is defined for all $u \in C_b^\infty(\mathbb{R}^n)$. Therefore we can extend the definition of $p(x, D_x)$ to $C_b^\infty(\mathbb{R}^n)$.

THEOREM 4.23 *Let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Then $p(x, D_x)$ defined by (4.23) for $u \in C_b^\infty(\mathbb{R}^n)$ is a bounded operator*

$$p(x, D_x)\colon C_b^\infty(\mathbb{R}^n) \to C_b^\infty(\mathbb{R}^n).$$

Proof: Consider $a_x(x', \xi) = p(x, \xi)u(x + x')$. Since

$$|\partial_{x'}^\alpha \partial_\xi^\beta p(x,\xi)u(x+x')| \leq C_{\alpha,\beta}|p|^{(m)}_{|\beta|}\|u\|_{C_b^{|\alpha|}}\langle\xi\rangle^m,$$

we have $a_x(x', \xi) \in \mathcal{A}^m_0$ with $|a_x|_{\mathcal{A}^m_0,k} \leq C|p|^{(m)}_k\|u\|_{C_b^k}$ uniformly in $x \in \mathbb{R}^n$. Thus

$$\sup_{x\in\mathbb{R}^n}|p(x,D_x)u| \leq C|p|^{(m)}_{2(l+l')}\|u\|_{C_b^{2(l+l')}} \tag{4.24}$$

by Theorem 4.9. In order to estimate $\partial_x^\alpha p(x, D_x)u$ we observe that $a(x, x', \xi, \xi') := p(x, \xi)u(x + x') \in \mathcal{A}^m_0(\mathbb{R}^{2n} \times \mathbb{R}^{2n})$ and therefore

$$\begin{aligned}
\partial_{x_j} p(x,D_x)u &= \text{Os-}\iint e^{-ix'\cdot\xi}(\partial_{x_j}p)(x,\xi)u(x+x')dx'đ\xi \\
&\quad + \text{Os-}\iint e^{-ix'\cdot\xi}p(x,\xi)(\partial_{x_j}u)(x+x')dx'đ\xi \\
&= (\partial_{x_j}p)(x,D_x)u + p(x,D_x)\partial_{x_j}u
\end{aligned}$$

by (4.15) in Theorem 4.13. Applying this formula successively and using (4.24), it follows immediately that $\|p(x, D_x)u\|_{C_b^k} \leq C|p|^{(m)}_{2(l+l')+k}\|u\|_{C_b^{2(l+l')+k}}$, which proves the theorem. ∎

Using this extension, we can prove that the symbol of a pseudodifferential operator is uniquely defined by its values on $C_b^\infty(\mathbb{R}^n)$:

Exercise 8 *Show that for all $x, \xi \in \mathbb{R}^n$*

$$e^{-ix\cdot\xi}p(x,D_x)e^{i\cdot\xi} = p(x,\xi). \tag{4.25}$$

Corollary 4.24 *Let $p, q \in S^\infty_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Then $p(x, D_x)u = q(x, D_x)u$ for all $u \in C_b^\infty(\mathbb{R}^n)$ implies $p(x, \xi) = q(x, \xi)$.*

More generally, if $P\colon C_b^\infty(\mathbb{R}^n) \to C_b^\infty(\mathbb{R}^n)$ is an arbitrary bounded linear map, then we can *define* a symbol of P by

$$p(x,\xi) := e^{-ix\cdot\xi} P e^{i\cdot\cdot\xi}$$

This gives an alternative approach for determining the symbol of $p_1(x,D_x)p_2(x,D_x)$:

Exercise 9 *Calculate*

$$e^{-ix\cdot\xi} p_1(x,D_x)p_2(x,D_x)e^{i\cdot\cdot\xi}$$

without using Theorem 4.16.

4.8 Adjoints of Pseudodifferential Operators and Operators in (x,y)-Form

Definition 4.25 Let $A, A^*\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. Then A^* is called *formally adjoint* of A if

$$(Au, v) = (u, A^*v) \qquad \text{for all } u, v \in \mathcal{S}(\mathbb{R}^n).$$

The adjoint of an operator plays an important role in many purposes. (For example: solvability of equations).

In the following, we will prove that every pseudodifferential operator $p(x,D_x)$ possesses a formal adjoint $p^*(x,D_x)$, which is again a pseudodifferential operator (of the same order). This will allows us to extend $p(x,D_x)$, up to now only defined on $\mathcal{S}(\mathbb{R}^n)$, to the a linear operator defined on $\mathcal{S}'(\mathbb{R}^n)$.

Now we calculate the formal adjoint of $p(x,D_x)$:

$$\begin{aligned}
(p(x,D_x)u, v) &= \iint e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)\, đ\xi \overline{v(x)} dx \\
&= \iint e^{ix\cdot\xi} p(x,\xi)\overline{v(x)} dx \hat{u}(\xi)\, đ\xi \\
&= \int \hat{u}(\xi) \overline{\int e^{-ix\cdot\xi}\overline{p(x,\xi)} v(x) dx}\, đ\xi,
\end{aligned}$$

where have used Fubini's theorem. (Note that $v, \hat{u} \in \mathcal{S}(\mathbb{R}^n)$ and therefore $e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi)\overline{v(x)} \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$ w.r.t. (x,ξ).)

Lemma 4.26 *Let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$. Then*

$$w(\xi) := \int e^{-ix\cdot\xi} p(x,\xi) v(x) dx \in \mathcal{S}(\mathbb{R}^n).$$

Proof: Exercise. - It is also a consequence of Lemma 4.28 below. ■

Hence we can use $(\mathcal{F}[u], v) = (u, \mathcal{F}^{-1}[v])$ and get

$$(p(x,D_x)^*v)(x) = \iint e^{i(x-y)\cdot\xi} \overline{p(y,\xi)} v(y) dy\, đ\xi. \tag{4.26}$$

This operator is a pseudodifferential operator in the so called *y-form* (also called R-form), which is a special case of operators of the form

$$p(x, D_x, x)u := \iint e^{i(x-y)\cdot\xi} p(x, y, \xi) u(y) dy đ\xi \tag{4.27}$$

for $u \in \mathcal{S}(\mathbb{R}^n)$, where $p \in S^m_{1,0}(\mathbb{R}^{2n} \times \mathbb{R}^n)$, i.e.,

$$|\partial_\xi^\alpha \partial_x^\beta \partial_y^\gamma p(x, y, \xi)| \leq C_{\alpha,\beta,\gamma}(1 + |\xi|)^{m-|\alpha|}$$

for all $\alpha, \beta, \gamma \in \mathbb{N}_0^n$. As before

$$p(x, D_x, x)u = \text{Os--}\iint e^{-ix'\cdot\xi} p(x, x + x', \xi) u(x + x') dx' đ\xi$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$.

THEOREM 4.27 *Let $p \in S^m_{1,0}(\mathbb{R}^{2n} \times \mathbb{R}^n)$, $m \in \mathbb{R}$. Then $p(x, D_x, x) = p_L(x, D_x)$, where*

$$p_L(x, \xi) = \text{Os--}\iint e^{-iy\cdot\eta} p(x, x + y, \xi + \eta) dy đ\eta \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n).$$

Moreover,

$$p_L(x, \xi) \sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} \partial_\xi^\alpha D_y^\alpha p(x, y, \xi)|_{y=x}.$$

Proof: Using

$$u(x + x') = \text{Os--}\iint e^{-iy\cdot\eta} u(x + x' + y) dy đ\eta = \text{Os--}\iint e^{i(x+x'-y)\cdot\eta} u(y) dy đ\eta,$$

$$\begin{aligned} p(x, D_x, x)u &= \text{Os--}\iiiint e^{-ix'\cdot\xi} e^{i(x+x'-y)\cdot\eta} p(x, x + x', \xi) u(y) dy đ\eta dx' đ\xi \\ &= \text{Os--}\iint e^{-i(x-y)\cdot\eta} \left(\text{Os--}\iint e^{-ix'\cdot(\xi-\eta)} p(x, x + x', \xi) dx' đ\xi \right) u(y) dy đ\eta \\ &= \text{Os--}\iint e^{-i(x-y)\cdot\eta} p_L(x, \xi) u(y) dy đ\eta. \end{aligned}$$

Hence application of Theorem 4.15 finishes the proof. ■

As a direct consequence we obtain:

Lemma 4.28 *Let $p \in S^m_{1,0}(\mathbb{R}^{2n} \times \mathbb{R}^n)$, $m \in \mathbb{R}$. Then $p(x, D_x, x)\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a bounded operator. Moreover, if the definition (4.27) is replaced by*

$$p(x, D_x, x)u := \text{Os--}\iint e^{-ix'\cdot\xi} p(x, x + x', \xi) u(x + x') dx' đ\xi$$

for $u \in C_b^\infty(\mathbb{R}^n)$, then $p(x, D_x, x)\colon C_b^\infty(\mathbb{R}^n) \to C_b^\infty(\mathbb{R}^n)$ is a bounded operator.

Proof: Since $p(x, D_x, x)u = p_L(x, D_x)u$ for all $u \in C_b^\infty(\mathbb{R}^n)$, the lemma is a consequence of the corresponding statements for $p_L(x, D_x)$. ∎

Corollary 4.29 *If $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, then the formally adjoint of $p(x, D_x)$ is $p^*(x, D_x)$ where*

$$p^*(x,\xi) = \mathrm{Os}\text{–}\iint e^{-iy\cdot\xi}\overline{p(x+y,\xi+\eta)}dy đ\eta \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n).$$

Moreover,

$$p^*(x,\xi) \sim \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha \overline{p(x,\xi)}.$$

Proof: The corollary is a direct consequence of (4.26) and the latter theorem. ∎

Definition 4.30 Let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Then we define $p(x, D_x) \colon \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ by

$$\langle p(x,D_x)u, v\rangle := \langle u, p^*(x,D_x)v)\rangle \qquad u \in \mathcal{S}'(\mathbb{R}^n), v \in \mathcal{S}(\mathbb{R}^n).$$

4.9 L^2-Continuity and Bessel Potential Spaces

THEOREM 4.31 *Let $p \in S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Then $p(x, D_x)$ (defined on $\mathcal{S}(\mathbb{R}^n)$) extends to a bounded operator $p(x, D_x) \colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.*

The proof of Theorem 4.31 is divided into several parts.

Lemma 4.32 *Theorem 4.31 holds for $p \in S^{-n-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$.*

Proof: If $p \in S^{-n-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, then $p(x,\xi) \in L^1(\mathbb{R}^n)$ with respect to ξ and therefore

$$\begin{aligned} p(x,D_x)f &= \int e^{ix\cdot\xi}p(x,\xi)\hat{f}(\xi)đ\xi = \iint e^{i(x-y)\cdot\xi}p(x,\xi)đ\xi f(y)dy \\ &= \int k(x,x-y)f(y)dy \end{aligned}$$

by Fubini's theorem, where $k(x,z) = \mathcal{F}^{-1}_{\xi\mapsto z}[p(x,\xi)]$. This kernel satisfies

$$|z^\alpha k(x,z)| = \left|\int e^{iz\cdot\xi}\partial_\xi^\alpha p(x,\xi)đ\xi\right| \leq C|p|^{(-n-1)}_{|\alpha|}$$

since $|\partial_\xi^\alpha p(x,\xi)| \leq |p|^{(-n-1)}_{|\alpha|}\langle\xi\rangle^{-n-1-|\alpha|} \in L^1(\mathbb{R}^n)$ w.r.t. ξ. Hence $|(1+|z|^2)^n k(x,z)| \leq C|p|^{(-n-1)}_{2n}$ and therefore

$$g(z) := \sup_{x\in\mathbb{R}^n} |k(x,z)| \leq (1+|z|^2)^{-n} \in L^1(\mathbb{R}^n).$$

Thus

$$\|p(x,D_x)f\|_2 \le C\left\|\int g(x-y)|f(y)|dy\right\|_2 \le C\|g\|_1\|f\|_2$$

because of $\|g * f\|_2 \le \|g\|_1\|f\|_2$, cf. Example 3.18.1. ■

Lemma 4.33 *Theorem 4.31 holds for* $p \in S^{-m}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ *with* $m > 0$.

Proof: In order to prove $\|p(x,D_x)f\|_2 \le C\|f\|_2$ for $f \in \mathcal{S}(\mathbb{R}^n)$, it is sufficient to show

$$\|p^*(x,D_x)p(x,D_x)f\|_2 \le C\|f\|_2$$

since

$$\|p(x,D_x)f\|_2^2 = (p^*(x,D_x)p(x,D_x)f,f) \le \|p^*(x,D_x)p(x,D_x)f\|_2\|f\|_2.$$

But, if $p \in S^{-m}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, then $p^*(x,D_x)p(x,D_x) = p'(x,D_x)$ with $p' \in S^{-2m}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Hence, using the previous lemma, we obtain by mathematical inductions that

$$\|p(x,D_x)f\|_2 \le C\|f\|_2$$

for all $p \in S^{-m_k}_{1,0}$ with $m_k = (n+1)/2^k$, $k \in \mathbb{N}$. This proves the lemma since for every $m > 0$ there is a $k \in \mathbb{N}$ such that $-m_k > -m$. ■

In order to finish the proof of Theorem 4.31, we need

Lemma 4.34 *If* $p \in S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ *with* $p(x,\xi) \in \mathbb{R}$, $x,\xi \in \mathbb{R}^n$, *and* $F \in C^\infty(\mathbb{R})$, *then* $F(p(x,\xi)) \in S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$.

Proof: First of all $|p(x,\xi)| \le R$ for some $R > 0$ and F is bounded on compact sets. Hence $|F(p(x,\xi))| \le \sup_{|z|\le R}|F(z)|$. Moreover,

$$\begin{aligned}
|\partial_{\xi_j}F(p(x,\xi))| &\le \sup_{|z|\le R}|F'(z)||\partial_{\xi_j}p(x,\xi)| \le C\langle\xi\rangle^{-1},\\
|\partial_{x_j}F(p(x,\xi))| &\le \sup_{|z|\le R}|F'(z)||\partial_{x_j}p(x,\xi)| \le C.
\end{aligned}$$

Finally, the estimate of $|\partial_\xi^\alpha\partial_x^\alpha F(p(x,\xi))|$ can be proved using an induction and the chain rule as before. ■

Proof of Theorem 4.31: If $p(x,\xi) \in S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, then $|p(x,\xi)| \le M$ for $M := |p|_0^{(0)} \ge 0$. Therefore $p'(x,\xi) := M^2 - p(x,\xi)\overline{p(x,\xi)} \in S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ and $p'(x,\xi) \ge 0$. Now let $F \in C^\infty(\mathbb{R})$ be defined by $F(x) = (1+x^2)^{\frac{1}{2}}$ for $x \in \mathbb{R}$. Then $q(x,\xi) := F(p'(x,\xi)) \in S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ and

$$\begin{aligned}
q^*(x,D_x)q(x,D_x)f &= \mathrm{OP}(F(p'(x,\xi))^2)f + r(x,D_x)f\\
&= (1+M^2)f - \mathrm{OP}(\overline{p(x,\xi)}p(x,\xi))f + r(x,D_x)f\\
&= (1+M^2)f - p^*(x,D_x)p(x,D_x)f + r'(x,D_x)f
\end{aligned}$$

for $f \in \mathcal{S}(\mathbb{R}^n)$, where $r, r' \in S^{-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Hence

$$\begin{aligned}
\|p(x,D_x)f\|_2^2 &\leq (p^*(x,D_x)p(x,D_x)f,f) + (q^*(x,D_x)q(x,D_x)f,f) \\
&\leq (1+M^2)\|f\|_2^2 + (r'(x,D_x)f,f).
\end{aligned}$$

Since $r' \in S^{-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $\|r'(x,D_x)f\|_2 \leq C\|f\|_2$ due to Lemma 4.33. Hence $\|p(x,D_x)f\|_2^2 \leq (1+M^2+C^2)\|f\|_2^2$ for all $f \in \mathcal{S}(\mathbb{R}^n)$. ■

Remark 4.35 Checking the previous proofs, it can be observed that $\|p(x,D_x)\|_{\mathcal{L}(L^2(\mathbb{R}^n))}$ depends only on $|p|_k^{(0)}$ for a suitably large $k \in \mathbb{N}$. Since the mapping $p \mapsto p(x,D_x)$ is linear, we have

$$\|p(x,D_x)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C|p|_k^{(0)}; \tag{4.28}$$

this can seen as follows: Let $q(x,\xi) = p(x,\xi)/|p|_k^{(0)}$. Then $|q|_k^{(0)} = 1$ and $\|q\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C$ independent of $p \in S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Hence multiplication by $|p|_k^{(0)}$ yields (4.28).

Definition 4.36 Let $s \in \mathbb{R}$. Then the *Bessel potential space* $H^s_2(\mathbb{R}^n)$ is defined as

$$\begin{aligned}
H^s_2(\mathbb{R}^n) &= \{u \in \mathcal{S}'(\mathbb{R}^n) \colon \langle D_x\rangle^s u \in L^2(\mathbb{R}^n)\}, \\
\|u\|_{s,2} &:= \|\langle D_x\rangle^s u\|_2.
\end{aligned}$$

Here a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ is said to be in $L^2(\mathbb{R}^n)$ if there is a $U \in L^2(\mathbb{R}^n)$ such that $\langle u,v\rangle = (U,v)_{L^2}$ for all $v \in \mathcal{S}(\mathbb{R}^n)$ (where $\langle .,.\rangle$ denotes the duality product and $(.,.)_{L^2}$ denotes the L^2-scalar product). For simplicity we will identify U and u.

Note: By Riesz' representation theorem, $u \in \mathcal{S}'(\mathbb{R}^n)$ is an $L^2(\mathbb{R}^n)$ if and only if there is a constant $C > 0$ such that $|\langle u,v\rangle| \leq C\|v\|_2$ for all $v \in \mathcal{S}(\mathbb{R}^n)$.

Remark 4.37 By definition $\langle D_x\rangle^s$ is an isomorphism from $H^s_2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. Hence $H^s_2(\mathbb{R}^n)$ normed by $\|.\|_{2,s}$ is Banach space. Moreover,

$$(u,v)_{H^s_2} := (\langle D_x\rangle^s u, \langle D_x\rangle^s v)_{L^2}, \qquad u,v \in H^s_2(\mathbb{R}^n)$$

is scalar product on $H^s_2(\mathbb{R}^n)$ and $\|u\|_{2,s}^2 = (u,u)_{H^s_2}$. Thus $H^s_2(\mathbb{R}^n)$ is a Hilbert space.

Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ and $\langle D_x\rangle^s \colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ for all $s \in \mathbb{R}$, $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s_2(\mathbb{R}^n)$ for all $s \in \mathbb{R}$.

Finally, $H^{s_1}_2(\mathbb{R}^n) \subseteq H^{s_2}_2(\mathbb{R}^n)$ for $s_1 \geq s_2$ since

$$\|u\|_{s_2,2} = \|\langle D_x\rangle^{s_2}u\|_2 = \|\langle D_x\rangle^{s_2-s_1}\langle D_x\rangle^{s_1}u\|_2 \leq C\|\langle D_x\rangle^{s_1}u\|_2 = C\|u\|_{s_1,2},$$

where we have used that $\langle\xi\rangle^{s_2-s_1} \in S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ and therefore $\langle D_x\rangle^{s_2-s_1} \colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded operator. – The parameter $s \in \mathbb{R}$ of $H^s_2(\mathbb{R}^n)$ determines the regularity of the functions $u \in H^s_2(\mathbb{R}^n)$ ("how many derivatives of u are in $L^2(\mathbb{R}^n)$"), cf. Lemma 4.42, below.

Lemma 4.38 *Let* $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$*. Then* $p(x, D_x) \colon H_2^{s+m}(\mathbb{R}^n) \to H_2^s(\mathbb{R}^n)$ *(defined on* $\mathcal{S}(\mathbb{R}^n)$*) extends a bounded linear mapping. Moreover, there is a* $k \in \mathbb{N}_0$ *such that* $\|p(x, D_x)\|_{\mathcal{L}(H_2^{s+m}(\mathbb{R}^n), H_2^s(\mathbb{R}^n))} \leq C_{s,m}|p|_k^{(m)}$*.*

Proof: Since $\langle D_x \rangle^{s+m} \colon H_2^{s+m}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $\langle D_x \rangle^{-s} \colon L^2(\mathbb{R}^n) \to H_2^s(\mathbb{R}^n)$ are linear isomorphisms, $p(x, D_x) \colon H_2^{s+m}(\mathbb{R}^n) \to H_2^s(\mathbb{R}^n)$ is a linear bounded operator if and only if

$$q(x, D_x) := \langle D_x \rangle^s p(x, D_x) \langle D_x \rangle^{-s-m} \colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is a bounded operator. Because of Theorem 4.16, $q \in S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Hence the first statement is a consequence of Theorem 4.31. The second statement is a consequence of Remark 4.35 and Proposition 4.5. ∎

Corollary 4.39 (Elliptic regularity) *Let* $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ *be an elliptic symbol and* $f \in H_2^s(\mathbb{R}^n)$*,* $s, m \in \mathbb{R}$*. Then, if* $u \in H_2^{-\infty}(\mathbb{R}^n) := \bigcup_{s \in \mathbb{R}} H_2^s(\mathbb{R}^n)$ *is a solution of the* pseudodifferential equation

$$p(x, D_x)u = f,$$

then $u \in H_2^{s+m}(\mathbb{R}^n)$*.*

Proof: Let $u \in H_2^{-\infty}(\mathbb{R}^n)$. Then $u \in H_2^{s+m-N}(\mathbb{R}^n)$ for some $N \in \mathbb{N}$. By the same construction as in Theorem 4.21, there is an $q_N(x, \xi) \in S^{-m}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$q_N(x, D_x) = \left(\sum_{k=0}^{N-1} (-r(x, D_x))^k \right) q(x, D_x),$$

where q is the same as in Corollary 4.20. (I.e., $p(x, D_x)q(x, D_x) = I + r(x, D_x)$ and $q(x, D_x)p(x, D_x) = I + r'(x, D_x)$ where $r, r' \in S^{-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Then as in Theorem 4.21

$$q_N(x, D_x)p(x, D_x) = I + r_N(x, D_x),$$

where $r_N \in S^{-N}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Therefore

$$q_N(x, D_x)f \quad = \quad q_N(x, D_x)p(x, D_x)u = u + r_N(x, D_x)u.$$

Since $q_N(x, D_x)f \in H_2^{s+m}(\mathbb{R}^n)$ and $r_N(x, D_x)u \in H_2^{s+m}(\mathbb{R}^n)$ by Lemma 4.38. ∎

Remark 4.40 The latter corollary says that the solution u is "as smooth as the right-hand side allows". This property is also called *elliptic regularity.*

Definition 4.41 Let $m \in \mathbb{N}_0$. Then the *Sobolev space* $W_2^m(\mathbb{R}^n)$ of order m is the space of all $u \in L^2(\mathbb{R}^n)$ such that the distributional derivative $\partial^\alpha u \in L^2(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le m$. Moreover, $W_2^m(\mathbb{R}^n)$ is normed by

$$\|u\|_{W_2^m} := \left(\sum_{|\alpha|\le m} \|\partial_x^\alpha u\|_2^2 \right)^{\frac{1}{2}}.$$

Lemma 4.42 *Let $m \in \mathbb{N}_0$. Then $W_2^m(\mathbb{R}^n) = H_2^m(\mathbb{R}^n)$ with equivalent norms.*

Proof: If $f \in H_2^m(\mathbb{R}^n)$, then

$$D_x^\alpha f = \mathrm{OP}(\xi^\alpha \langle \xi \rangle^{-m}) \langle D_x \rangle^m f,$$

where $q_\alpha(\xi) := \xi^\alpha \langle \xi \rangle^{-m} \in S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ if $|\alpha| \le m$. Hence

$$\left(\sum_{|\alpha|\le m} \|D_x^\alpha f\|_2^2 \right)^{\frac{1}{2}} \le C_m \|\langle D_x \rangle^m f\|_2 = \|f\|_{m,2}.$$

Conversely, if $f \in W_2^m(\mathbb{R}^n)$, $\partial_x^\alpha f \in L^2(\mathbb{R}^n)$ for all $|\alpha| \le m$. Moreover,

$$\langle \xi \rangle = \frac{\langle \xi \rangle^2}{\langle \xi \rangle} = \frac{1}{\langle \xi \rangle} + \sum_{j=1}^n \xi_j \frac{\xi_j}{\langle \xi \rangle}.$$

Thus

$$\langle \xi \rangle^m = \sum_{k=1}^N p_k(\xi) q_k(\xi),$$

where $q_k(\xi)$ is a polynomial of order m and $p_k \in S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $N \in \mathbb{N}$. Hence

$$\|f\|_{m,2} = \|\langle D_x \rangle^m f\|_2 \le \left\| \sum_{k=1}^N p_k(D_x) q_k(D_x) f \right\|_2 \le C_m \sum_{k=1}^N \|q_k(D_x) f\|_2 \le C_m \|f\|_{W_2^m},$$

where we have used that $p_k(D_x)\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are bounded operators and that $\|q_k(D_x) f\|_2 \le C\|f\|_{W_2^m}$ since $q_k(D_x)$ is a differential operator of order m. ■

Lemma 4.43 *Let $s > \frac{n}{2}$. Then*

$$\|u\|_\infty \le C \|\langle D_x \rangle^s u\|_2, \qquad \|u\|_2 \le C \|\langle x \rangle^s u\|_\infty$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$. In particular, $H_2^s(\mathbb{R}^n) \hookrightarrow C_b^0(\mathbb{R}^n)$.

Proof: See Exercise Sheet 8. ■

Corollary 4.44 *Let*

$$|u|''_{k,\mathcal{S}} := \sup_{|\alpha|+|\beta|\leq k} \|x^\alpha D_x^\beta u\|_2$$

for $u \in \mathcal{S}(\mathbb{R}^n)$, $k \in \mathbb{N}_0$. Then $|.|''_{k,\mathcal{S}}$, $k \in \mathbb{N}$, is a decreasing sequence of semi-norms on $\mathcal{S}(\mathbb{R}^n)$ which is equivalent to the semi-norms $|.|_{k,\mathcal{S}}$ defined above. More precisely,

$$|u|''_{k,\mathcal{S}} \leq C_k |u|_{k+2n,\mathcal{S}} \quad \textit{and} \quad |u|_{k,\mathcal{S}} \leq C_k |u|''_{k+2n,\mathcal{S}}$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$ and $k \in \mathbb{N}_0$.

Proof: Since $\langle D_x \rangle^{2n} = (1-\Delta)^n$ is a differential operator of order $2n$,

$$|u|_{k,\mathcal{S}} = \sup_{|\alpha|+|\beta|\leq k} \|x^\alpha D_x^\beta u\|_\infty \leq \sup_{|\alpha|+|\beta|\leq k} \|\langle D_x \rangle^{2n} x^\alpha D_x^\beta u\|_2 \leq C|u|''_{k+2n,\mathcal{S}}$$

by Lemma 4.43. Similarly, since $\langle x \rangle^2 = (1+|x|^2)^n$ is a polynomial of order $2n$,

$$|u|''_{k,\mathcal{S}} = \sup_{|\alpha|+|\beta|\leq k} \|x^\alpha D_x^\beta u\|_2 \leq \sup_{|\alpha|+|\beta|\leq k} \|\langle x \rangle^{2n} x^\alpha D_x^\beta u\|_\infty \leq C|u|_{k+2n,\mathcal{S}}.$$

■

Remark 4.45 Note that by definition $H_2^s(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ for all $s \in \mathbb{R}$. Therefore it remains to check that the extension of $p(x, D_x)$ defined on $\mathcal{S}(\mathbb{R}^n)$ to $H_2^s(\mathbb{R}^n)$ coincides with the restriction of $p(x, D_x)$ defined on $\mathcal{S}'(\mathbb{R}^n)$ by duality.

Let P_X denote the operator $p(x, D_x)$ defined on $X = \mathcal{S}(\mathbb{R}^n)$, $X = \mathcal{S}'(\mathbb{R}^n)$, or $X = H_2^{s+m}(\mathbb{R}^n)$. Moreover, let $f \in H_2^{s+m}(\mathbb{R}^n)$. Then there is a sequence $f_k \in \mathcal{S}(\mathbb{R}^n)$ such that $\lim_{k\to\infty} f_k = f$ in $H_2^{s+m}(\mathbb{R}^n)$. Then

$$P_{H_2^{s+m}} f = \lim_{k\to\infty} P_{H_2^{s+m}} f_k = \lim_{k\to\infty} P_{\mathcal{S}} f_k \qquad \text{in } H_2^s(\mathbb{R}^n).$$

Moreover, $f_k \to_{k\to\infty} f$ in $H_2^{s+m}(\mathbb{R}^n)$ implies $f_k \to_{k\to\infty} f$ in $\mathcal{S}'(\mathbb{R}^n)$ since

$$|\langle f_k - f, \varphi \rangle| = |(\langle D_x \rangle^{s+m}(f_k - f), \langle D_x \rangle^{-s-m} \varphi)| \leq \|f_k - f\|_{H_2^{s+m}} \|\langle D_x \rangle^{-s-m} \varphi\|_2 \to_{k\to\infty} 0$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Therefore

$$\langle P_{\mathcal{S}'} f, \varphi \rangle = \langle f, p^*(x, D_x)\varphi \rangle = \lim_{k\to\infty} \langle f_k, p^*(x, D_x)\varphi \rangle = \lim_{k\to\infty} \langle P_{\mathcal{S}} f_k, \varphi \rangle = \langle P_{H_2^{s+m}} f, \varphi \rangle$$

since $P_{\mathcal{S}} f_k \to_{k\to\infty} P_{H_2^{s+m}} f$ in $H_2^s(\mathbb{R}^n)$ implies $P_{\mathcal{S}} f_k \to_{k\to\infty} P_{H_2^{s+m}} f$ in $\mathcal{S}'(\mathbb{R}^n)$ as above.

4.10 Summary and Final Remarks

In this chapter we have studied pseudodifferential symbols $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}^n$, and the relation to the associated operators $p(x, D_x)$, first defined on $\mathcal{S}(\mathbb{R}^n)$ and later extended to larger spaces.

Finally, we note the following structural properties: The symbol space $S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ is a topological vector space, which topology is defined by a sequence of (increasing and separating) semi-norms (a Fréchet space). The space of continuous linear mappings $T\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ denoted by $\mathcal{L}(\mathcal{S}(\mathbb{R}^n))$ is also a topological vector space if equipped with the topology of pointwise convergence. In these terms

$$\mathrm{OP}\colon S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) \to \mathcal{L}(\mathcal{S}(\mathbb{R}^n))\colon p \mapsto p(x, D_x)$$

is a homomorphism of topological vector spaces (i.e., linear and continuous). Moreover, on the level of symbol we have the pointwise multiplication of symbols:

$$\cdot\colon S^{m_1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) \times S^{m_2}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) \to S^{m_1+m_2}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) : (p_1, p_2) \mapsto p_1(x,\xi) \cdot p_2(x,\xi),$$

which is a continuous bilinear mapping. On the level of operators we have the composition. Therefore the question arises how these operations on the symbol and on the operator level are related. We have proved that there is a continuous bilinear mapping

$$\#\colon S^{m_1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) \times S^{m_2}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) \to S^{m_1+m_2}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) : (p_1, p_2) \mapsto p_1 \# p_2$$

such that

$$\mathrm{OP}(p_1 \# p_2) = \mathrm{OP}(p_1) \circ \mathrm{OP}(p_2).$$

Although in general $(p_1 \# p_2)(x,\xi) \neq p_1(x,\xi) \cdot p_2(x,\xi)$, we have obversed that

$$(p_1 \# p_2)(x,\xi) = p_1(x,\xi) \cdot p_2(x,\xi) + r(x,\xi), \qquad r \in S^{m_1+m_2-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n).$$

Hence the relation $(p_1 \# p_2)(x,\xi) \equiv p_1(x,\xi) \cdot p_2(x,\xi)$ holds modulo symbols of lower order. This relation shows that inversion of an operator modulo lower order operators coincides with inversion of the symbol (for large $|\xi|$). This was used to construct a parametrix for an elliptic pseudodifferential operator.

After studying compositions of pseudodifferential operators, it was shown that every pseudodifferential operator $p(x, D_x)$ possesses a formal adjoint $p^*(x, D_x)$ and

$${}^*\colon S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) \to S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)\colon p \mapsto p^*$$

is a continuous *involution*, i.e. $(p^*)^* = p$, which coincides with complex conjugation modulo lower order terms:

$$p^*(x,\xi) = \overline{p(x,\xi)} + r(x,\xi), \qquad r \in S^{m-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n).$$

Hence the pseudodifferential operators are a rich class of operators with many algebraic properties. These properties make pseudodifferential operators to an important tool in *index theory*, cf. [?] or [?].

Finally, we study the pseudodifferential operator $p(x, D_x)$ on $C_b^\infty(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$, and as operator between Bessel potential spaces. We note that

$$\mathcal{S}(\mathbb{R}^n) \subset C_b^\infty(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n), \qquad \mathcal{S}(\mathbb{R}^n) \subset H_2^s(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n).$$

In order to study $p(x, D_x)$ (initially defined on $\mathcal{S}(\mathbb{R}^n)$) on these spaces, the operator has to be extended in a suitable way. But, since all spaces are contained in $\mathcal{S}'(\mathbb{R}^n)$, it remains to check that the extension is consistent with the restriction of $p(x, D_x)$ defined on $\mathcal{S}'(\mathbb{R}^n)$ by duality to the corresponding space. This was done for the Bessel potential spaces. But it remains to prove this for the definition of $p(x, D_x)$ on $C_b^\infty(\mathbb{R}^n)$ due to (4.23).

5 Some Extensions and Applications

5.1 Resolvents and Parameter-Elliptic Differential Operators

The study parabolic initial value problems

$$\begin{aligned} \partial_t u(x,t) + Au(x,t) &= 0 \qquad \text{for } (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) &= u_0 \qquad \text{for } x \in \mathbb{R}^n, \end{aligned}$$

where A is a suitable elliptic operator as $-\Delta$, can be reduced to the resolvent equation

$$(\lambda + A)u(x) = f \qquad \text{for } x \in \mathbb{R}^n.$$

This is the content of the theory of analytic semi-groups, cf. e.g. [?],[RR93]. We only state one of the main results

THEOREM 5.1 *Let X be a Banach space and let $A\colon \mathcal{D}(A) \subseteq X \to X$ be a linear, closed, and densely defined operator. If $\lambda + A$ is invertible for every $\lambda \in \lambda_0 + \Sigma_\delta$, $\Sigma_\delta := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \delta\}$, where $\delta \in (-\pi/2, \pi)$ and $\lambda_0 \in \mathbb{R}$, and satisfies*

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{C}{|\lambda - \lambda_0|} \qquad \text{for } \lambda \in \lambda_0 + \Sigma_\delta, \tag{5.1}$$

then

$$e^{-At} := \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda + A)^{-1} d\lambda \qquad \text{for } t \in \Sigma_{\delta - \pi/2}$$

is a bounded linear operator, which depends analytically on $t \in \Sigma_{\delta-\pi/2}$ and which satisfies

1. *$\frac{d}{dt} e^{-At} = -Ae^{-At}$ for $t \in \Sigma_{\delta-\pi/2}$,*
2. *$\lim_{t\to 0} e^{-At} u_0 = u_0$ for all $u_0 \in X$,*
3. *$\|e^{-At}\|_{\mathcal{L}(X)} \leq Ce^{\lambda_0 t}$ for $t \in \Sigma_{\delta-\pi/2}$.*

Here Γ is the negatively orientated boundary of $(\lambda_0 + \Sigma_\delta) \setminus B_1(0)$.

This theorem is a consequence of the characterization of the so called *analytic and strongly continuous semi-group*, cf. [?, Theorem 3.1, Chapter 1] or [RR93, Theorem 11.17].

In the following let $a(x,\xi) = \sum_{|\alpha|\leq m} c_\alpha(x)\xi^\alpha$, $c_\alpha \in C_b^\infty(\mathbb{R}^n)$, $m \in \mathbb{N}$. Moreover, let $a_0(x,\xi) = \sum_{|\alpha|=m} c_\alpha(x)\xi^\alpha$ denote its *principle symbol*

Definition 5.2 Let $\delta \in (0,\pi)$ and $a(x,\xi)$ a differential symbol as above. Then a is said to be *(uniformly) parameter elliptic* on Σ_δ if

$$\overline{\{a_0(x,\xi) : x \in \mathbb{R}^n, |\xi| = 1\}} \subset \Sigma_{\pi-\delta} \quad \text{and} \quad |a_0(x,\xi)| \geq c > 0 \qquad x \in \mathbb{R}^n, |\xi| = 1$$

Example 5.3 Let $a(x,\xi) = |\xi|^2$. Then $m = 2$ and $|a_0(x,\xi)| = 1$ for all $x \in \mathbb{R}^n$ and $|\xi| = 1$. Hence a is parameter elliptic on Σ_δ for every $\delta \in (0,\pi)$.

Moreover, if $A(x) \in C_b^\infty(\mathbb{R}^n)^{n\times n}$ is a real symmetric matrix, which is uniformly positive definite, i.e., $\xi^T A(x)\xi \geq c > 0$, then $a(x,\xi) = \xi^T A(x)\xi$ is parameter elliptic on Σ_δ for every $\delta \in (0,\pi)$.

Finally, if $n = 1$, $a(x,\xi) = i\xi$ is uniformly elliptic on Σ_δ only if $\delta \in (0,\pi/2)$.

Proposition 5.4 Let a be parameter elliptic on Σ_δ, $\delta \in (0,\pi)$, and let $a_0(x,\xi)$ be its principal symbol. Then

$$|\partial_\xi^\alpha \partial_x^\beta (\lambda + a_0(x,\xi))^{-1}| \leq C(1 + |\lambda|^{\frac{1}{m}} + |\xi|)^{-m-|\alpha|} \tag{5.2}$$

for all $\lambda \in \Sigma_\delta$, $x,\xi \in \mathbb{R}^n$, such that $|\lambda| + |\xi|^m \geq 1$.

Proof: The proof relies essentially on the following statement:

Claim:

$$|\partial_\xi^\alpha \partial_x^\beta (\lambda + a_0(x,\xi))^{-1}| \leq C \tag{5.3}$$

for all $\lambda \in \Sigma_\delta$ and $\xi \in \mathbb{R}^n$ with $\max(|\xi|^m, |\lambda|) = 1$.

Proof of the claim: First let $\max(|\xi|^m, |\lambda|) = |\xi| = 1$. Since $c_\alpha \in C_b^\infty(\mathbb{R}^n)$, the set

$$A := \overline{\{-a_0(x,\xi) : x \in \mathbb{R}^n, |\xi| = 1\}}$$

is bounded. Moreover, $A \subset -\Sigma_{\pi-\delta} = \mathbb{C} \setminus \overline{\Sigma_\delta}$. Hence $\operatorname{dist}(A, \partial\Sigma_\delta) > 0$ since A is compact and $\partial\Sigma_\delta$ is closed. Thus

$$|(\lambda + a_0(x,\xi))^{-1}| \leq \frac{1}{\operatorname{dist}(A, \partial\Sigma_\delta)}$$

for all $|\xi| = 1$ and $\lambda \in \Sigma_\delta$. Therefore

$$|\partial_\xi^\alpha \partial_x^\beta (\lambda + a_0(x,\xi))^{-1}| \leq C_{\alpha,\beta} \qquad \text{for } |\xi| = 1, x \in \mathbb{R}^n, \lambda \in \Sigma_\delta,$$

by the chain and product rule since all derivatives of $a_0(x,\xi)$ are uniformly bounded on $|\xi| = 1$, $x \in \mathbb{R}^n$.

Now we consider the case $\max(|\xi|^m, |\lambda|) = |\lambda| = 1$. Because of the homogeneity of the principal symbol, i.e., $a_0(x, r\xi) = r^m a_0(x,\xi)$ for $r > 0$,

$$\overline{\{a_0(x,\xi) : x \in \mathbb{R}^n, |\xi| \leq 1\}} = \bigcup_{0 \leq r \leq 1} r^m \cdot \overline{\{a_0(x,\xi) : x \in \mathbb{R}^n, |\xi| = 1\}} \subset \Sigma_{\pi-\delta} \cup \{0\},$$

where $\bigcup_{0\leq r\leq 1} r^m \cdot \overline{\{a_0(x,\xi) : x \in \mathbb{R}^n, |\xi| = 1\}}$ is a compact set since it is the image of $[0,1] \times \overline{\{a_0(x,\xi) : x \in \mathbb{R}^n, |\xi| = 1\}}$ under the multiplication $(r,z) \mapsto r^m \cdot z$. Now let

$$A := \overline{\{-a_0(x,\xi) : x \in \mathbb{R}^n, |\xi| \leq 1\}}.$$

Then $A \subset \mathbb{C} \setminus \Sigma_\delta$ and $\operatorname{dist}(A, \partial\Sigma_\delta \cap \{|\lambda| = 1\}) > 0$. Thus

$$|(\lambda + a_0(x,\xi))^{-1}| \leq \frac{1}{\operatorname{dist}(A, \partial\Sigma_\delta \cap \{|\lambda| = 1\})}$$

and therefore

$$|\partial_\xi^\alpha \partial_x^\beta (\lambda + a_0(x,\xi))^{-1}| \leq C_{\alpha,\beta}$$

for all $|\xi| \leq 1$ and $\lambda \in \Sigma_\delta$ with $|\lambda| = 1$. This completes the proof of the claim.

Finally, let $a_\lambda(x,\xi) := (\lambda + a_0(x,\xi))^{-1}$. Then using the homogeneity

$$\partial_\xi^\alpha \partial_x^\beta a_\lambda(x,\xi) = r^{-m-|\alpha|} \partial_\xi^\alpha \partial_x^\beta a_{r^{-m}\lambda}\left(x, \frac{\xi}{r}\right),$$

(5.3) implies

$$|\partial_\xi^\alpha \partial_x^\beta (\lambda + a_0(x,\xi))^{-1}| \leq C \max(|\xi|, |\lambda|^{\frac{1}{m}})^{-m-|\alpha|} \leq C'(|\lambda|^{\frac{1}{m}} + |\xi|)^{-m-|\alpha|}$$

for all $\xi \in \mathbb{R}^n$ and $\lambda \in \Sigma_\delta$, which proves the proposition. ■

Lemma 5.5 *Let $a(x,\xi)$ be a parameter-elliptic symbol on Σ_δ, $\delta \in (0,\pi)$, as defined above. Then $A = a(x,D_x)\colon \mathcal{D}(A) \to H_2^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, with $\mathcal{D}(A) = H_2^{s+m}(\mathbb{R}^n) \subset H_2^s(\mathbb{R}^n)$ is closed and densely defined linear operator. Moreover, there is an $R > 0$ such that $\lambda + A$ is invertible for all $\lambda \in \Sigma_\delta$ with $|\lambda| \geq R > 0$ and*

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(H_2^s(\mathbb{R}^n))} \leq C|\lambda|^{-1}.$$

Proof: Let $\lambda \in \Sigma_\delta$ with $|\lambda| \geq 1$. Moreover, let $a_\lambda(x,\xi) = (\lambda + a_0(x,\xi))^{-1}$ and let $m' \in [0,m]$. Then

$$|\partial_\xi^\alpha \partial_x^\beta a_\lambda(x,\xi)| \leq C\left(1 + |\lambda|^{\frac{1}{m}} + |\xi|\right)^{-m-|\alpha|} \leq C|\lambda|^{-\frac{m-m'}{m}} \langle\xi\rangle^{-m'-|\alpha|} \tag{5.4}$$

for all $\lambda \in \Sigma_\delta$, $|\lambda| \geq 1$, and $\xi \in \mathbb{R}^n$. Hence $a_\lambda \in S^{-m'}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ with $|a_\lambda|_k^{(-m')} \leq C_k|\lambda|^{-\frac{m-m'}{m}}$ uniformly in $\lambda \in \Sigma_\delta$ with $|\lambda| \geq 1$ for every $k \in \mathbb{N}$. Therefore

$$\|a_\lambda(x,D_x)\|_{\mathcal{L}(H_2^s(\mathbb{R}^n), H_2^{s+m'}(\mathbb{R}^n))} \leq C|\lambda|^{-\frac{m-m'}{m}} \tag{5.5}$$

uniformly in $\lambda \in \Sigma_\delta$ with $|\lambda| \geq 1$ because of Lemma 4.38. Moreover, because of the asymptotic expansion stated in Theorem 4.16 and the fact all terms depend continuously on p_1 and p_2

$$(\lambda + a_0(x,D_x))a_\lambda(x,D_x) = \lambda a_\lambda(x,D_x) + a_0(x,D_x)a_\lambda(x,D_x) = I + r_\lambda(x,D_x),$$

where

$$|r_\lambda|_k^{(0)} \leq C_k |a_0|_{n(k)}^{(m)} |a_\lambda|_{n(k)}^{(-m+1)} \leq C_k|\lambda|^{-\frac{1}{m}}.$$

(More precisely, since $a_0(x,D_x)$ is a differential operator

$$r_\lambda(x,\xi) = \sum_{|\alpha|\le m, \alpha\neq 0} \frac{1}{\alpha!}\partial_\xi^\alpha a_0(x,\xi) D_x^\alpha a_\lambda(x,\xi).)$$

Thus

$$\|r_\lambda(x,D_x)\|_{\mathcal{L}(H_2^s(\mathbb{R}^n))} \le C|\lambda|^{-\frac{1}{m}}$$

for all $|\lambda|\ge 1$. Furthermore,

$$(\lambda + a(x,D_x))a_\lambda(x,D_x) = I + r_\lambda(x,D_x) + R'_\lambda,$$

where $R'_\lambda := \sum_{|\alpha|<m} c_\alpha(x) D_x^\alpha a_\lambda(x,D_x)$ satisfies

$$\|R'_\lambda\|_{\mathcal{L}(H_2^s(\mathbb{R}^n))} \le C\|a_\lambda(x,D_x)\|_{\mathcal{L}(H_2^s(\mathbb{R}^n),H_2^{s+m-1}(\mathbb{R}^n))} \le C|\lambda|^{-\frac{1}{m}}$$

due to (5.5) for $m' = m-1$.

Hence there is an $R>0$ such that $\|r_\lambda(x,D_x) + R'_\lambda\|_{\mathcal{L}(H_2^s(\mathbb{R}^n))} \le \frac{1}{2}$ for all $|\lambda|\ge R$. Therefore $I + r_\lambda(x,D_x) + R'_\lambda$ is invertible and

$$\|(I + r_\lambda(x,D_x) + R'_\lambda)^{-1}\|_{\mathcal{L}(H_2^s(\mathbb{R}^n))} \le 2$$

for all $|\lambda|\ge R$. Therefore $\lambda + a(x,D_x)$ is invertible for all $|\lambda|\ge R$ and

$$\begin{aligned} &\|(\lambda + a(x,D_x))^{-1}\|_{\mathcal{L}(H_2^s(\mathbb{R}^n))} \\ &\le \|a_\lambda(x,D_x)\|_{\mathcal{L}(H_2^s(\mathbb{R}^n))}\|(I + r_\lambda(x,D_x) + R'_\lambda)^{-1}\|_{\mathcal{L}(H_2^s(\mathbb{R}^n))} \le C|\lambda|^{-1}. \end{aligned}$$

■

Remark 5.6 The latter lemma implies that the spectrum of $-A$, where A is a parameter elliptic differential operator, is contained in a *key-hole region* $(\mathbb{C}\setminus\Sigma_\delta)\cup B_R(0)$ for some $R>0$ and that the resolvent $(\lambda + A)^{-1}$ decays as $|\lambda|^{-1}$ as $|\lambda|\to\infty$ in Σ_δ.

Corollary 5.7 *Let $A = a(x,D_x)$ and $\delta\in(\pi/2,\pi)$ be as in Lemma 5.5. Moreover, let $\omega := \sup\{\operatorname{Re}\lambda : \lambda\in\sigma(-A)\}$. Then for every $\lambda_0>\omega$ there is a $\delta'\in(0,\delta)$ such that A satisfies the conditions of Theorem 5.1. In particular, A generates a strongly continuous, analytic semi-group e^{-At}, $t\ge 0$, which satisfies*

$$\|e^{-At}\|_{\mathcal{L}(H_2^s(\mathbb{R}^n))} \le Ce^{\lambda_0 t}, \qquad t\ge 0.$$

Proof: First of all we choose $\delta'\in(\pi/2,\pi)$ so close to $\pi/2$ such that $\overline{\lambda_0 + \Sigma_{\delta'}}\subset\rho(-A)$. (Such a δ' exists since $\sigma(-A)\cap\{\operatorname{Re}\lambda = \omega\}$ is bounded.)

Because of Lemma 5.5,

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(H_2^s(\mathbb{R}^n))} \leq \frac{C}{|\lambda|} \leq \frac{C'}{|\lambda - \lambda_0|}$$

for all $\lambda \in \Sigma_\delta$ with $|\lambda| \geq R' := \max(R, |\lambda_0| + 1)$. Moreover, since $\delta' < \delta$ there is an $R'' \geq R'$ such that $(\lambda_0 + \Sigma_{\delta'}) \setminus B_{R''}(0) \subset \Sigma_\delta \setminus B_{R''}(0)$.

On the other hand, $(\lambda + A)^{-1}$ is a continuous (even holomorphic) function on the compact set $\overline{(\lambda_0 + \Sigma_\delta) \cap B_{R''}(0)} \subset \rho(-A)$. Hence

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(H_2^s(\mathbb{R}^n))} \leq C \leq \frac{C'}{|\lambda - \lambda_0|}$$

for all $\lambda \in \lambda_0 + \Sigma_\delta$ with $|\lambda| \leq R''$. Altogther we have proved (5.1). Therefore Theorem 5.1 implies the statement of the corollary. ■

5.2 Kernel Representation of a Pseudodifferential Operator

A classical theorem due to Schwartz states that for every bounded linear operator

$$K\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$$

there is a unique $k \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\langle Ku, v\rangle_{\mathcal{S}(\mathbb{R}^n)} = \langle k, u\otimes v\rangle_{\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n)}, \tag{5.6}$$

, where $(u\otimes v)(x,y) = u(x)v(y)$, cf. e.g. [Tre67, Corollary to Theorem 51.6]. Here $k \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ is called *kernel* of K. Formally $Ku = \int_{\mathbb{R}^n} k(x,y)u(y)dy$.

Obviously, this result applies to pseudodifferential operators since $p(x,D_x)\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ for $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ and $m \in \mathbb{R}$ arbitrary.

For the case that $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ with $m \leq -n-1$, we have seen in the proof of Lemma 4.32 that

$$p(x,D_x)f = \int_{\mathbb{R}^n} k'(x,x-y)f(y)dy,$$

where $k'(x,z) = \mathcal{F}^{-1}_{\xi\mapsto x}[p(x,.)]$ is a continuous function in $z \in \mathbb{R}^n$. This shows that in this case the kernel of $K = p(x,D_x)$ due to (5.6) is the continuous function

$$k(x,y) = k'(x,x-y).$$

In general the kernel of a pseudodifferential operator does not have to be continuous. For example if $p(x,\xi) = 1$ and therefore $p(x,D_x) = I$, the kernel is the distribution $k \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ defined by

$$\langle k, u(x)v(y)\rangle_{\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n)} = \int_{\mathbb{R}^n} u(x)v(x)dx$$

Formally "$k(x,y) = \delta_0(x-y)$" and $p(x,D_x)f = \delta_0 * f$, where $*$ denotes the convolution.

The main result of this section gives a precise statement of the form of the Schwartz kernel of a pseudodifferential operator.

THEOREM 5.8 *Let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$. Then the Schwartz kernel $K \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ of $p(x,D_x)$ is a smooth function on $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$. More precisely,*

$$\langle K, u\otimes v\rangle_{\mathcal{S}(\mathbb{R}^n\times\mathbb{R}^n)} = \int_{\mathbb{R}^n\times\mathbb{R}^n} k(x,x-y)u(x)v(y)d(x,y),$$

for all $u,v \in \mathcal{S}(\mathbb{R}^n)$ with $\operatorname{supp} u \cap \operatorname{supp} v = \emptyset$ where k is a smooth function in $x,z \in \mathbb{R}^n$, $z \neq 0$, which satisfies

$$|\partial_x^\beta \partial_y^\alpha k(x,z)| \leq C_{\alpha,\beta,N}|z|^{-n-m-|\alpha|-N} \tag{5.7}$$

if $n+m+|\alpha|+N > 0$.

In order to prove this theorem we need some preliminaries.

Dyadic decomposition

In this section we introduce an important tool in modern harmonic analysis and theory of function spaces - the dyadic decomposition of the phase space.

Let $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi_0(\xi) = 1$ for $|\xi| \leq 1$ and $\varphi_0(\xi) = 0$ for $|\xi| \geq 2$. Moreover, let $\varphi_j(\xi) = \varphi_0(2^{-j}\xi) - \varphi_0(2^{-j+1}\xi)$ for $j \in \mathbb{N}$. Then

$$\operatorname{supp} \varphi_j \subseteq \{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}, \qquad j \in \mathbb{N}$$

and

$$\sum_{j=0}^{\infty} \varphi_j(\xi) = \sum_{j=0}^{k} \varphi_j(\xi) = \varphi_0(2^{-k}\xi) = 1$$

for $\xi \in \mathbb{R}^n$ and $k \in \mathbb{N}$ such that $|\xi| \leq 2^k$. Hence φ_j, $j \in \mathbb{N}$, is a partition of unity on $\mathbb{R}^n$ subordinated to the dyadic spheres $\{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$, $j \in \mathbb{N}$, and $\overline{B_2(0)}$.

Remark 5.9 The asymptotic behavior of a function $f(\xi)$ as $|\xi| \to \infty$ can be described with the aid of this partition of unity an alternative way as follows:

$$|f(\xi)| \leq C\langle\xi\rangle^m \quad \Leftrightarrow \quad \sup_{\xi \in \mathbb{R}^n} |\varphi_j(\xi) f(\xi)| \leq C' 2^{jm} \quad \text{for all } j \in \mathbb{N}_0$$

where $m \in \mathbb{R}$ and $C' > 0$ does not depend on j. - See exercises.

Obviously, $\varphi_j \in S^{-\infty}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ since $\varphi_j \in C_0^\infty(\mathbb{R}^n)$. Moreover,

$$\sum_{j=0}^{N} \varphi_j(\xi) = \varphi_0(2^{-N}\xi) \to_{N\to\infty} 1 \qquad \text{pointwise and}$$

$$\sum_{j=0}^{N} \partial_\xi^\alpha \varphi_j(\xi) \to_{N\to\infty} 0 \qquad \text{uniformly if } \alpha \neq 0.$$

Hence

$$\sum_{j=0}^{\infty} \varphi_j(D_x) f = f \qquad \text{in } \mathcal{S}(\mathbb{R}^n)$$

since $|\sum_{j=0}^{N} \varphi_j(D_x) f - f|''_{k,\mathcal{S}} \to_{N\to\infty} 0$ for all $k \in \mathbb{N}$, where $|.|''_{k,\mathcal{S}}$, $k \in \mathbb{N}$, is the equivalent sequence of semi-norms on $\mathcal{S}(\mathbb{R}^n)$ which was defined in Section 4.10 by replacing $\|.\|_\infty$ by $\|.\|_2$.

As a direct consequence, we can decompose a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ as

$$f = \sum_{j=0}^{\infty} \varphi_j(D_x) f$$

where the series converges to f in $\mathcal{S}'(\mathbb{R}^n)$ since

$$\lim_{N\to\infty}\langle\sum_{j=0}^{N}\varphi_j(D_x)f,g\rangle=\lim_{N\to\infty}\langle f,\sum_{j=0}^{N}\varphi_j(D_x)g\rangle=\langle f,g\rangle$$

for all $g\in\mathcal{S}(\mathbb{R}^n)$.

Remark 5.10 Here $f_j:=\varphi_j(D_x)f$, $j\in\mathbb{N}$, is smooth since it is the convolution of a tempered distribution with $\mathcal{F}^{-1}[\varphi_j]\in\mathcal{S}(\mathbb{R}^n)$, cf. exercise sheet 11. By the Paley-Wiener theorem, cf. [KG74, Theorem 5.4, Chapter I] or [Ray91], f_j is even real analytic since $\operatorname{supp}\hat{f}_j$ is compact.

With the aid of the dyadic decomposition we decompose a pseudodifferential operator as

$$p(x,D_x)f=\sum_{j=0}^{\infty}p(x,D_x)\varphi_j(D_x)f=\sum_{j=0}^{\infty}p_j(x,D_x)f,\qquad f\in\mathcal{S}(\mathbb{R}^n),$$

where $p_j(x,\xi)=p(x,\xi)\varphi_j(\xi)\in S^{-\infty}_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$ and the series converges in $\mathcal{S}(\mathbb{R}^n)$ since $p(x,D_x)\colon\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n)$ is continuous. Moreover, since $p_j(x,\xi)$ is compactly supported in ξ,

$$p_j(x,D_x)f=\int_{\mathbb{R}^n}k_j(x,x-y)f(y)dy,$$

where $k_j(x,z)=\mathcal{F}^{-1}_{\xi\mapsto z}[p_j(x,\xi)]$ as in the proof of Lemma 4.32.

Lemma 5.11 *Let $p\in S^m_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$, $m\in\mathbb{R}$ and let $k_j(x,z)$ be defined as above. Then*

$$|\partial_x^\beta\partial_z^\alpha k_j(x,z)|\le C_{\alpha,\beta,M}|z|^{-M}2^{j(n+m-M+|\alpha|)}\tag{5.8}$$

for all $\alpha,\beta\in\mathbb{N}_0^n$, $M\in\mathbb{N}_0$, where $C_{\alpha,\beta,M}$ does not depend on $j\in\mathbb{N}_0$.

Proof: First of all,

$$z^\gamma\partial_x^\beta D_z^\alpha k_j(x,z)=\int_{\mathbb{R}^n}e^{ix\cdot\xi}D_\xi^\gamma[\xi^\alpha\partial_x^\beta p_j(x,\xi)]đ\xi$$

for all $\alpha,\beta,\gamma\in\mathbb{N}_0^n$. We now make the most direct estimates on the above integral. First, the integrand is supported in the ball $\{|\xi|\le 2^{j+1}\}$, which has volume bounded by a multiple of 2^{nj}. Second, since the support is also limited by the condition $2^{j-1}\le|\xi|$ (when $j\ne 0$) and $c2^j\le\langle\xi\rangle\le C2^j$ on $\{2^{j-1}\le|\xi|\le 2^{j+1}\}$,

$$\left|D_\xi^\gamma[\xi^\alpha\partial_x^\beta p_j(x,\xi)]\right|\le C_{\alpha,\beta,\gamma}2^{j(m+|\alpha|-|\gamma|)}$$

due to the symbol estimates of $\xi^\alpha\partial_x^\beta p_j(x,\xi)\in S^{m+|\alpha|}_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$. Hence

$$|z^\gamma D_x^\beta D_z^\alpha k_j(x,z)|\le C_{\alpha,\beta,\gamma}2^{j(n+m+|\alpha|-M)},\qquad\text{whenever }|\gamma|=M.$$

Taking the supremum over all γ with $|\gamma| = M$, gives (5.8), and proves the lemma. ■

Proof of Theorem 5.8: Let $K \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ be the Schwartz kernel of $p(x, D_x)$. Then

$$\begin{aligned} \langle K, u \otimes v \rangle_{\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)} &= \langle p(x, D_x)u, v \rangle_{\mathcal{S}(\mathbb{R}^n)} = \sum_{j=0}^{\infty} \langle p_j(x, D_x)u, v \rangle \\ &= \sum_{j=0}^{\infty} \int_{\mathbb{R}^n \times \mathbb{R}^n} k_j(x, x-y) u(y) v(x) d(x, y) \end{aligned}$$

for all $u, v \in \mathcal{S}(\mathbb{R}^n)$, where $(u \otimes v)(x, y) = u(x)v(y)$. Let $k(x, z) = \sum_{j=0}^{\infty} k_j(x, z)$ for $x, z \in \mathbb{R}^n$, $z \neq 0$. It is sufficient to show that

$$\sum_{j=0}^{\infty} |\partial_x^\beta \partial_z^\alpha k_j(x, z)|$$

satisfies the estimates given by the right side of (5.7) for $z \neq 0$. This implies that

$$\langle K, u \otimes v \rangle_{\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)} = \int_{\mathbb{R}^{2n}} k(x, x-y) u(y) v(x) d(x, y)$$

for all $u, v \in \mathcal{S}(\mathbb{R}^n)$ with $\operatorname{supp} u \cap \operatorname{supp} v = \emptyset$, which implies the statement.

First we consider the case when $0 < |z| \leq 1$. We brake the above sum into two parts: the first where $2^j \leq |z|^{-1}$, the second where $2^j > |z|^{-1}$. In order to estimate the first sum we use (5.8) with $M = 0$:

$$\sum_{2^j \leq |z|^{-1}} |\partial_x^\beta \partial_z^\alpha k_j(x, z)| \leq C_{\alpha,\beta} \sum_{2^j \leq |z|^{-1}} 2^{j(n+m+|\alpha|)},$$

where

$$\sum_{2^j \leq |z|^{-1}} 2^{j(n+m+|\alpha|)} = \begin{cases} O(|z|^{-n-m-|\alpha|}) & \text{when } n+m+|\alpha| > 0, \\ O(\log(|z|^{-1}) + 1) & \text{when } n+m+|\alpha| \leq 0. \end{cases}$$

In either cases we get the estimate

$$\sum_{2^j \leq |z|^{-1}} |\partial_x^\beta \partial_z^\alpha k_j(x, z)| \leq C_{\alpha,\beta} |z|^{-n-m-|\alpha|-N}$$

under the restriction that $|z| \leq 1$, $N \geq 0$, and $n + m + |\alpha| + N \geq 0$.

Next, for the second sum, we choose $M > n + m + |\alpha|$. Because of (5.8), we get the estimate

$$\sum_{2^j > |z|^{-1}} |\partial_x^\beta \partial_z^\alpha k_j(x, z)| \leq C_{\alpha,\beta,M} |z|^{-M} \sum_{2^j > |z|^{-1}} 2^{j(n+m+|\alpha|-M)} \leq C'_{\alpha,\beta,M} |z|^{-n-m-|\alpha|}.$$

The last term is certainly $O(|z|^{-n-m-|\alpha|-N})$, if $N \geq 0$, since $|z| \leq 1$.

Finally, we consider the situation $|z| \geq 1$. If $M > n + m + |\alpha| + N$, then (5.8) shows that

$$\sum_{j=0}^{\infty} |\partial_x^\beta \partial_z^\alpha k_j(x,z)| \leq C_{\alpha,\beta,M} |z|^{-M} \sum_{j=0}^{\infty} 2^{j(n+m-M+|\alpha|)} \leq C'_{\alpha,\beta,M} |z|^{-M}.$$

But $|z|^{-M} \leq |z|^{-n-m-|\alpha|-N}$ since $|z| \geq 1$. Hence the proof is complete. ■

We have the following consequences of Theorem 5.8:

1. If $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $m < 0$, then $\sup_{x\in\mathbb{R}^n} |k(x,z)| \in L^1(\mathbb{R}^n)$. Therefore

   $$\|p(x,D_x)f\|_q \leq \sup_{x\in\mathbb{R}^n} |k(x,z)| \|f\|_q, \qquad f \in L^q(\mathbb{R}^n),$$

   for all $1 \leq q \leq \infty$; cf. exercises.

2. Let $p \in S^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Then, because of the boundedness $p(x,D_x)\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and the kernel estimates

   $$\begin{aligned} \sup_{x\in\mathbb{R}^n} |k(x,z)| &\leq C|z|^{-n}, \\ \sup_{x\in\mathbb{R}^n} |\nabla_{x,z} k(x,z)| &\leq C|z|^{-n-1}, \end{aligned}$$

   a general result from the theory of *singular integral operators*, cf. [Ste93, Chapter 5, Section 5.1] or [Tay91, §0.11], implies that

   $$p(x,D_x)\colon L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$$

   is a bounded operator for all $1 < q < \infty$. – Note that the latter statement is in general false if $q = 1$ or $q = \infty$.

3. If $\varphi, \psi \in C_b^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$, then $\varphi(x)p(x,D_x)\psi(x)$ is a smoothing operator in the sense that

   $$\varphi(x)p(x,D_x)\psi(x)\colon \mathcal{S}'(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n).$$

   This statement can be proved as follows: If $f \in \mathcal{S}'(\mathbb{R}^n)$, then there is an $N \in \mathbb{N}$ such that $f' := \langle x\rangle^{-2N} \langle D_x\rangle^{-2N} f \in L^2(\mathbb{R}^n)$, cf. Exercise sheet 12. Hence

   $$\varphi(x)p(x,D_x)\psi f = \varphi(x) \int_{\mathbb{R}^n} k(x,x-y)\psi(y)\langle D_y\rangle^{2N} \langle y\rangle^{2N} f'(y)dy$$

   since $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$ implies $\varphi(x)k(x,x-y)\psi(y) = 0$ in a neighborhood of the diagonal $x = y$. Moreover,

   $$\varphi(x)p(x,D_x)(\psi f) = \int_{\mathbb{R}^n} \left[\langle D_y\rangle^{2N} (\varphi(x)k(x,x-y)\psi(y))\right] \langle y\rangle^{2N} f'(y)dy,$$

   where Theorem 5.8 implies that $\langle D_y\rangle^{2N} \varphi(x)k(x,x-y)\psi(y)$ is a smooth kernel which is rapidly decreasing in y. This implies the statement.

4. If $f \in \mathcal{S}'(\mathbb{R}^n)$, then the *singular support* of f denoted by $\operatorname{sing\,supp} f$ is the complement of the set U of all $x \in \mathbb{R}^n$ such that $f|_V$ coincides with a smooth function for an open neighborhood V of x, i.e.,

$$\langle f, \varphi \rangle = \int_{\mathbb{R}^n} \tilde{f}(x)\varphi(x)dx \qquad \text{for all } \varphi \in C_0^\infty(V),$$

where $\tilde{f} \in C^\infty(\mathbb{R}^n)$. It is easy to show that U is open and therefore $\operatorname{sing\,supp} f$ is closed.

If $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, it holds that

$$\operatorname{sing\,supp} p(x, D_x)f \subseteq \operatorname{sing\,supp} f \qquad \text{for all } f \in \mathcal{S}'(\mathbb{R}^n)$$

An operator satisfying this property is called *pseudo-local.* The statement can be proved using the definitions and the previous statement.

5.3 Boundedness on Hölder-Zygmund and Besov Spaces

Definition 5.12 Let $s > 0$. Then the *(Hölder-)Zygmund space* $C^s_*(\mathbb{R}^n)$ is defined as

$$\begin{aligned} C^s_*(\mathbb{R}^n) &= \{u \in C^0_b(\mathbb{R}^n) : \|\varphi_j(D_x)u\|_\infty \leq C2^{-js}, j \in \mathbb{N}_0\}, \\ \|u\|_{C^s_*} &= \sup_{j\in\mathbb{N}_0} 2^{js}\|\varphi_j(D_x)u\|_\infty. \end{aligned}$$

Remark 5.13 If $s \notin \mathbb{N}$, then $C^s_*(\mathbb{R}^n)$ coincides with the usual Hölder space $C^s(\mathbb{R}^n)$ normed by

$$\|u\|_{C^s} = \max_{|\alpha|\leq[s]} \|D^\alpha u\|_\infty + \max_{|\alpha|\leq[s]} \sup_{x,y\in\mathbb{R}^n, x\neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^{s-[s]}}.$$

This is a consequence of Theorem 5.22 below.

THEOREM 5.14 *Let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$, and let $s > 0$ such that $s+m > 0$. Then $p(x, D_x)\colon C^{s+m}_*(\mathbb{R}^n) \to C^s_*(\mathbb{R}^n)$ is a bounded linear operator.*

Lemma 5.15 *Let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$, and let $p_j(x, D_x) = p(x, D_x)\varphi_j(D_x)$ be defined as above. Then*

$$\|p_j(x, D_x)\|_{\mathcal{L}(L^q(\mathbb{R}^n))} \leq C2^{jm}, \qquad j \in \mathbb{N}_0, \tag{5.9}$$

for $1 \leq q \leq \infty$ where C does not dependent on j.

Proof: As seen in Section 5.2,

$$p_j(x, D_x)f = \int_{\mathbb{R}^n} k_j(x, x-y)f(y)dy,$$

where the k_j satisfy (5.8). According to these estimates

$$\int_{\mathbb{R}^n} |k_j(x,z)|dz \leq C \left(\int_{|z|\leq 2^{-j}} 2^{j(n+m)}dz + \int_{|z|>2^{-j}} |z|^{-n-1}2^{j(m+1)}dz \right),$$

the first coming from (5.8) for $M = 0$, and the second from the case $M = n+1$. Hence we get by a simple calculation

$$\int_{\mathbb{R}^n} |k_j(x,z)|dz \leq C2^{jm},$$

which proves (5.9) since

$$\left\| \int_{\mathbb{R}^n} k_j(x, x-y)f(y)dy \right\|_q \leq \sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} |k_j(x,z)|dz\|f\|_q.$$

■

Remark 5.16 If $p_j(D_x, x) = \varphi_j(D_x)p(D_x, x)$ is the corresponding operator in y-form, then

$$p_j(D_x, x)f = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\xi\cdot(x-y)}p_j(y,\xi)f(y)đ\xi dy = \int_{\mathbb{R}^n} k_j(y, x-y)f(y)dy,$$

where $k_j(y,z) = \mathcal{F}^{-1}_{\xi\mapsto z}[p_j(y,\xi)]$ is the same function as for the operator in x-form. Moreover,

$$\|p_j(D_x, x)\|_{\mathcal{L}(L^q(\mathbb{R}^n))} \leq C2^{jm}, \qquad j \in \mathbb{N}_0 \tag{5.10}$$

holds for $1 \leq q \leq \infty$. This inequality is proved in precisely the same way as (5.9).

Lemma 5.17 *Let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$, and let $p_i(x, D_x)$ be defined as above. Then for every $l \in \mathbb{N}$*

$$\|\varphi_j(D_x)p_i(x, D_x)\|_{\mathcal{L}(L^q(\mathbb{R}^n))} \leq C_l 2^{\min(i,j)m}2^{-|i-j|l}, \qquad i, j \in \mathbb{N}_0, \tag{5.11}$$

for $1 \leq q \leq \infty$ where C_l does not dependent on i, j.

Proof: Similarly as in the proof of Lemma 4.42,

$$1 = \left(\frac{1}{1+|\xi|^2} + \sum_{j=1}^n \frac{\xi_j}{1+|\xi|^2}\xi_j\right)^l = \sum_{|\alpha|\leq l} p^{(\alpha)}(\xi)\xi^\alpha, \tag{5.12}$$

where $p^{(\alpha)} \in S^{-l}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. Hence

$$\varphi_j(D_x)p_i(x, D_x) = \varphi_j(D_x)\sum_{|\alpha|\leq l} p^{(\alpha)}(D_x)D^\alpha_x p_i(x, D_x) = \sum_{|\alpha|\leq l} p^{(\alpha)}(D_x)\varphi_j(D_x)D^\alpha_x p_i(x, D_x)$$

since $\varphi_j(D_x)$ and $p^{(\alpha)}(D_x)$ commute. Since $p^{(\alpha)}(\xi) \in S^{-l}_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$ and $D^\alpha_x p_i(x, D_x) = q(x, D_x)\varphi_i(D_x) = q_i(x, D_x)$ with $q \in S^{m+|\alpha|}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, we can now invoke (5.9) for $p^{(\alpha)}_j(D_x) = p^{(\alpha)}(D_x)\varphi_j(D_x)$ and $q_i(x, D_x)$ to get

$$\begin{aligned}&\|\varphi_j(D_x)p_i(x, D_x)\|_{\mathcal{L}(L^q(\mathbb{R}^n))}\\ &\quad\leq \sum_{|\alpha|\leq l}\|p^{(\alpha)}(D_x)\varphi_j(D_x)\|_{\mathcal{L}(L^q(\mathbb{R}^n))}\|q_i(x, D_x)\|_{\mathcal{L}(L^q(\mathbb{R}^n))} \leq C2^{-jl}\cdot 2^{i(l+m)},\end{aligned}$$

which proves (5.11) for the case $i \leq j$.

In order to prove the case $i > j$, we use some kind of symmetry in i and j. First of all, we have $p(x, D_x) = (p(x, D_x)^*)^* = (p^*(x, D_x))^* = q(D_x, x)$ for $q(y,\xi) = \overline{p^*(y,\xi)}$. Hence

$$\varphi_j(D_x)p_i(x, D_x) = \varphi_j(D_x)q(D_x, x)\varphi_i(D_x) = \sum_{|\alpha|\leq l} q_j(D_x, x)D^\alpha_x p^{(\alpha)}(D_x)\varphi_i(D_x).$$

Now using (5.10) for $q_j(D_x,x)D_x^\alpha$ and $p^{(\alpha)}(D_x)\varphi_i(D_x) = p_i^{(\alpha)}(x,D_x)$, we conclude in the same way as before that

$$\begin{aligned}
&\|\varphi_j(D_x)p_i(x,D_x)\|_{\mathcal{L}(L^q(\mathbb{R}^n))} \\
&\leq \sum_{|\alpha|\leq l} \|q_j(x,D_x)D_x^\alpha\|_{\mathcal{L}(L^q(\mathbb{R}^n))}\|p^{(\alpha)}(D_x)\varphi_i(D_x)\|_{\mathcal{L}(L^q(\mathbb{R}^n))} \leq C2^{-il}\cdot 2^{j(l+m)},
\end{aligned}$$

which proves (5.11) for the case $i > j$.

■

Proof of Theorem 5.14: First of all,

$$f = \sum_{j=0}^\infty f_j,$$

where $f_j = \varphi_j(D_x)f$, $f \in \mathcal{S}'(\mathbb{R}^n)$. Since $\operatorname{supp}\varphi_j \cap \operatorname{supp}\varphi_k = \emptyset$ of $|j-k|>1$,

$$p(x,D_x)f = \sum_{k=0}^\infty p_k(x,D_x)f = \sum_{k=0}^\infty p_k(x,D_x)(f_{k-1}+f_k+f_{k+1}) = \sum_{k=0}^\infty p_k(x,D_x)\tilde{f}_k,$$

where $\tilde{f}_k = f_{k-1}+f_k+f_{k+1}$ and we have set $f_{-1}=0$. If $f \in C_*^{s+m}(\mathbb{R}^n)$, then

$$2^{(s+m)k}\|\tilde{f}_k\|_\infty \leq 2^{(s+m)k}\left(\|f_{k-1}\|_\infty + \|f_k\|_\infty + \|f_{k+1}\|_\infty\right) \leq C\|f\|_{C_*^{s+m}}$$

Now using (5.11),

$$2^{sj}\|\varphi_j(D_x)p_k(x,D_x)\tilde{f}_k\|_\infty \quad\leq\quad C2^{sj+\min(j,k)m-|j-k|l-k(s+m)}2^{k(s+m)}\|\tilde{f}_k\|_\infty,$$

where

$$2^{sj+\min(j,k)m-|j-k|l-k(s+m)} = \begin{cases} 2^{j(s-l)+k(l-s)} = 2^{-|j-k|(l-s)} & \text{if } j\geq k, \\ 2^{-k(m+s+l)+j(m+s+l)} = 2^{-|j-k|(m+s+l)} & \text{if } j<k. \end{cases}$$

Choosing $l \in \mathbb{N}_0$ so large that $l-s\geq 1$ and $m+s+l\geq 1$,

$$\begin{aligned}
\|p(x,D_x)f\|_{C_*^s} &= \sup_{j\in\mathbb{N}_0} 2^{sj}\|\varphi_j(D_x)p(x,D_x)f\|_\infty \\
&\leq C\sup_{j\in\mathbb{N}_0}\sum_{k=0}^\infty 2^{-|j-k|}2^{k(s+m)}\|\tilde{f}_k\|_\infty \leq C\|f\|_{C_*^{s+m}}.
\end{aligned}$$

■

THEOREM 5.18 *Let $0<s<1$. Then $C_*^s(\mathbb{R}^n) = C^s(\mathbb{R}^n)$ with equivalent norms, where $C^s(\mathbb{R}^n)$ is normed by*

$$2\|f\|_{C^s} = \|f\|_\infty + \sup_{x,y\in\mathbb{R}^n}\frac{|f(x)-f(y)|}{|x-y|^s}$$

Proof: First let $f \in C^s(\mathbb{R}^n)$. Then

$$\sup_{x\in\mathbb{R}^n} |f(x-y) - f(x)| \leq \|f\|_{C^s} |y|^s$$

for all $y \in \mathbb{R}^n$. Moreover, $\varphi_j(D_x)$ for $j \in \mathbb{N}$ is the convolution with $\psi_{2^{-j}}(x) := 2^{jn}\psi(2^j x)$, $\psi(x) = \mathcal{F}^{-1}_{\xi\mapsto x}[\varphi_0(\xi) - \varphi_0(2^{-1}\xi)]$, where

$$\int_{\mathbb{R}^n} \psi_{2^{-j}}(y)dy = \int_{\mathbb{R}^n} \psi(y)dy = \mathcal{F}[\psi](0) = 0.$$

Hence

$$\varphi_j(D_x)f = \int_{\mathbb{R}^n} f(x-y)\psi_{2^{-j}}(y)dy = \int_{\mathbb{R}^n} (f(x-y) - f(y))\psi_{2^{-j}}(y)dy$$

and therefore

$$\|\varphi_j(D_x)f\|_\infty \leq \|f\|_{C^s} \int_{\mathbb{R}^n} |y|^s \psi_{2^{-j}}(y)dy = 2^{-js}\|f\|_{C^s} \int_{\mathbb{R}^n} |z|^s\psi(z)dz = C2^{-js}\|f\|_{C^s}$$

for all $j \in \mathbb{N}$. The latter inequality implies $\|f\|_{C^s_*} \leq \|f\|_{C^s}$ since also $\|\varphi_0(D_x)f\|_\infty \leq C\|f\|_\infty$.

Conversely let $f \in C^s_*(\mathbb{R}^n)$. Then $f = \sum_{j=0}^\infty f_j$, $f_j = \varphi_j(D_x)f$, where the sum converges in $C^0_b(\mathbb{R}^n)$ since $\|f_j\|_\infty \leq 2^{-sj}\|f\|_{C^s_*}$. Now, if $|y| \leq 1$,

$$f(x-y) - f(y) = \sum_{2^j \leq |y|^{-1}} (f_j(x-y) - f_j(y)) + \sum_{2^j > |y|^{-1}} (f_j(x-y) - f_j(y)).$$

For the first sum we use the mean value theorem to conclude that

$$|f_j(x-y) - f_j(y)| \leq |y|\|\nabla f_j\|_\infty.$$

Moreover, since

$$\partial_{x_k} f_j = \partial_{x_k}\varphi_{j-1}(D_x)f_j + \partial_{x_k}\varphi_{j-1}(D_x)f_j + \partial_{x_k}\varphi_{j+1}(D_x)f_j$$

we can use (5.9) for $p(x,\xi) = \xi_k \in S^1_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $k = 1, \ldots, n$ to obtain

$$\begin{aligned}
\sum_{2^j \leq |y|^{-1}} |f_j(x-y) - f_j(y)| &\leq C \sum_{2^j \leq |y|^{-1}} |y| 2^j \|\nabla f_j\|_\infty \\
&\leq C|y| \sum_{2^j \leq |y|^{-1}} 2^{j(1-s)} \|f\|_{C^s_*} \leq C|y|^s\|f\|_{C^s_*}
\end{aligned}$$

if $2^j \leq |y|^{-1}$.

The second sum is simply estimated by

$$\sum_{2^j > |y|^{-1}} |f_j(x-y) - f_j(y)| \leq C \sum_{2^j > |y|^{-1}} \|f_j\|_\infty \leq C\|f\|_{C^s_*} \sum_{2^j > |y|^{-1}} 2^{-js} = C|y|^s\|f\|_{C^s_*}$$

Altogether $\|f\|_{C^s} \leq C\|f\|_{C^s_*}$. ∎

We now come to a generalization of the Zygmund-Hölder spaces. Let X be an arbitrary Banach space. Then the vector-valued variant of the classical spaces $\ell^q(\mathbb{N}_0)$, $1 \le q \le \infty$, are defined as

$$\begin{aligned}
\ell^q(\mathbb{N}_0, X) &= \{(a_k)_{k\in\mathbb{N}_0} : a_k \in X, \|(a_k)_{k\in\mathbb{N}_0}\|_{\ell^q(\mathbb{N}_0,X)} < \infty\}, \quad \text{where} \\
\|(a_k)_{k\in\mathbb{N}_0}\|_{\ell^q(\mathbb{N}_0,X)} &= \left(\sum_{k=0}^{\infty} \|a_k\|_X^q\right)^{\frac{1}{q}} \quad \text{for } 1 \le q < \infty, \\
\|(a_k)_{k\in\mathbb{N}_0}\|_{\ell^\infty(\mathbb{N}_0,X)} &= \sup_{k\in\mathbb{N}_0} \|a_k\|_X.
\end{aligned}$$

Definition 5.19 Let $1 \le q, r \le \infty$ and let $s \in \mathbb{R}$. Then the *Besov space* $B^s_{qr}(\mathbb{R}^n)$ is defined as

$$\begin{aligned}
B^s_{qr}(\mathbb{R}^n) &= \{u \in \mathcal{S}'(\mathbb{R}^n) : (2^{sj}\varphi_j(D_x)u)_{j\in\mathbb{N}_0} \in \ell^r(\mathbb{N}_0, L^q(\mathbb{R}^n))\}, \\
\|u\|_{B^s_{qr}} &= \|(\varphi_j(D_x)u)_{j\in\mathbb{N}_0}\|_{\ell^r(\mathbb{N}_0,L^q(\mathbb{R}^n))} \\
&= \begin{cases} \left(\sum_{j=0}^{\infty} \|2^{sj}\varphi_j(D_x)u\|_{L^q}^r\right)^{\frac{1}{r}} & \text{if } 1 \le r < \infty, \\ \sup_{j\in\mathbb{N}_0} \|2^{sj}\varphi_j(D_x)u\|_{L^q} & \text{if } r = \infty. \end{cases}
\end{aligned}$$

Here s is called *order* of $B^s_{q,r}(\mathbb{R}^n)$, q is called *integral exponent*, and r is called *sum exponent*.

Remark 5.20 1. If $s > 0$, then $B^s_{\infty\infty}(\mathbb{R}^n) = C^s_*(\mathbb{R}^n)$.

2. For the case $q = r = 2$ it can be shown that $B^s_{22}(\mathbb{R}^n) = H^s_2(\mathbb{R}^n)$, cf. [BL76].

3. The Besov space $B^s_{qq}(\mathbb{R}^{n-1})$, $s > 0$, appear naturally as the *trace spaces* of Bessel potential: If $s > \frac{1}{q}$, $1 < q < \infty$, then there is a bounded trace operator

   $$\gamma \colon H^s_q(\mathbb{R}^n) \to B^{s-\frac{1}{q}}_{qq}(\mathbb{R}^{n-1})$$

   such that $(\gamma u)(x') = u(x', 0)$ for all $u \in \mathcal{S}(\mathbb{R}^n)$. For a proof of this fact and for people which interested in Bessel potential and Besov spaces we refer to [BL76].

THEOREM 5.21 *Let $p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$, and let $s \in \mathbb{R}$, $1 \le q, r \le \infty$. Then $p(x, D_x) \colon B^{s+m}_{qr}(\mathbb{R}^n) \to B^s_{qr}(\mathbb{R}^n)$ is a bounded linear operator.*

The proof is a simple modification of the proof of 5.14. For the convenience of the reader we include the complete modified proof here:

Proof: First of all,

$$f = \sum_{j=0}^{\infty} f_j,$$

where $f_j = \varphi_j(D_x)f$, $f \in \mathcal{S}'(\mathbb{R}^n)$. Since $\operatorname{supp}\varphi_j \cap \operatorname{supp}\varphi_k = \emptyset$ of $|j-k|>1$,

$$p(x,D_x)f = \sum_{k=0}^{\infty} p_k(x,D_x)f = \sum_{k=0}^{\infty} p_k(x,D_x)(f_{k-1}+f_k+f_{k+1}) = \sum_{k=0}^{\infty} p_k(x,D_x)\tilde{f}_k,$$

where $\tilde{f}_k = f_{k-1}+f_k+f_{k+1}$ and we have set $f_{-1}=0$. If $f \in B^{s+m}_{qr}(\mathbb{R}^n)$, then

$$\begin{aligned}
\|(2^{(s+m)k}\tilde{f}_k)_{k\in\mathbb{N}_0}\|_{\ell^r(\mathbb{N}_0,L^q(\mathbb{R}^n))} &= \|(2^{(s+m)k}(f_{k-1}+f_k+f_{k+1}))_{k\in\mathbb{N}_0}\|_{\ell^r(\mathbb{N}_0,L^q(\mathbb{R}^n))} \\
&\leq C\left\|\left(2^{(s+m)k}\varphi_k(D_x)f\right)_{k\in\mathbb{N}_0}\right\|_{\ell^r(\mathbb{N}_0,L^q(\mathbb{R}^n))} = C\|f\|_{B^{s+m}_{qr}}
\end{aligned}$$

Now using (5.11),

$$2^{sj}\|\varphi_j(D_x)p_k(x,D_x)\tilde{f}_k\|_{L^q} \leq C2^{sj+\min(j,k)m-|j-k|l-k(s+m)}2^{k(s+m)}\|\tilde{f}_k\|_{L^q},$$

where

$$2^{sj+\min(j,k)m-|j-k|l-k(s+m)} = \begin{cases} 2^{j(s-l)+k(l-s)} = 2^{-|j-k|(l-s)} & \text{if } j \geq k, \\ 2^{-k(m+s+l)+j(m+s+l)} = 2^{-|j-k|(m+s+l)} & \text{if } j < k. \end{cases}$$

Choosing $l \in \mathbb{N}$ so large that $l-s \geq 1$ and $m+s+l \geq 1$,

$$2^{sj}\|\varphi_j(D_x)p(x,D_x)f\|_{L^q} \leq C\sum_{k=0}^{\infty} 2^{-|j-k|}2^{k(s+m)}\|\tilde{f}_k\|_{L^q}$$

Now let $a_j = 2^{-|j|}$, $b_j = 2^{(s+m)j}\|\tilde{f}_j\|_{L^q}$ for $j \in \mathbb{N}_0$ and $a_j = b_j = 0$ for $j \in \mathbb{Z}\setminus\mathbb{N}_0$. Then

$$\begin{aligned}
\|p(x,D_x)f\|_{B^s_{qr}} &\leq C\,\|(a_j)_{j\in\mathbb{Z}} * (b_j)_{j\in\mathbb{Z}}\|_{\ell^r} \leq C\|(a_j)_{j\in\mathbb{Z}}\|_{\ell^1}\|(b_j)_{j\in\mathbb{Z}}\|_{\ell^r} \\
&= C\|(2^{(s+m)j}\tilde{f}_j)_{j\in\mathbb{N}_0}\|_{\ell^r(\mathbb{N}_0,L^q(\mathbb{R}^n))} \leq C\|f\|_{B^{s+m}_{qr}},
\end{aligned}$$

where the sequence $(c_j)_{j\in\mathbb{Z}} = (a_j)_{j\in\mathbb{Z}} * (b_j)_{j\in\mathbb{Z}}$ is the convolution of $(a_j)_{j\in\mathbb{Z}}$ and $(b_j)_{j\in\mathbb{Z}}$ defined by

$$c_j = \sum_{k=-\infty}^{\infty} a_{j-k}b_k.$$

Here we have used the discrete version of Young's inequality $\|(a_j)_{j\in\mathbb{Z}} * (b_j)_{j\in\mathbb{Z}}\|_{\ell^r} \leq \|(a_j)_{j\in\mathbb{Z}}\|_{\ell^1}\|(b_j)_{j\in\mathbb{Z}}\|_{\ell^r}$, which can be proved in the same way as for the usual convolution using Hölder's inequality. ■

For completeness we state the following theorem, which shows that the Besov spaces can be considered as generalization of Hölder spaces:

THEOREM 5.22 *Let $1 \leq q < \infty$ and let $0 < s < 1$. Then*

$$\|f\|'_{B^s_{qq}} = \left(\|f\|_q^q + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^q}{|x-y|^{n+sq}}dxdy\right)^{\frac{1}{q}}$$

is an equivalent norm on $B^s_{qq}(\mathbb{R}^n)$.

See [BL76] for the proof.

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