Elementary Topology
A First Course
Textbook in Problems

O. Y. Viro, O. A. Ivanov,
N. Y. Netsvetaev, V. M. Kharlamov

Abstract. This book includes basic material on general topology, introduces algebraic topology via the fundamental group and covering spaces, and provides a background on topological and smooth manifolds. It is written mainly for students with a limited experience in mathematics, but determined to study the subject actively. The material is presented in a concise form, proofs are omitted. Theorems, however, are formulated in detail, and the reader is expected to treat them as problems.

Foreword

Genre, Contents and Style of the Book

The core of the book is the material usually included in the Topology part of the two year Geometry lecture course at the Mathematical Department of St. Petersburg University. It was composed by Vladimir Abramovich Rokhlin in the sixties and has almost not changed since then.

We believe this is the minimum topology that must be mastered by any student who has decided to become a mathematician. Students with research interests in topology and related fields will surely need to go beyond this book, but it may serve as a starting point. The book includes basic material on general topology, introduces algebraic topology via its most classical and elementary part, the theory of the fundamental group and covering spaces, and provides a background on topological and smooth manifolds. It is written mainly for students with a limited experience in mathematics, but who are determined to study the subject actively.

The core material is presented in a concise form; proofs are omitted. Theorems, however, are formulated in detail. We present them as problems and expect the reader to treat them as problems. Most of the theorems are easy to find elsewhere with complete proofs. We believe that a serious attempt to prove a theorem must be the first reaction to its formulation. It should precede looking for a book where the theorem is proved.

On the other hand, we want to emphasize the role of formulations. In the early stages of studying mathematics it is especially important to take each formulation seriously. We intentionally force a reader to think about each simple statement. We hope that this will make the book inconvenient for mere skimming.

The core material is enhanced by many problems of various sorts and additional pieces of theory. Although they are closely related to the main material, they can be (and usually are) kept outside of the standard lecture course. These enhancements can be recognized by wider margins, as the next paragraph.

> The problems, which do not comprise separate topics and are intended exclusively to be exercises, are typeset with small face. Some of them are very easy and included just to provide additional examples. Few problems are difficult. They are to indicate relations with other parts of mathematics, show possible

directions of development of the subject, or just satisfy an ambitious reader. Problems, whose solutions seem to be the most difficult (from the authors' viewpoint), are marked with a star, as in many other books.

Further, we want to deliver additional pieces of theory (with respect to the core material) to more motivated and advanced students. Maybe, a mathematician, who does not work in the fields geometric in flavor, can afford the luxury not to know some of these things. Maybe, students studying topology can postpone this material to their graduate study. We would like to include this in graduate lecture courses. However, quite often it does not happen, because most of the topics of this sort are rather isolated from the contents of traditional graduate courses. They are important, but more related to the material of the very first topology course. In the book these topics are intertwined with the core material and exercises, but are distinguishable: they are typeset, like these lines, with large face, large margins, theorems and problems in them are numerated in a special manner described below.

Exercises and illustrative problems to the additional topics are typeset with even wider margins and marked in a different way.

Thus, the whole book contains four layers:

- the core material,
- exercises and illustrative problems to the core material,
- additional topics,
- exercises and illustrative problems to additional topics.

The text of the core material is typeset with large face and smallest margins.

The text of problems elaborating on the core material is typeset with small face and larger margins.

The text of additional topics is typeset is typeset with large face and slightly smaller margins as the problems elaborating on the core material.

The text of problems illustrating additional topics is typeset with small face and the largest margins.

Therefore the book looks like a Russian folklore doll, *matreshka* composed of several dolls sitting inside each other. We apologize for being nonconventional in this and hope that it may help some readers and does not irritate the others too much.

The whole text of the book is divided into sections. Each section is divided into subsections. Subsections are not numerated. Each of them is devoted to a single topic and consists of definitions, commentaries, theorems, exercises, problems, and riddles.

By a *riddle* we mean a problem of a special sort: its solution is not contained in the formulation. One has to guess a solution, rather than deduce it.

0.A. Theorems, exercises, problems and riddles belonging to the core material are marked with pairs consisting of the number of section and a letter separated with a dot. The letter identifies the item inside the section.

> ***0.1.*** Exercises, problems, and riddles, which are not included in the core, but are closely related to it (and typeset with small face) are marked with pairs consisting of the number of the section and the number of the item inside the section. The numbers in the pair are separated also by a dot.

Theorems, exercises, problems and riddles related to additional topics are enumerated independently inside each section and denoted similarly.

> ***0:A.*** The only difference is that the components of pairs marking the items are separated by a colon (rather than dot).

We assume that the reader is familiar with naive set theory, but anticipate that this familiarity may be superficial. Therefore at points where set theory is especially crucial we make set-theoretic digressions maintained in the same style as the rest of the book.

Advice to the Reader

Since the book contains a summary of elementary topology, you may use the book while preparing for an examination (especially, if the exam reduces to solving a collection of problems). However, if you attend lectures on the subject, it would be much wiser to read the book prior to the lectures and prove theorems before the lecturer gives the proofs.

We think that a reader who is able to prove statements of the core of the book, does not need to solve all the other problems. It would be reasonable instead to look through formulations and concentrate on the most difficult problems. The more difficult the theorems of the main text seem to you, the more carefully you should consider illustrative problems, and the less time you should waste with problems marked with stars.

Keep in mind that sometimes a problem which seems to be difficult is followed by easier problems, which may suggest hints or serve as technical lemmas. A chain of problems of this sort is often concluded with a problem which suggests a return to the theorem, once you are armed with the lemmas.

Most of our illustrative problems are easy to invent, and, moreover, if you study the subject seriously, it is always worthwhile to invent problems of this sort. To develop this style of studying mathematics while solving our problems one should attempt to invent one's own problems and solve them (it does not matter if they are similar to ours or not). Of course, some problems presented in this book are not easy to invent.

Contents

Part 1

General Topology

Although it may seem unexpected, the goal of this part of the book is to teach the language of mathematics. More specifically, one of its most important components: the language of set-theoretic topology, which treats the basic notions related to continuity. The term *general topology* means: this is the topology that is needed and used by most mathematicians.

As a research field, it was completed a long time ago. Its permanent usage in the capacity of a common mathematical language has polished its system of definitions and theorems. Nowadays studying general topology really resembles studying a language rather than mathematics: one needs to learn a lot of new words, while proofs of all theorems are extremely simple. On the other hand, the theorems are numerous. It is not surprising: they play the role of rules regulating usage of words.

We have to warn students, for whom this is one of the first mathematical subjects. Do not hurry to fall in love with it too seriously, do not let an imprinting happen. This field may seam to be charming, but it is not very active. It hardly provides as much room for exciting new research as most other fields.

CHAPTER 1

Generalities of General Topology

1. Topology in a Set

Definition of Topological Space

Let X be a set. Let Ω be a collection of its subsets such that:

(a) the union of a family of sets, which are elements of Ω, belongs to Ω;
(b) the intersection of a finite family of sets, which are elements of Ω, belongs to Ω;
(c) the empty set $\varnothing$ and the whole X belong to Ω.

Then

- Ω is called a *topological structure* or just a *topology*[1] in X;
- the pair (X, Ω) is called a *topological space*;
- an element of X is called a *point* of this topological space;
- an element of Ω is called an *open set* of the topological space (X, Ω).

The conditions in the definition above are called the *axioms of topological structure.*

Simplest Examples

A *discrete topological space* is a set with the topological structure which consists of all the subsets.

1.A. Check that this is a topological space, i.e., all axioms of topological structure hold true.

An *indiscrete topological space* is the opposite example, in which the topological structure is the most meager. It consists only of X and $\varnothing$.

1.B. This is a topological structure, is it not?

Here are less trivial examples.

1.1. Let X be the ray $[0, +\infty)$, and Ω consists of $\varnothing$, X, and all the rays $(a, +\infty)$ with $a \geq 0$. Prove that Ω is a topological structure.

[1]Thus Ω is important: it is called by the same word as the whole branch of mathematics. Of course, this does not mean that Ω coincides with the subject of topology, but everything in this subject is related to Ω.

1.2. Let X be a plane. Let Σ consist of $\varnothing$, X, and all open disks with center at the origin. Is this a topological structure?

1.3. Let X consist of four elements: $X = \{a, b, c, d\}$. Which of the following collections of its subsets are topological structures in X, i.e., satisfy the axioms of topological structure:

(a) $\varnothing$, X, $\{a\}$, $\{b\}$, $\{a, c\}$, $\{a, b, c\}$, $\{a, b\}$;
(b) $\varnothing$, X, $\{a\}$, $\{b\}$, $\{a, b\}$, $\{b, d\}$;
(c) $\varnothing$, X, $\{a, c, d\}$, $\{b, c, d\}$?

The space of *1.1* is called an *arrow*. We denote the space of *1.3* (a) by $4pT$. It is a sort of toy space made of 4 points. Both of these spaces, as well as the space of *1.2*, are not important, but provide good simple examples.

The Most Important Example: Real Line

Let X be the set $\mathbb{R}$ of all real numbers, Ω be the set of unions of all intervals (a, b) with $a, b \in \mathbb{R}$.

1.C. Check if Ω satisfies the axioms of topological structure.

This is the topological structure which is always meant when $\mathbb{R}$ is considered as a topological space (unless other topological structure is explicitly specified). This space is called usually the *real line* and the structure is referred to as the *canonical* or *standard* topology in $\mathbb{R}$.

1.4. Let X be $\mathbb{R}$, and Ω consists of empty set and all the infinite subsets of $\mathbb{R}$. Is Ω a topological structure?

1.5. Let X be $\mathbb{R}$, and Ω consists of empty set and complements of all finite subsets of $\mathbb{R}$. Is Ω a topological structure?

The space of *1.5* is denoted by $\mathbb{R}_{T_1}$ and called the *line with T_1-topology*.

1.6. Let (X, Ω) be a topological space and Y be the set obtained from X by adding a single element a. Is

$$\{\{a\} \cup U \; : \; U \in \Omega\} \cup \{\varnothing\}$$

a topological structure in Y?

Using New Words: Points, Open and Closed Sets

Recall that, for a topological space (X, Ω), elements of X are called *points*, and elements of Ω are called *open sets*.[2]

1.D. Reformulate the axioms of topological structure using the words *open set* wherever possible.

A set $F \in X$ is said to be *closed* in the space (X, Ω) if its complement $X \smallsetminus F$ is open (i.e., $X \smallsetminus F \in \Omega$).

[2]The letter Ω stands for the letter O which is the initial of the words with the same meaning: *Open* in English, *Otkrytyj* in Russian, *Offen* in German, *Ouvert* in French.

Set-Theoretic Digression. De Morgan Formulas

1.E. Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be an arbitrary family of subsets of a set X. Prove that

$$X \smallsetminus \bigcup_{\lambda\in\Lambda} A_\lambda = \bigcap_{\lambda\in\Lambda}(X \smallsetminus A_\lambda) \tag{1}$$

$$X \smallsetminus \bigcap_{\lambda\in\Lambda} A_\lambda = \bigcup_{\lambda\in\Lambda}(X \smallsetminus A_\lambda). \tag{2}$$

Formula (2) is deduced from (1) in one step, is it not? These formulas are nonsymmetric cases of a single formulation, which contains in a symmetric way sets and their complements, unions and intersections.

1.7. Riddle. Find such a formulation.

Being Open or Closed

1.F Properties of Closed Sets. Prove that:

(a) the intersection of any collection of closed sets is closed;
(b) union of any finite number of closed sets is closed;
(c) empty set and the whole space (i.e., the underlying set of the topological structure) are closed.

Notice that the property of being closed is not a negation of the property of being open.

1.G. Find examples of sets, which

(a) are both open, and closed simultaneously;
(b) are neither open, nor closed.

1.8. Give an explicit description of closed sets in

(a) a discrete space;
(b) an indiscrete space;
(c) the arrow;
(d) $4pT$;
(e) $\mathbb{R}_{T_1}$.

1.H. Prove that a closed segment $[a,b]$ is closed in $\mathbb{R}$.

Concepts of closed and open sets are similar in a number of ways. The main difference is that the intersection of an infinite collection of open sets does not have to be necessarily open, while the intersection of any collection of closed sets is closed. Along the same lines, the union of an infinite collection of closed sets is not necessarily closed, while the union of any collection of open sets is open.

1.9. Prove that the half-open interval $[0, 1)$ is neither open nor closed in $\mathbb{R}$, but can be presented as either the union of closed sets or intersection of open sets.

1.10. Prove that every open set of the real line is a union of disjoint open intervals.

1.11. Prove that the set $A = \{0\} \cup \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is closed in $\mathbb{R}$.

Cantor Set

Let K be the set of real numbers which can be presented as sums of series of the form $\sum_{k=1}^{\infty} \frac{a_k}{3^k}$ with $a_k = 0$ or 2. In other words, K is the set of real numbers which in the positional system with base 3 are presented as $0.a_1a_2 \dots a_k \dots$ without digit 1.

1:A. Find a geometric description of K.

1:A:1. Prove that

(a) K is contained in $[0, 1]$,
(b) K does not intersect $\left(\frac{1}{3}, \frac{2}{3}\right)$,
(c) K does not intersect $\left(\frac{3s+1}{3^k}, \frac{3s+2}{3^k}\right)$ for any integers k and s.

1:A:2. Present K as $[0, 1]$ with an infinite family of open intervals removed.

1:A:3. Try to draw K.

The set K is called the *Cantor set.* It has a lot of remarkable properties and is involved in numerous problems below.

1:B. Prove that K is a closed set in the real line.

Characterization of Topology in Terms of Closed Sets

1.12. Prove that if a collection $\mathcal{F}$ of subsets of X satisfies the following conditions:

(a) the intersection of any family of sets from $\mathcal{F}$ belongs to $\mathcal{F}$;
(b) the union of any finite number sets from $\mathcal{F}$ belongs to $\mathcal{F}$;
(c) $\varnothing$ and X belong to $\mathcal{F}$,

then $\mathcal{F}$ is the set of all closed sets of a topological space (which one?).

1.13. List all collections of subsets of a three-element set such that there exist topologies, in which these collections are complete sets of closed sets.

Topology and Arithmetic Progressions

1.14*. Consider the following property of a subset F of the set $\mathbb{N}$ of natural numbers: there exists $N \in \mathbb{N}$ such that F does not contain an arithmetic progression of length greater than N. Prove, that subsets with this property together with the whole $\mathbb{N}$ form a collection of closed subsets in some topology in $\mathbb{N}$.

Solving this problem, you probably are not able to avoid the following combinatorial theorem.

1.15 Van der Waerden's Theorem*. For every $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that for any $A \subset \{1, 2, \dots, N\}$, either A or $\{1, 2, \dots, N\} \smallsetminus A$ contains an arithmetic progression of length n.

Neighborhoods

By a *neighborhood* of a point one means any open set containing this point. Analysts and French mathematicians (following N. Bourbaki) prefer a wider notion of neighborhood: they use this word for any set containing a neighborhood in the sense above.

1.16. Give an explicit description of all neighborhoods of a point in

(a) a discrete space;
(b) an indiscrete space;
(c) the arrow;
(d) $4pT$.

2. Bases

Definition of Base

Usually the topological structure is presented by describing its part, which is sufficient to recover the whole structure. A collection Σ of open sets is called a *base* for a topology if each nonempty open set is a union of sets of Σ. For instance, all intervals form a base for the real line.

2.1. Are there different topological structures with the same base?

2.2. Find some bases of topology of

(a) a discrete space;
(b) an indiscrete space;
(c) the arrow;
(d) $4pT$.

Try to choose the bases as small as possible.

2.3. Describe all topological structures having exactly one base.

Bases for Plane

2.4. Prove that any base of the canonical topology of $\mathbb{R}$ can be diminished.

Consider the following three collections of subsets of $\mathbb{R}^2$:

- Σ^2 which consists of all possible open disks (i.e., disks without its boundary circles);
- Σ^∞ which consists of all possible open squares (i.e., squares without their sides and vertices) with sides parallel to the coordinate axis;
- Σ^1 which consists of all possible open squares with sides parallel to the bisectors of the coordinate angles.

(Squares of Σ^∞ and Σ^1 are defined by inequalities $\max\{|x-a|, |y-b|\} < \rho$ and $|x-a| + |y-b| < \rho$ respectively.)

2.5. Prove that every element of Σ^2 is a union of elements of Σ^∞.

2.6. Prove that intersection of any two elements of Σ^1 is a union of elements of Σ^1.

2.7. Prove that each of the collections Σ^2, Σ^∞, Σ^1 is a base for some topological structure in $\mathbb{R}^2$, and that the structures defined by these collections coincide.

When a Collection of Sets is a Base

2.A. *A collection Σ of open sets is a base for the topology, iff for any open set U and any point $x \in U$ there is a set $V \in \Sigma$ such that $x \in V \subset U$.*

2.B. *A collection Σ of subsets of a set X is a base for some topology in X, iff X is a union of sets of Σ and intersection of any two sets of Σ is a union of sets in Σ.*

2.C. Show that the second condition in *2.B* (on intersection) is equivalent to the following: the intersection of any two sets of Σ contains, together with any of its points, some set of Σ containing this point (cf. *2.A*).

Subbases

Let (X, Ω) be a topological space. A collection Δ of its open subsets is called a *subbase* for Ω, provided the collection

$$\Sigma = \{V \mid V = \cap_{i=1}^{k} W_i,\ W_i \in \Delta, k \in \mathbb{N}\}$$

of all finite intersections of sets belonging to Δ is a base for Ω.

2.8. Prove that for any set X a collection Δ of its subsets is a subbase of a topology in X, iff $\Delta \neq \varnothing$ and $X = \cup_{W \in \Delta} W$.

Infinity of the Set of Prime Numbers

2.9. Prove that all infinite arithmetic progressions consisting of natural numbers form a base for some topology in $\mathbb{N}$.

2.10. Using this topology prove that the set of all prime numbers is infinite.

(Hint: otherwise the set $\{1\}$ would be open (?!))

Hierarchy of Topologies

If Ω_1 and Ω_2 are topological structures in a set X such that $\Omega_1 \subset \Omega_2$ then Ω_2 is said to be *finer* than Ω_1, and Ω_1 *coarser* than Ω_2. For instance, among all topological structures in the same set the indiscrete topology is the coarsest topology, and the discrete topology is the finest one, is it not?

2.11. Show that T_1-topology (see Section 1) is coarser than the canonical topology in the real line.

2.12. Riddle. Let Σ_1 and Σ_2 be bases for topological structures Ω_1 and Ω_2 in a set X. Find necessary and sufficient condition for $\Omega_1 \subset \Omega_2$ in terms of the bases Σ_1 and Σ_2 without explicit referring to Ω_1 and Ω_2 (cf. *2.7*).

Bases defining the same topological structure are said to be *equivalent.*

2.D. Riddle. Formulate a necessary and sufficient condition for two bases to be equivalent without mentioning of topological structures (cf. *2.7*. Bases Σ^2, Σ^∞, and Σ^1 must satisfy the condition you are to find).

3. Metric Spaces

Definition and First Examples

A function $\rho : X \times X \to \mathbb{R}_+ = \{ x \in \mathbb{R} \mid x \geq 0 \}$ is called a *metric* (or *distance*) in X, if

(a) $\rho(x, y) = 0$, iff $x = y$;
(b) $\rho(x, y) = \rho(y, x)$ for every $x, y \in X$;
(c) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for every $x, y, z \in X$.

The pair (X, ρ), where ρ is a metric in X, is called a *metric space*. The condition (c) is *triangle inequality*.

3.A. Prove that for any set X

$$\rho : X \times X \to \mathbb{R}_+ : \quad (x, y) \mapsto \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{if } x \neq y \end{cases}$$

is a metric.

3.B. Prove that $\mathbb{R} \times \mathbb{R} \to \mathbb{R}_+ : (x, y) \mapsto |x - y|$ is a metric.

3.C. Prove that $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ : (x, y) \mapsto \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$ is a metric.

Metrics *3.B* and *3.C* are always meant when $\mathbb{R}$ and $\mathbb{R}^n$ are considered as metric spaces unless another metric is specified explicitly. Metric *3.B* is a special case of metric *3.C*. These metrics are called *Euclidean*.

Further Examples

3.1. Prove that $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ : (x, y) \mapsto \max_{i=1,\dots,n} |x_i - y_i|$ is a metric.

3.2. Prove that $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ : (x, y) \mapsto \sum_{i=1}^n |x_i - y_i|$ is a metric.

Metrics in $\mathbb{R}^n$ introduced in *3.C–3.2* are included in infinite series of the metrics

$$\rho^{(p)} : \ (x, y) \mapsto \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad p \geq 1.$$

3.3. Prove that $\rho^{(p)}$ is a metric for any $p \geq 1$.

3.3.1 ***Hölder Inequality.*** Prove that

$$\sum_{i=1}^n x_i y_i \leq \left(\sum_{i=1}^n x_i^p \right)^{1/p} \left(\sum_{i=1}^n y_i^q \right)^{1/q}$$

if $x_i, y_i \geq 0$, $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Metric of *3.C* is $\rho^{(2)}$, metric of *3.2* is $\rho^{(1)}$, and metric of *3.1* can be denoted by $\rho^{(\infty)}$ and adjoined to the series since

$$\lim_{p \to +\infty} \left(\sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} = \max a_i,$$

for any positive $a_1, a_2, \dots, a_n$.

3.4. Riddle. How is this related to Σ^2, Σ^∞, and Σ^1 from Section 2?

For a real number $p \geq 1$ denote by $l^{(p)}$ the set of sequences $x = \{x_i\}_{i=1,2,\dots}$ such that the series $\sum_{i=1}^{\infty} |x|^p$ converges.

3.5. Prove that for any two elements $x, y \in l^{(p)}$ the series $\sum_{i=1}^{\infty} |x_i - y_i|^p$ converges and that

$$(x, y) \mapsto \left(\sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad p \geq 1$$

is a metric in $l^{(p)}$.

Balls and Spheres

Let (X, ρ) be a metric space, let a be its point, and let r be a positive real number. The sets

$$D_r(a) = \{ x \in X \mid \rho(a, x) < r \}, \tag{3}$$

$$D_r[a] = \{ x \in X \mid \rho(a, x) \leq r \}, \tag{4}$$

$$S_r(a) = \{ x \in X \mid \rho(a, x) = r \} \tag{5}$$

are called, respectively, *open ball*, *closed ball*, and *sphere* of the space (X, ρ) with center at a and radius r.

Subspaces of a Metric Space

If (X, ρ) is a metric space and $A \subset X$, then the restriction of metric ρ to $A \times A$ is a metric in A, and $(A, \rho|_{A \times A})$ is a metric space. It is called a *subspace* of (X, ρ).

The ball $D_1[0]$ and sphere $S_1(0)$ in $\mathbb{R}^n$ (with Euclidean metric, see *3.C*) are denoted by symbols D^n and S^{n-1} and called *n-dimensional ball* and $(n-1)$-*dimensional sphere*. They are considered as metric spaces (with the metric restricted from $\mathbb{R}^n$).

3.D. Check that D^1 is the segment $[-1, 1]$; D^2 is a disk; S^0 is the pair of points $\{-1, 1\}$; S^1 is a circle; S^2 is a sphere; D^3 is a ball.

The last two statements clarify the origin of terms *sphere* and *ball* (in the context of metric spaces).

Some properties of balls and spheres in arbitrary metric space resemble familiar properties of planar disks and circles and spatial balls and spheres.

3.E. Prove that for points x and a of any metric space and any $r > \rho(a, x)$

$$D_{r - \rho(a,x)}(x) \subset D_r(a).$$

Surprising Balls

However in other metric spaces balls and spheres may have rather surprising properties.

3.6. What are balls and spheres in $\mathbb{R}^2$ with metrics of *3.1* and *3.2* (cf. *3.4*)?

3.7. Find $D_1[a]$, $D_{\frac{1}{2}}[a]$, and $S_{\frac{1}{2}}(a)$ in the space of *3.A*.

3.8. Find a metric space and two balls in it such that the ball with the smaller radius contains the ball with the bigger one and does not coincide with it.

3.9. What is the minimal number of points in the space which is required to be constructed in *3.8*.

3.10. Prove that in *3.8* the big radius does not exceed double the smaller radius.

Segments (What Is Between)

3.11. Prove that the segment with end points $a, b \in \mathbb{R}^n$ can be described as

$$\{ x \in \mathbb{R}^n \mid \rho(a, x) + \rho(x, b) = \rho(a, b) \},$$

where ρ is the Euclidean metric.

3.12. How do the sets defined as in *3.11* look like with ρ of *3.1* and *3.2*? (Consider the case $n = 2$ if it appears to be easier.)

Bounded Sets and Balls

A subset A of a metric space (X, ρ) is said to be *bounded*, if there is a number $d > 0$ such that $\rho(x, y) < d$ for any $x, y \in A$. The greatest lower bound of such d is called the *diameter* of A and denoted by $\operatorname{diam}(A)$.

3.F. Prove that a set A is bounded, iff it is contained in a ball.

3.13. What is the relation between the minimal radius of such a ball and $\operatorname{diam}(A)$?

Norms and Normed Spaces

Let X be a vector space (over $\mathbb{R}$). Function $X \to \mathbb{R}_+ : x \mapsto \|x\|$ is called a *norm* if

(a) $\|x\| = 0$, iff $x = 0$;
(b) $\|\lambda x\| = |\lambda| \|x\|$ for any $\lambda \in \mathbb{R}$ and $x \in X$;
(c) $\|x + y\| \leq \|x\| + \|y\|$ for any $x, y \in X$.

3.14. Prove that if $x \mapsto \|x\|$ is a norm then

$$\rho : X \times X \to \mathbb{R}_+ : (x, y) \mapsto \|x - y\|$$

is a metric.

The vector space equipped with a specified norm is called a *normed space*. The metric defined by the norm as in *3.14* turns the normed space into the metric one in a canonical way.

3.15. Look through the problems of this section and figure out which of the metric spaces involved are, in fact, normed vector spaces.

3.16. Prove that every ball in the normed space is a convex[3] set symmetric with respect to the center of the ball.

3.17*. Prove that every convex closed bounded set in $\mathbb{R}^n$, which is symmetric with respect to its center and is not contained in any affine space except $\mathbb{R}^n$ itself, is the unit ball with respect to some norm, and that this norm is uniquely defined by this ball.

Metric Topology

3.G. *The collection of all open balls in the metric space is a base for some topology (cf. 2.A, 2.B and 3.E).*

This topology is called *metric topology*. It is said to be *induced* by the metric. This topological structure is always meant whenever the metric space is considered as a topological one (for instance, when one says about open and closed sets, neighborhoods, etc. in this space).

3.H. Prove that the standard topological structure in $\mathbb{R}$ introduced in Section 1 is induced by metric $(x, y) \mapsto |x - y|$.

3.18. What topological structure is induced by the metric of *3.A*?

3.I. *A set is open in a metric space, iff it contains together with any its point a ball with center at this point.*

3.19. Prove that a closed ball is closed (with respect to the metric topology).

3.20. Find a closed ball, which is open (with respect to the metric topology).

3.21. Find an open ball, which is closed (with respect to the metric topology).

3.22. Prove that a sphere is closed.

3.23. Find a sphere, which is open.

Metrizable Topological Spaces

A topological space is said to be *metrizable* if its topological structure is induced by some metric.

3.J. An indiscrete space is not metrizable unless it consists of a single point (it has too few open sets).

3.K. A finite space is metrizable iff it is discrete.

3.24. Which topological spaces described in Section 1 are metrizable?

[3]Recall that a set A is said to be *convex* if for any $x, y \in A$ the segment connecting x, y is contained in A. Of course, this definition is based on the notion of *segment,* so it makes sense only for subsets of spaces, where the notion of segment connecting two point is defined. This is the case in vector and affine spaces over $\mathbb{R}$

Equivalent Metrics

Two metrics in the same set are said to be *equivalent* if they induce the same topology.

3.25. Are the metrics of *3.C*, *3.1*, and *3.2* equivalent?

3.26. Prove that metrics ρ_1, ρ_2 in X are equivalent if there are numbers $c, C > 0$ such that

$$c\rho_1(x, y) \le \rho_2(x, y) \le C\rho_1(x, y)$$

for any $x, y \in X$.

3.27. Generally speaking the inverse is not true.

3.28. Riddle. Hence the condition of the equivalence of metrics formulated in *3.26* can be weakened. How?

3.29*. Prove that the following two metrics ρ_1, ρ_C in the set of all continuous functions $[0, 1] \to \mathbb{R}$ are not equivalent:[4]

$$\rho_1(f, g) = \int_0^1 \big|f(x) - g(x)\big|dx; \qquad \rho_C(f, g) = \max_{x\in[0,1]} \big|f(x) - g(x)\big|.$$

Is it true that topological structure defined by one of them is finer than another?

Ultrametric

A metric ρ is called an *ultrametric* if it satisfies to *ultrametric triangle inequality*:

$$\rho(x, y) \le \max\{\rho(x, z),\ \rho(z, y)\}$$

for any x, y, z.

A metric space (X, ρ) with ultrametric ρ is called an *ultrametric space.*

3.30. Check that only one metric in *3.A–3.2* is ultrametric. Which one?

3.31. Prove that in an ultrametric space all triangles are isosceles (i.e., for any three points a, b, c two of the three distances $\rho(a, b)$, $\rho(b, c)$, $\rho(a, c)$ are equal).

3.32. Prove that in a ultrametric space spheres are not only closed (cf. *3.22*) but also open.

The most important example of ultrametric is *p-adic metric* in the set $\mathbb{Q}$ of all rational numbers. Let p be a prime number. For $x, y \in \mathbb{Q}$, present the difference $x - y$ as $\frac{r}{s}p^\alpha$, where r, s, and α are integers, and r, s are relatively prime with p. Put $\rho(x, y) = p^{-\alpha}$.

3.33. Prove that this is an ultrametric.

[4]Indexes in the notations allude to the spaces these metrics are defining.

Operations with Metrics

3.34. Prove that if $\rho : X \times X \to \mathbb{R}_+$ is a function which satisfies conditions (a) and (c) of the definition of metric then the function

$$(x, y) \mapsto \rho(x, y) + \rho(y, x)$$

is a metric in X.

3.35. Prove that if ρ_1, ρ_2 are metrics in X then $\rho_1 + \rho_2$ and $\max\{\rho_1, \rho_2\}$ are also metrics. Are the functions $\min\{\rho_1, \rho_2\}$, $\frac{\rho_1}{\rho_2}$, and $\rho_1\rho_2$ metrics?

3.36. Prove that if $\rho : X \times X \to \mathbb{R}_+$ is a metric then

(a) function

$$(x, y) \mapsto \frac{\rho(x, y)}{1 + \rho(x, y)}$$

is a metric;

(b) function

$$(x, y) \mapsto f\big(\rho(x, y)\big)$$

is a metric, if f satisfies the following conditions:

(1) $f(0) = 0$,
(2) f is a monotone increasing function, and
(3) $f(x + y) \le f(x) + f(y)$ for any $x, y \in \mathbb{R}$.

3.37. Prove that metrics ρ and $\dfrac{\rho}{1 + \rho}$ are equivalent.

Distance Between Point and Set

Let (X, ρ) be a metric space, $A \subset X$, $b \in X$. The $\inf\{\, \rho(b, a) \mid a \in A \,\}$ is called a *distance from the point b to the set A* and denoted by $\rho(b, A)$.

3.L. Let A be a closed set. Prove that $\rho(b, A) = 0$, iff $b \in A$.

3.38. Prove that $|\rho(x, A) - \rho(y, A)| \le \rho(x, y)$ for any set A and points x, y of the same metric space.

Distance Between Sets

Let A and B be bounded subsets in the metric space (X, ρ). Put

$$d_\rho(A, B) = \max\Big\{\sup_{a\in A} \rho(a, B),\ \sup_{b\in B} \rho(b, A)\Big\}.$$

This number is called the *Hausdorff distance* between A and B.

3.39. Prove that the Hausdorff distance in the set of all bounded subsets of a metric space satisfies the conditions (b) and (c) of the definition of metric.

3.40. Prove that for every metric space the Hausdorff distance is a metric in the set of its closed bounded subsets.

Let A and B be bounded polygons in the plane[5]. Put

$$d_\Delta(A,B) = S(A) + S(B) - 2S(A \cap B),$$

where $S(C)$ is the area of polygon C.

3.41. Prove that d_Δ is a metric in the set of all plane bounded polygons.

We will call d_Δ the *area metric.*

3.42. Prove that in the set of all bounded plane polygons the area metric is *not* equivalent to the Hausdorff metric.

3.43. Prove that in the set of convex bounded plane polygons the area metric is equivalent to the Hausdorff metric.

[5]Although we assume that the notion of bounded polygon is well-known from elementary geometry, recall the definition. A bounded plane polygon is a set of the points of a simple closed polygonal line and the points surrounded by this line. By a simple closed polygonal line we mean a cyclic sequence of segments such that each of them starts at the point where the previous one finishes and these are the only pairwise intersections of the segments.

4. Subspaces

Let (X, Ω) be a topological space, and $A \subset X$. Denote by Ω_A the collection of sets $A \cap V$, where $V \in \Omega$.

4.A. *Ω_A is a topological structure in A.*

The pair (A, Ω_A) is called a *subspace* of the space (X, Ω). The collection Ω_A is called the *subspace topology* or the *relative topology* or the topology *induced* on A by Ω, and its elements are called *open sets* in A.

4.B. The canonical topology in $\mathbb{R}^1$ and the topology induced on $\mathbb{R}^1$ as a subspace of $\mathbb{R}^2$ coincide.

> ***4.1. Riddle.*** How to construct a base for the topology induced on A using the base for the topology in X?
>
> ***4.2.*** Describe the topological structures induced
>
> (a) on the set $\mathbb{N}$ of natural numbers by the topology of the real line;
> (b) on $\mathbb{N}$ by the topology of the arrow;
> (c) on the two-point set $\{1, 2\}$ by the topology of $\mathbb{R}_{T_1}$;
> (d) on the same set by the topology of the arrow.
>
> ***4.3.*** Is the half-open interval $[0, 1)$ open in the segment $[0, 2]$ considered as a subspace of the real line?

4.C. *A set is closed in a subspace, iff it is the intersection of the subspace and a closed subset of the ambient space.*

Relativity of Openness

Sets, which are open in the subspace, are not necessarily open in the ambient space.

4.D. The unique open set in $\mathbb{R}^1$, which is also open in $\mathbb{R}^2$, is the empty set $\varnothing$.

However:

4.E. Open sets of an open subspace are open in the ambient space, i.e., if $A \in \Omega$ then $\Omega_A \subset \Omega$.

The same relation holds true for closed sets. Sets, which are closed in the subspace, are not necessarily closed in the ambient space. However:

4.F. Closed sets of the closed subspace are closed in the ambient space.

> ***4.4.*** Prove that a set U is open in X, iff every its point has a neighborhood V in X such that $U \cap V$ is open in V.
>
> It allows one to say that the property of being open is a local property.
>
> ***4.5.*** Show that the property of being closed is not a local property.

4.G Transitivity of Induced Topology. *Let (X, Ω) be a topological space, and $X \supset A \supset B$. Then $(\Omega_A)_B = \Omega_B$, i.e., the topology induced on B by the topology induced on A coincides with the topology induced on B directly.*

4.6. Let (X, ρ) be a metric space, and $A \subset X$. Then the topology in A generated by metric $\rho|_{A \times A}$ coincides with the topology induced on A by the topology in X generated by metric ρ. (To prove this statement you need to prove two inclusions. Which of them is less obvious?)

Agreement on Notations of Topological Spaces

Different topological structures in the same set are not considered simultaneously very often. That is why a topological space is usually denoted by the same symbol as the set of its points, i.e., instead of (X, Ω) one writes just X. The same is applied for metric spaces: instead of (X, ρ) one writes just X.

5. Position of a Point with Respect to a Set

This section is devoted to a further expansion of the vocabulary needed when one speaks of phenomena in a topological space.

Interior, Exterior and Boundary Points

Let X be a topological space, $A \subset X$, and $b \in X$. The point b is called

- an *interior* point of the set A if it has a neighborhood contained in A;
- an *exterior* point of the set A if it has a neighborhood disjoint with A;
- a *boundary* point of the set A if any its neighborhood intersects both A and the complement of A.

Interior and Exterior

The *interior* of a set A in a topological space X is the maximal (with respect to inclusion) open in X set contained in A, i.e., an open set, which contains any other open subset of A. It is denoted $\operatorname{Int} A$ or, going into details, $\operatorname{Int}_X A$.

5.A. *Every subset of a topological space has interior. It is the union of all open sets contained in this set.*

5.B. *The interior of a set is the union of its interior points.*

5.C. *A set is open, iff it coincides with its interior.*

5.D. Prove that in $\mathbb{R}$:
(a) $\operatorname{Int}[0,1) = (0,1)$,
(b) $\operatorname{Int}\mathbb{Q} = \varnothing$ and
(c) $\operatorname{Int}(\mathbb{R} \smallsetminus \mathbb{Q}) = \varnothing$.

5.1. Find the interior of $\{a, b, d\}$ in space $4pT$.

The *exterior* of a set is the maximal open set disjoint from A. It is obvious that the exterior of A is $\operatorname{Int}(X \smallsetminus A)$.

Closure

The *closure* of a set A is the minimal closed set containing A. It is denoted $\operatorname{Cl} A$ or, going into details, $\operatorname{Cl}_X A$.

5.E. *Every subset of topological space has closure. It is the intersection of all closed sets containing this set.*

5.2. Prove that if A is a subspace of X, and $B \subset A$, then $\operatorname{Cl}_A B = (\operatorname{Cl}_X B) \cap A$. Is it true that $\operatorname{Int}_A B = (\operatorname{Int}_X B) \cap A$?

A point b is called an *adherent point* for a set A if all of its neighborhood intersect A.

5.F. *The closure of a set is the set of its adherent points.*

5.G. *A set A is closed, iff $A = \operatorname{Cl} A$.*

5.H. *The closure of a set is the complement of its exterior. In formulas: $\operatorname{Cl} A = X \smallsetminus \operatorname{Int}(X \smallsetminus A)$, where X is the space and $A \subset X$.*

5.I. Prove that in $\mathbb{R}$:

(a) $\operatorname{Cl}[0,1) = [0,1]$,
(b) $\operatorname{Cl} \mathbb{Q} = \mathbb{R}$,
(c) $\operatorname{Cl}(\mathbb{R} \smallsetminus \mathbb{Q}) = \mathbb{R}$.

5.3. Find the closure of $\{a\}$ in $4pT$.

Frontier

The *frontier* of a set A is the set $\operatorname{Cl} A \smallsetminus \operatorname{Int} A$. It is denoted by $\operatorname{Fr} A$ or, more precisely, $\operatorname{Fr}_X A$.

5.4. In $4pT$ find the frontier of $\{a\}$.

5.J. The frontier of a set is the set of its boundary points.

5.K. Prove that a set A is closed, iff $\operatorname{Fr} A \subset A$.

5.5. Prove that $\operatorname{Fr} A = \operatorname{Fr}(X \smallsetminus A)$. Find a formula for $\operatorname{Fr} A$, which is symmetric with respect to A and $X \smallsetminus A$.

5.6. The frontier of a set A equals the intersection of the closure of A and the closure of the complement of A:

$$\operatorname{Fr} A = \operatorname{Cl} A \cap \operatorname{Cl}(X \smallsetminus A).$$

Closure and Interior with Respect to a Finer Topology

5.7. Let Ω_1, Ω_2 be topological structure in X, and $\Omega_1 \subset \Omega_2$. Let Cl_i denote the closure with respect to Ω_i. Prove that $\operatorname{Cl}_1 A \supset \operatorname{Cl}_2 A$ for any $A \subset X$.

5.8. Formulate and prove an analogous statement about interior.

Properties of Interior and Closure

5.9. Prove that if $A \subset B$ then $\operatorname{Int} A \subset \operatorname{Int} B$.

5.10. Prove that $\operatorname{Int} \operatorname{Int} A = \operatorname{Int} A$.

5.11. Is it true that for any sets A and B the following equalities hold true:

$$\operatorname{Int}(A \cap B) = \operatorname{Int} A \cap \operatorname{Int} B, \tag{6}$$

$$\operatorname{Int}(A \cup B) = \operatorname{Int} A \cup \operatorname{Int} B? \tag{7}$$

5.12. Give an example in which one of that equalities does not hold true.

5.13. In the example that you have found solving the previous problem an inclusion of one hand side into another one holds true. Does this inclusion hold true for any A and B?

5.14. Study the operator Cl in a way suggested by the investigation of Int undertaken in *5.9–5.13*.

5.15. Find $\mathrm{Cl}\{1\}$, $\mathrm{Int}[0,1]$, and $\mathrm{Fr}(2,+\infty)$ in the arrow.

5.16. Find $\mathrm{Int}\big((0,1]\cup\{2\}\big)$, $\mathrm{Cl}\big(\{\frac{1}{n}\mid n\in\mathbb{N}\}\big)$, and $\mathrm{Fr}\,\mathbb{Q}$ in $\mathbb{R}$.

5.17. Find $\mathrm{Cl}\,\mathbb{N}$, $\mathrm{Int}(0,1)$, and $\mathrm{Fr}[0,1]$ in $\mathbb{R}_{T_1}$. How to find the closure and interior of a set in this space?

5.18. Prove that a sphere contains the frontier of the open ball with the same center and radius.

5.19. Find an example in which a sphere is disjoint from the closure of the open ball with the same center and radius.

Let A be a subset, and b be a point of the metric space (X,ρ). Recall (see Section 3) that the distance $\rho(b,A)$ from the point b to the set A is the $\inf\{\,\rho(b,a)\mid a\in A\,\}$.

5.L. Prove that $b\in\mathrm{Cl}\,A$, iff $\rho(b,A)=0$.

5.20 The Kuratowski Problem. How many pairwise distinct sets can one obtain out of a single set using operators Cl and Int?

The following problems will help you to solve problem *5.20*.

5.20.1. Find a set $A\subset\mathbb{R}$ such that the sets A, $\mathrm{Cl}\,A$, and $\mathrm{Int}\,A$ would be pairwise distinct.

5.20.2. Is there a set $A\subset\mathbb{R}$ such that

(a) A, $\mathrm{Cl}\,A$, $\mathrm{Int}\,A$, $\mathrm{Cl}\,\mathrm{Int}\,A$ are pairwise distinct;
(b) A, $\mathrm{Cl}\,A$, $\mathrm{Int}\,A$, $\mathrm{Int}\,\mathrm{Cl}\,A$ are pairwise distinct;
(c) A, $\mathrm{Cl}\,A$, $\mathrm{Int}\,A$, $\mathrm{Cl}\,\mathrm{Int}\,A$, $\mathrm{Int}\,\mathrm{Cl}\,A$ are pairwise distinct?

If you find such sets, keep on going in the same way, and when fail, try to formulate a theorem explaining the failure.

5.20.3. Prove that $\mathrm{Cl}\,\mathrm{Int}\,\mathrm{Cl}\,\mathrm{Int}\,A=\mathrm{Cl}\,\mathrm{Int}\,A$.

5.21*. Find three sets in the real line, which have the same frontier. Is it possible to increase the number of such sets?

Recall that a set $A\subset\mathbb{R}^n$ is said to be *convex* if together with any two points it contains the whole interval connecting them (i.e., for any $x,y\in A$ any point z belonging to the segment $[x,y]$ belongs to A).

Let A be a convex set in $\mathbb{R}^n$.

5.22. Prove that $\mathrm{Cl}\,A$ and $\mathrm{Int}\,A$ are convex.

5.23. Prove that A contains a ball, unless A is not contained in an $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$.

5.24. When is $\mathrm{Fr}\,A$ convex?

Characterization of Topology by Closure or Interior Operations

5.25*. Let in the set of all subset of a set X exist an operator Cl_* which has the following properties:

(a) $\mathrm{Cl}_* \varnothing = \varnothing$;
(b) $\mathrm{Cl}_* A \supset A$;
(c) $\mathrm{Cl}_*(A \cup B) = \mathrm{Cl}_* A \cup \mathrm{Cl}_* B$;
(d) $\mathrm{Cl}_* \mathrm{Cl}_* A = \mathrm{Cl}_* A$.

Prove that $\Omega = \{ U \subset X \mid \mathrm{Cl}_*(X \smallsetminus U) = X \smallsetminus U \}$ is a topological structure, and $\mathrm{Cl}_* A$ is the closure of a set A in the space (X, Ω).

5.26. Find an analogous system of axioms for Int.

Dense Sets

Let A and B be sets in a topological space X. A is said to be *dense in B* if $\mathrm{Cl}\, A \supset B$, and *everywhere dense* if $\mathrm{Cl}\, A = X$.

5.M. *A set is everywhere dense, iff it intersects any nonempty open set.*

5.N. The set $\mathbb{Q}$ is everywhere dense in $\mathbb{R}$.

5.27. Give a characterization of everywhere dense sets in an indiscrete space, in the arrow and in $\mathbb{R}_{T_1}$.

5.28. Prove that a topological space is a discrete space, iff it has a unique everywhere dense set (which is the entire space, of course).

5.29. Is it true that the union of everywhere dense sets is everywhere dense, and that the intersection of everywhere dense sets is everywhere dense?

5.30. Prove that the intersection of two open everywhere dense sets is everywhere dense.

5.31. Which condition in the previous problem is redundant?

5.32*. Prove that in $\mathbb{R}$ a countable intersection of open everywhere dense sets is everywhere dense. Is it possible to replace $\mathbb{R}$ here by an arbitrary topological space?

5.33*. Prove that $\mathbb{Q}$ cannot be presented as a countable intersection of open sets dense in $\mathbb{R}$.

5.34. Formulate a necessary and sufficient condition on the topology of a space which has an everywhere dense point. Find spaces satisfying the condition in Section 1.

Nowhere Dense Sets

A set is called *nowhere dense* if its exterior is everywhere dense.

5.35. Can a set be everywhere dense and nowhere dense simultaneously?

5.O. A set A is nowhere dense in X, iff any neighborhood of any point $x \in X$ contains a point y such that the complement of A contains y together with one of its neighborhoods.

5.36. Riddle. What can you say about the interior of a nowhere dense set?

5.37. Is $\mathbb{R}$ nowhere dense in $\mathbb{R}^2$?

5.38. Prove that if A is nowhere dense then $\operatorname{Int}\operatorname{Cl} A = \varnothing$.

5.39. Prove that the frontier of a closed set is nowhere dense. Is this true for the boundary of an open set; boundary of an arbitrary set?

5.40. Prove that a finite union of nowhere dense sets is nowhere dense.

5.41. Prove that in $\mathbb{R}^n$ $(n \geq 1)$ every proper algebraic set (i.e., a set defined by algebraic equations) is nowhere dense.

5.42. Prove that for every set A there exists a maximal open set B in which A is dense. The extreme cases $B = X$ and $B = \varnothing$ mean that A is either everywhere dense or nowhere dense respectively.

Limit Points and Isolated Points

A point b is called a *limit point* of a set A if any neighborhood of b intersects $A \smallsetminus \{b\}$.

5.P. Every limit point of a set is its adherent point.

5.43. Give an example proving that an adherent point may be not a limit one.

A point b is called an *isolated point* of a set A if $b \in A$ and there exists a neighborhood of b disjoint with $A \smallsetminus \{b\}$.

5.Q. *A set A is closed, iff it contains all its limit points.*

5.44. Find limit and isolated points of the sets $(0,1] \cup \{2\}$, $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ in $\mathbb{Q}$ and in $\mathbb{R}$.

5.45. Find limit and isolated points of the set $\mathbb{N}$ in $\mathbb{R}_{T_1}$.

Locally Closed Sets

A subset A of a topological space X is called *locally closed* if each of its points has a neighborhood U such that $A \cap U$ is closed in U (cf. *4.4–4.5*).

5.46. Prove that the following conditions are equivalent:

(a) A is locally closed in X;
(b) A is an open subset of its closure $\operatorname{Cl}_X A$;
(c) A is the intersection of open and closed subsets of X.

6. Set-Theoretic Digression. Maps

Maps and the Main Classes of Maps

A *mapping* f of a set X to a set Y is a triple consisting of X, Y, and a rule,[6] which assigns to every element of X exactly one element of Y. There are other words with the same meaning: *map, function.*

If f is a mapping of X to Y then one writes $f : X \to Y$, or $X \xrightarrow{f} Y$. The element b of Y assigned by f to an element a of X is denoted by $f(a)$ and called the *image* of a under f. One writes $b = f(a)$, or $a \overset{f}{\mapsto} b$, or $f : a \mapsto b$.

A mapping $f : X \to Y$ is called a *surjective map*, or just a *surjection* if every element of Y is an image of at least one element of X. A mapping $f : X \to Y$ is called an *injective map*, *injection*, or *one-to-one map* if every element of Y is an image of not more than one element of X. A mapping is called a *bijective map*, *bijection*, or *invertible* if it is surjective and injective.

Image and Preimage

The *image* of a set $A \subset X$ under a map $f : X \to Y$ is the set of images of all points of A. It is denoted by $f(A)$. Thus

$$f(A) = \{f(x) \,:\, x \in A\}.$$

The image of the entire set X (i.e., $f(X)$) is called the *image* of f. The *preimage* of a set $B \subset Y$ under a map $f : X \to Y$ is the set of elements of X whose images belong to B. It is denoted by $f^{-1}(B)$. Thus

$$f^{-1}(B) = \{a \in X \,:\, f(a) \in B\}.$$

Be careful with these terms: their etymology can be misleading. For example, the image of the preimage of a set B can differ from B. And even if it does not differ, It may happen that the preimage is not the only set with this property. Hence, the preimage *cannot* be defined as a set whose image is a given set.

6.A. $f\big(f^{-1}(B)\big) = B$, iff B is contained in the image of f.

6.B. $f\big(f^{-1}(B)\big) \subset B$ for any map $f : X \to Y$ and $B \subset Y$.

6.C. Let $f : X \to Y$ and $B \subset Y$ such that $f\big(f^{-1}(B)\big) = B$. Then the following statements are equivalent:

(a) $f^{-1}(B)$ is the unique subset of X whose image equals B;
(b) for any $a_1, a_2 \in f^{-1}(B)$ the equality $f(a_1) = f(a_2)$ implies $a_1 = a_2$.

6.D. A map $f : X \to Y$ is an injection, iff for any $B \subset Y$ such that $f\big(f^{-1}(B)\big) = B$ the preimage $f^{-1}(B)$ is the unique subset of X whose image equals B.

---

[6] Of course, the rule (as everything in the set theory) may be thought of as a set. Namely, one considers a set of ordered pairs (x, y) with $x \in X$, $y \in Y$ such that the rule assigns y to x. This set is called the *graph* of f. It is a subset of the set $X \times Y$ of all ordered pairs (x, y).

6.E. $f^{-1}\big(f(A)\big) \supset A$ for any map $f : X \to Y$ and $A \subset X$.

6.F. $f^{-1}\big(f(A)\big) = A$, iff $f(A) \cap f(X \smallsetminus A) = \varnothing$.

6.1. Do the following equalities hold true for any $A, B \subset Y$ and any $f : X \to Y$:

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B), \tag{8}$$

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B), \tag{9}$$

$$f^{-1}(Y \smallsetminus A) = X \smallsetminus f^{-1}(A)? \tag{10}$$

6.2. Do the following equalities hold true for any $A, B \subset X$ and any $f : X \to Y$:

$$f(A \cup B) = f(A) \cup f(B), \tag{11}$$

$$f(A \cap B) = f(A) \cap f(B), \tag{12}$$

$$f(X \smallsetminus A) = Y \smallsetminus f(A)? \tag{13}$$

6.3. Give examples in which two of the equalities above are false.

6.4. Replace the false equalities of *6.2* by correct inclusions.

6.5. What simple condition on $f : X \to Y$ should be imposed in order to make correct all the equalities of *6.2* for any $A, B \subset X$?

6.6. Prove that for any map $f : X \to Y$, and subsets $A \subset X$, $B \subset Y$:

$$B \cap f(A) = f\big(f^{-1}(B) \cap A\big).$$

Identity and Inclusion

The *identity map* of a set X is the map $X \to X$ defined by formula $x \mapsto x$. It is denoted by id_X, or just id, when there is no ambiguity. If A is a subset of X then the map $A \to X$ defined by formula $x \mapsto x$ is called an *inclusion map*, or just *inclusion*, of A into X and denoted by $\mathrm{in} : A \to X$, or just in, when A and X are clear.

6.G. The preimage of a set B under an inclusion $\mathrm{in} : A \to X$ is $B \cap A$.

Composition

The *composition* of mappings $f : X \to Y$ and $g : Y \to Z$ is the mapping $g \circ f : X \to Z$ defined by formula $x \mapsto g\big(f(x)\big)$.

6.H. $h \circ (g \circ f) = (h \circ g) \circ f$ for any maps $f : X \to Y$, $g : Y \to Z$, and $h : Z \to U$.

6.I. $f \circ (\mathrm{id}_X) = f = (\mathrm{id}_X) \circ f$ for any $f : X \to Y$.

6.J. The composition of injections is injective.

6.K. If the composition $g \circ f$ is injective then f is injective.

6.L. The composition of surjections is surjective.

6.M. If the composition $g \circ f$ is surjective then g is surjective.

6.N. The composition of bijections is a bijection.

6.7. Let a composition $g \circ f$ be bijective. Is then f or g necessarily bijective?

Inverse and Invertible

A map $g : Y \to X$ is said to be *inverse* to a map $f : X \to Y$ if $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. A map, for which an inverse map exists, is said to be *invertible.*

6.O. A mapping is invertible, iff it is a bijection.

6.P. If an inverse map exists then it is unique.

Submappings

If $A \subset X$ and $B \subset Y$ then for every $f : X \to Y$ such that $f(A) \subset B$ there is mapping $\mathrm{ab}(f) : A \to B$ defined by formula $x \mapsto f(x)$ and called an *abbreviation* of the mapping f to A, B, or *submapping*, or *submap*. If $B = Y$ then $\mathrm{ab}\, f : A \to Y$ is denoted by $f|_A$ and called the *restriction* of f to A. If $B \neq Y$ then $\mathrm{ab}\, f : A \to B$ is denoted by $f|_{A,B}$ or even simply $f\,|$.

6.Q. The restriction of a map $f : X \to Y$ to $A \subset X$ is the composition of inclusion $\mathrm{in}\, A :\to X$ and f. In other words, $f|_A = f \circ \mathrm{in}$.

6.R. Any abbreviation (including any restriction) of injections is injective.

6.S. If a restriction of a mapping is surjective then the original mapping is surjective.

7. Continuous Maps

Definition and Main Properties of Continuous Maps

Let X, Y be topological spaces. A map $f : X \to Y$ is said to be *continuous* if the preimage of any open subset of Y is an open subset of X.

7.A. *A map is continuous, iff the preimage of any closed set is closed.*

7.B. *The identity map of any topological space is continuous.*

7.1. Let Ω_1, Ω_2 be topological structures in X. Prove that the identity mapping of X

$$\mathrm{id} : (X, \Omega_1) \to (X, \Omega_2)$$

is continuous, iff $\Omega_2 \subset \Omega_1$.

7.2. Let $f : X \to Y$ be a continuous map. Is it continuous with respect to

(a) a finer topology in X and the same topology in Y,
(b) a coarser topology in X and the same topology in Y,
(c) a finer topology in Y and the same topology in X,
(d) a coarser topology in Y and the same topology in X?

7.3. Let X be a discrete space and Y an arbitrary space. Which maps $X \to Y$ and $Y \to X$ are continuous?

7.4. Let X be an indiscrete space and Y an arbitrary space. Which maps $X \to Y$ and $Y \to X$ are continuous?

7.C. *Let A be a subspace of X. The inclusion* $\mathrm{in} : A \to X$ *is continuous.*

7.D. The topology Ω_A induced on $A \subset X$ by the topology of X is the coarsest topology in A such that the inclusion mapping $\mathrm{in} : A \to X$ is continuous with respect to it.

7.5. Riddle. The statement *7.D* admits a natural generalization with the inclusion map replaced by an arbitrary map $f : A \to X$ of an arbitrary set A. Find this generalization.

7.E. *A composition of continuous maps is continuous.*

7.F. *A submap of a continuous map is continuous.*

7.G. *A map $f : X \to Y$ is continuous, iff* $\mathrm{ab}\, f : X \to f(X)$ *is continuous.*

7.H. Any constant map (i.e., a map with image consisting of a single point) is continuous.

Reformulations of Definition

7.6. Prove that a mapping $f : X \to Y$ is continuous, iff

$$\mathrm{Cl}\, f^{-1}(A) \subset f^{-1}(\mathrm{Cl}\, A)$$

for any $A \subset Y$.

7.7. Formulate and prove similar criteria of continuity in terms of $\mathrm{Int}\, f^{-1}(A)$ and $f^{-1}(\mathrm{Int}\, A)$. Do the same for $\mathrm{Cl}\, f(A)$ and $f(\mathrm{Cl}\, A)$.

7.8. Let Σ be a base for topology in Y. Prove that a map $f : X \to Y$ is continuous, iff $f^{-1}(U)$ is open for any $U \in \Sigma$.

More Examples

7.9. Is the mapping $f : [0,2] \to [0,2]$ defined by formula

$$f(x) = \begin{cases} x, & \text{if } x \in [0,1); \\ 3 - x, & \text{if } x \in [1,2] \end{cases}$$

continuous (with respect to the topology induced from the real line)?

7.10. Is the map f of segment $[0,2]$ (with the topology induced by the topology of the real line) into the arrow (see Section 1) defined by formula

$$f(x) = \begin{cases} x, & \text{if } x \in [0,1]; \\ x + 1, & \text{if } x \in (1,2] \end{cases}$$

continuous?

7.11. Give an explicit characterization of continuous mappings of $\mathbb{R}_{T_1}$ (see Section 1) to $\mathbb{R}$.

7.12. Which maps $\mathbb{R}_{T_1} \to \mathbb{R}_{T_1}$ are continuous?

7.13. Give an explicit characterization of continuous mappings of the arrow to itself.

7.14. Let f be a mapping of the set $\mathbb{Z}_+$ of nonnegative numbers onto $\mathbb{R}$ defined by formula

$$f(x) = \begin{cases} \frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Let $g : \mathbb{Z}_+ \to f(\mathbb{Z}_+)$ be its submap. Induce topology on $\mathbb{Z}_+$ and $f(\mathbb{Z}_+)$ from $\mathbb{R}$. Are f and the map g^{-1}, inverse to g, continuous?

Behavior of Dense Sets

7.15. Prove that the image of an everywhere dense set under a surjective continuous map is everywhere dense.

7.16. Is it true that the image of nowhere dense set under a continuous map is nowhere dense.

7.17*. Does there exist a nowhere dense set A of $[0,1]$ (with the topology induced out of the real line) and a continuous map $f : [0,1] \to [0,1]$ such that $f(A) = [0,1]$?

Local Continuity

A map f of a topological space X to a topological space Y is said to be *continuous at a point* $a \in X$ if for every neighborhood U of $f(a)$ there exists a neighborhood V of a such that $f(V) \subset U$.

7.I. *A map $f : X \to Y$ is continuous, iff it is continuous at each point of X.*

7.J. *Let X, Y be metric spaces, and $a \in X$. A map $f : X \to Y$ is continuous at a, iff for every ball with center at $f(a)$ there exists a ball with center at a whose image is contained in the first ball.*

7.K. *Let X, Y be metric spaces, and $a \in X$. A mapping $f : X \to Y$ is continuous at the point a, iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every point $x \in X$ inequality $\rho(x, a) < \delta$ implies $\rho\big(f(x), f(a)\big) < \varepsilon$.*

Theorem *7.K* means that continuity introduced above coincides with the one that is usually studied in Calculus.

Properties of Continuous Functions

7.18. Let $f, g : X \to \mathbb{R}$ be continuous. Prove that the mappings $X \to \mathbb{R}$ defined by formulas

$$x \mapsto f(x) + g(x), \tag{14}$$

$$x \mapsto f(x)g(x), \tag{15}$$

$$x \mapsto f(x) - g(x), \tag{16}$$

$$x \mapsto \big|f(x)\big|, \tag{17}$$

$$x \mapsto \max\{f(x), g(x)\}, \tag{18}$$

$$x \mapsto \min\{f(x), g(x)\} \tag{19}$$

are continuous.

7.19. Prove that if $0 \notin g(X)$ then a mapping $X \to \mathbb{R}$ defined by formula

$$x \mapsto \frac{f(x)}{g(x)}$$

is continuous.

7.20. Find a sequence of continuous functions $f_i : \mathbb{R} \to \mathbb{R}$, $(i \in \mathbb{N})$ such that the formula

$$x \mapsto \sup\{\, f_i(x) \mid i \in \mathbb{N}\}$$

defines a function $\mathbb{R} \to \mathbb{R}$ which is not continuous.

7.21. Let X be any topological space. Prove that a function $f : X \to \mathbb{R}^n$: $x \mapsto (f_1(x), \dots, f_n(x))$ is continuous, iff all the functions $f_i : X \to \mathbb{R}$ with $i = 1, \dots, n$ are continuous.

Real $p \times q$-matrices comprise a space $Mat(p \times q, \mathbb{R})$, which differs from $\mathbb{R}^{pq}$ only in the way of numeration of its natural coordinates (they are numerated by pairs of indices).

7.22. Let $f : X \to Mat(p \times q, \mathbb{R})$ and $g : X \to Mat(q \times r, \mathbb{R})$ be continuous maps. Prove that then

$$X \to Mat(p \times r, \mathbb{R}) : x \mapsto g(x)f(x)$$

is a continuous map.

Recall that $GL(n; \mathbb{R})$ is the subspace of $Mat(n \times n, \mathbb{R})$ consisting of all the invertible matrices.

7.23. Let $f : X \to GL(n; \mathbb{R})$ be a continuous map. Prove that $X \to GL(n; \mathbb{R}) : x \mapsto (f(x))^{-1}$ is continuous.

Special About Metric Case

7.L. *For every subset A of a metric space X the function defined by formula $x \mapsto \rho(x, A)$ (see Section 3) is continuous.*

> ***7.24.*** Prove that a topology of a metric space is the coarsest topology, with respect to which for every $A \subset X$ the function $X \to \mathbb{R}$ defined by formula $x \mapsto \rho(x, A)$ is continuous.

A mapping f of a metric space X into a metric space Y is called an *isometric embedding* if $\rho\big(f(a), f(b)\big) = \rho(a, b)$ for every $a, b \in X$. A bijection which is an isometric embedding is called an *isometry.*

7.M. Every isometric embedding is injective.

7.N. Every isometric embedding is continuous.

> A mapping $f : X \to X$ of a metric space X is called *contractive* if there exists $\alpha \in (0, 1)$ such that $\rho\big(f(a), f(b)\big) \le \alpha\rho(a, b)$ for every a, $b \in X$.
>
> ***7.25.*** Prove that every contractive mapping is continuous.
>
> Let X, Y be metric spaces. A mapping $f : X \to Y$ is said to be *Hölder* if there exist $C > 0$ and $\alpha > 0$ such that $\rho\big(f(a), f(b)\big) \le C\rho(a, b)^\alpha$ for every a, $b \in X$.
>
> ***7.26.*** Prove that every Hölder mapping is continuous.

Functions on Cantor Set and Square-Filling Curves

Recall that Cantor set K is the set of real numbers which can be presented as sums of series of the form $\sum_{k=1}^{\infty} \frac{a_k}{3^k}$ with $a_k = 0$ or 2.

7:A. Let γ_1 be a map $K \to I$ defined by

$$\sum_{k=1}^{\infty} \frac{a_k}{3^k} \mapsto \sum_{k=1}^{\infty} \frac{a_k}{2^{k+1}}.$$

Prove that $\gamma_1 : K \to I$ is a continuous surjection. Draw the graph of φ.

7:B. Prove that the function $K \to K$ defined by

$$\sum_{k=1}^{\infty} \frac{a_k}{3^k} \mapsto \sum_{k=1}^{\infty} \frac{a_{2k}}{3^k}$$

is continuous.

Denote by K^2 the set $\{(x, y) \in \mathbb{R}^2 : x \in K, y \in K\}$.

7:C. Prove that the map $\gamma_2 : K \to K^2$ defined by

$$\sum_{k=1}^{\infty} \frac{a_k}{3^k} \mapsto \left(\sum_{k=1}^{\infty} \frac{a_{2k-1}}{3^k}, \sum_{k=1}^{\infty} \frac{a_{2k}}{3^k} \right)$$

is a continuous surjection.

7:D. Prove that the map $\gamma_3 : K \to I^2$ defined as the composition of $\gamma_2 : K \to K^2$ and $K^2 \to I^2 : (x, y) \mapsto (\gamma_1(x), \gamma_1(y))$ is a continuous surjection.

7:E. Prove that the map $\gamma_3 : K \to I^2$ is a restriction of a continuous map. (Cf. *1:A:2*.)

The latter map is a continuous surjection $I \to I^2$. Thus, this is a curve filling the square. A curve with this property was first constructed by G. Peano in 1890. Though the construction sketched above is based on the same ideas as the original Peano's construction, they are slightly different. Since then a lot of other similar examples have been found. You may find a nice survey of them in a book by Hans Sagan, Space-Filling Curves, Springer-Verlag 1994. Here is a sketch of Hilbert's construction.

7:F. Prove that there exists a sequence of polygonal maps $f_k : I \to I^2$ such that

(a) f_k connects all centers of the squares forming the obvious subdivision of I^2 into 4^k equal squares with side $1/2^k$;
(b) $\operatorname{dist}(f_k(x), f_{k-1}(x)) \le \sqrt{2}/2^{k+1}$ for any $x \in I$ (here dist means the metric induced on I^2 from the standard Euclidean metric of R^2).

7:G. Prove that any sequence of paths $f_k : I \to I^2$ satisfying the conditions of *7:F* converges to a map $f : I \to I^2$ (i.e. for any $x \in I$ there exists a limit $f(x) = \lim_{k\to\infty} f_k(x)$) and this map is continuous and its image is dense in I^2.

7:H. [7] Prove that any continuous map $I \to I^2$ with dense image is surjective.

7:I. Generalize *7:C* – *7:E* *7:F* – *7:H* to obtain a continuous surjection of I onto I^n.

Sets Defined by Systems of Equations and Inequalities

7.O. *Let f_i $(i = 1, \dots , n)$ be continuous mappings $X \to \mathbb{R}$. Then the subset of X consisting of solutions of the system of equations*

$$f_1(x) = 0, \dots , f_n(x) = 0$$

is closed.

7.P. *Let f_i $(i = 1, \dots , n)$ be continuous mappings $X \to \mathbb{R}$. Then the subset of X consisting of solutions of the system of inequalities*

$$f_1(x) \ge 0, \dots , f_n(x) \ge 0$$

is closed, while the set consisting of solutions of the system of inequalities

$$f_1(x) > 0, \dots , f_n(x) > 0$$

is open.

---

[7]Although this problem can be solved using theorems well-known from Calculus, we have to mention that it would be more appropriate after Section 13. Cf. Problems *13.O*, *13.T*, *13.K*.

7.27. Where in *7.O* and *7.P* a finite system can be replaced by an infinite one.

Set-Theoretic Digression. Covers

A collection Γ of subsets of a set X is called a *cover* or a *covering* of X if X is a union of sets of belonging to Γ, i.e., $X = \bigcup_{A\in\Gamma} A$. In this case elements of Γ are said to *cover* X.

There is also a more general meaning of these words. A collection Γ of subsets of a set Y is called a *cover* or a *covering* of a set $X \subset Y$ if X is contained in the union of the sets belonging to Γ, i.e., $X \subset \bigcup_{A\in\Gamma} A$. In this case, sets belonging to Γ are also said to *cover* X.

Fundamental Covers

Consider a cover Γ of a topological space X. Each element of Γ inherits from X a topological structure. When are these structures sufficient for recovering the topology of X? In particular, under what conditions on Γ does continuity of a map $f : X \to Y$ follow from continuity of its restrictions to elements of Γ. To answer these questions, solve the problems *7.28*–*7.29* and *7.Q*–*7.V*.

7.28. Is this true for the following coverings:

(a) $X = [0, 2]$, $\Gamma = \{[0, 1], (1, 2]\}$;
(b) $X = [0, 2]$, $\Gamma = \{[0, 1], [1, 2]\}$;
(c) $X = \mathbb{R}$, $\Gamma = \{\mathbb{Q}, \mathbb{R} \smallsetminus \mathbb{Q}\}$;
(d) $X = \mathbb{R}$, Γ is a set of all one-point subsets of $\mathbb{R}$?

7.29. A cover of a topological space consisting of one-point subsets has the property described above, iff the space is discrete.

A cover Γ of a space X is said to be *fundamental* if a set $U \subset X$ is open, iff for every $A \in \Gamma$ the set $U \cap A$ is open in A.

7.Q. A covering Γ of a space X is fundamental, iff a set $U \subset X$ is open provided $U \cap A$ is open in A for every $A \in \Gamma$.

7.R. A covering Γ of a space X is fundamental, iff a set $F \subset X$ is closed provided $F \cap A$ is closed A for every $A \in \Gamma$.

A cover of a topological space is said to be *open* if it consists of open sets, and *closed* if it consists of closed sets. A cover of a topological space is said to be *locally finite* if every point of the space has a neighborhood intersecting only a finite number of elements of the cover.

7.S. *Every open cover is fundamental.*

7.T. *Every finite closed cover is fundamental.*

7.U. Every locally finite closed cover is fundamental.

7.V. *Let Γ be a fundamental cover of a topological space X. If the restriction of a mapping $f : X \to Y$ to each element of Γ is continuous then f is continuous.*

A cover Γ' is said to be a *refinement* of a cover Γ if every element of Γ' is contained in some element of Γ.

7.30. Prove that if a cover Γ' is a refinement of a cover Γ, and Γ' is fundamental then Γ is also fundamental.

7.31. Let Δ be a fundamental cover of a topological space X, and Γ be a cover of X such that $\Gamma_A = \{U \cap A \mid U \in \Gamma\}$ is a fundamental cover for subspace $A \subset X$ for every $A \in \Delta$. Prove that Γ is a fundamental cover.

7.32. Prove that the property of being fundamental is local, i.e., if every point of a space X has a neighborhood V such that $\Gamma_V = \{U \cap V \mid U \in \Gamma\}$ is fundamental, then Γ is fundamental.

8. Homeomorphisms

Definition and Main Properties of Homeomorphisms

An invertible mapping is called a *homeomorphism* if both this mapping and its inverse are continuous.

8.A. Find an example of a continuous bijection, which is not a homeomorphism.

8.B. Find a continuous bijection $[0, 1) \to S^1$, which is not a homeomorphism.

8.C. The identity map of a topological space is a homeomorphism.

8.D. *A composition of homeomorphisms is a homeomorphism.*

8.E. *The inverse of a homeomorphism is a homeomorphism.*

Homeomorphic Spaces

A topological space X is said to be *homeomorphic* to space Y if there exists a homeomorphism $X \to Y$.

8.F. *Being homeomorphic is an equivalence relation. (Cf. 8.C–8.E.)*

Role of Homeomorphisms

8.G. Let $f : X \to Y$ be a homeomorphism. Then $U \subset X$ is open (in X), iff $f(U)$ is open (in Y).

8.H. $f : X \to Y$ is a homeomorphism, iff f is a bijection and defines a bijection between the topological structures of X and Y.

8.I. Let $f : X \to Y$ be a homeomorphism. Then for every $A \subset X$

(a) A is closed in X, iff $f(A)$ is closed in Y;
(b) $f(\operatorname{Cl} A) = \operatorname{Cl} f(A)$;
(c) $f(\operatorname{Int} A) = \operatorname{Int} f(A)$;
(d) $f(\operatorname{Fr} A) = \operatorname{Fr} f(A)$;
(e) A is a neighborhood of a point $x \in X$, iff $f(A)$ is a neighborhood of the point $f(x)$;
(f) etc.

Therefore from the topological point of view homeomorphic spaces are completely identical: a homeomorphism $X \to Y$ establishes one-to-one correspondence between all phenomena in X and Y which can be expressed in terms of topological structures.

This phenomenon was used as a basis for a definition of the subject of topology in the first stages of its development, when the notion of topological space had not been developed yet. Then mathematicians studied only subspaces of Euclidean spaces, their continuous mappings and homeomorphisms. Felix

Klein in his famous Erlangen Program,[8] where he classified various geometries that had emerged up to that time, like Euclidean, Lobachevsky, affine, and projective geometries, defined topology as a part of geometry which deals with the properties preserved by homeomorphisms.

More Examples of Homeomorphisms

8.J. Let $f : X \to Y$ be a homeomorphism. Prove that for every $A \subset X$ the reduction $\mathrm{ab}(f) : A \to f(A)$ is also a homeomorphism.

8.K. Prove that every isometry (see Section 7) is a homeomorphism.

8.L. Prove that every nondegenerate affine transformation of $\mathbb{R}^n$ is a homeomorphism.

8.1. Prove that inversion

$$x \mapsto \frac{Rx}{|x|^2} : \mathbb{R}^n \smallsetminus \{0\} \to \mathbb{R}^n \smallsetminus \{0\}$$

is a homeomorphism.

8.2. Let $\mathbb{H} = \{\, z \in \mathbb{C} \mid \mathrm{Im} z > 0 \,\}$ be the upper half-plane. Prove that mapping $f : \mathbb{H} \to \mathbb{H}$ defined by $f(z) = \dfrac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}$, is a homeomorphism if $\begin{vmatrix} a & b \\ c & d \end{vmatrix} > 0$.

8.3. Prove that a bijection $\mathbb{R} \to \mathbb{R}$ is a homeomorphism, iff it is a monotone function.

8.4. Prove that every bijection of an indiscrete space onto itself is a homeomorphism. Prove that the same holds true for a discrete space and $\mathbb{R}_{T_1}$.

8.5. Find all homeomorphisms of the space $4pT$ (see Section 1) to itself.

8.6. Prove that every continuous bijection of the arrow onto itself is a homeomorphism.

8.7. Find two homeomorphic spaces X and Y and a continuous bijection $X \to Y$, which is not a homeomorphism.

8.8. Is $\gamma_2 : K \to K^2$ considered in Problem *7:C* a homeomorphism? Recall that K is the Cantor set, $K^2 = \{(x, y) \in \mathbb{R}^2 \,:\, x \in K, y \in K\}$ and γ_2 is defined by

$$\sum_{k=1}^{\infty} \frac{a_k}{3^k} \mapsto \left(\sum_{k=1}^{\infty} \frac{a_{2k-1}}{3^k}, \sum_{k=1}^{\infty} \frac{a_{2k}}{3^k} \right)$$

[8] In fact it was not assumed to be a program in the sense of being planned, although it became a kind of program. It was a sort of dissertation presented by Klein for getting the position as a professor at Erlangen University.

Examples of Homeomorphic Spaces

Below the homeomorphism relation is denoted by $\cong$. It is not a commonly accepted notation. In other textbooks any sign close to, but distinct from $=$, e. g. $\sim$, $\simeq$, $\approx$, is used.

8.M. $[0,1] \cong [a,b]$ for any $a < b$.

8.N. $[0,1) \cong [a,b) \cong (0,1] \cong (a,b]$ for any $a < b$.

8.O. $(0,1) \cong (a,b)$ for any $a < b$.

8.P. $(-1,1) \cong \mathbb{R}$.

8.Q. $[0,1) \cong [0,+\infty)$ and $(0,1) \cong (0,+\infty)$.

8.R. $S^1 \smallsetminus \{(0,1)\} \cong \mathbb{R}^1$.

8.S. $S^n \smallsetminus \{\text{point}\} \cong \mathbb{R}^n$.

8.9. Prove that the following plane figures are homeomorphic:

(a) the whole plane $\mathbb{R}^2$;
(b) open square $\{\,(x,y) \in \mathbb{R}^2 \mid x,y \in (0,1)\,\}$;
(c) open strip $\{\,(x,y) \in \mathbb{R}^2 \mid x \in (0,1)\,\}$;
(d) half-plane $\{\,(x,y) \in \mathbb{R}^2 \mid y > 0\,\}$;
(e) open half-strip $\{\,(x,y) \in \mathbb{R}^2 \mid x > 0,\, y \in (0,1)\,\}$;
(f) open disk $\{\,(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\,\}$;
(g) open rectangle $\{\,(x,y) \in \mathbb{R}^2 \mid a < x < b,\, c < y < d\,\}$;
(h) open quadrant $\{\,(x,y) \in \mathbb{R}^2 \mid x,y > 0\,\}$;
(i) $\{\,(x,y) \in \mathbb{R}^2 \mid y^2 + |x| > x\,\}$, i.e., plane cut along the ray $\{\,y = 0,\, x \geq 0\,\}$.

8.T. Prove that

(a) closed disk D^2 is homeomorphic to square $I^2 = \{\,(x,y) \in \mathbb{R}^2 \mid x,y \in [0,1]\,\}$;
(b) open disc $\operatorname{Int} D^2 = \{\,(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\,\}$ is homeomorphic to open square $\operatorname{Int} I^2 = \{\,(x,y) \in \mathbb{R}^2 \mid x,y \in (0,1)\,\}$;
(c) circle S^1 is homeomorphic to the boundary of square $\partial I^2 = I^2 \smallsetminus \operatorname{Int} I^2$.

8.U. Prove that

(a) every bounded closed convex set in the plane with nonempty interior is homeomorphic to D^2;
(b) every bounded open convex nonempty set in the plane is homeomorphic to the plane;
(c) boundary of every bounded convex set in the plane with nonempty interior is homeomorphic to S^1.

8.10. In which of the situations considered in *8.U* can the assumption that the set is bounded be omitted?

8.11. Classify up to homeomorphism all closed convex sets in the plane. (Make a list without repeats; prove that every such set is homeomorphic to one in the list; postpone a proof of nonexistence of homeomorphisms till Section 9.)

8.12*. Generalize the previous three problems to the case of sets in $\mathbb{R}^n$ with arbitrary n.

The latter four problems show that angles are not essential in topology, i.e., for a line or boundary of a domain the property of having angles is not preserved by homeomorphism. And now two more problems on this.

8.13. Prove that every closed simple (i.e., without self-intersections) polygon in $\mathbb{R}^2$ (and in $\mathbb{R}^n$ with $n > 2$) is homeomorphic to the circle S^1.

8.14. Prove that every non-closed simple finite unit polyline in $\mathbb{R}^2$ (and in $\mathbb{R}^n$ with $n > 2$) is homeomorphic to the segment $[0, 1]$.

8.15. Prove that $\mathbb{R}^2 \setminus \{\, |x|, |y| > 1 \,\} \cong I^2 \setminus \{(\pm 1, \pm 1), (\pm 1, \pm 1)\}$.

8.16. Prove that the following plane figures are homeomorphic to each other:

(a) $\{\, (x, y) \mid 0 \le x,\ y < 1 \,\}$;
(b) $\{\, (x, y) \mid 0 < x < 1,\ 0 \le y < 1 \,\}$;
(c) $\{\, (x, y) \mid 0 \le x \le 1,\ 0 \le y < 1 \,\}$;
(d) $\{\, (x, y) \mid x, y \ge 0 \,\}$;
(e) $\{\, (x, y) \mid x \ge 0 \,\}$;
(f) $\{\, (x, y) \mid x \ge y \ge 0 \,\}$;
(g) $\{\, (x, y) \mid x^2 + y^2 \le 1,\ x \ne 1 \,\}$.

8.17. Prove that the following plane figures are homeomorphic to each other:

(a) punctured plane $\mathbb{R}^2 \setminus \{(0, 0)\}$;
(b) punctured disc $\{\, (x, y) \mid 0 < x^2 + y^2 < 1 \,\}$;
(c) annulus $\{\, (x, y) \mid a < x^2 + y^2 < b \,\}$ where $0 < a < b$;
(d) plane without disc $\{\, (x, y) \mid x^2 + y^2 > 1 \,\}$;
(e) plane without square $\{\, (x, y) \mid 0 \le x, y \le 1 \,\}$;
(f) plane without segment $\mathbb{R}^2 \setminus [0, 1]$.

8.18. Let $X \subset \mathbb{R}^2$ be an union of several segments with a common end point. Prove that the complement $\mathbb{R}^2 \setminus X$ is homeomorphic to the punctured plane.

8.19. Let $X \subset \mathbb{R}^2$ simple non-closed finite polyline. Prove that its complement $\mathbb{R}^2 \setminus X$ is homeomorphic to the punctured plane.

8.20. Let $D_1, \dots, D_n \subset \mathbb{R}^2$ be pairwise disjoint closed discs. The complement of the union of its interior is said to be *plane with n holes*. Prove that any two planes with n holes are homeomorphic, i.e., dislocation of discs $D_1, \dots, D_n$ does not affect on the topological type of $\mathbb{R}^2 \setminus \cup_{i=1}^n \operatorname{Int} D_i$.

8.21. Prove that for continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f < g$, the space between their graphs $\{\, (x, y) \in \mathbb{R}^2 \mid f(x) \le y \le g(x) \,\}$ is homeomorphic to a closed strip $\{\, (x, y) \mid y \in [0, 1] \,\}$.

8.22. Prove that a mug (with handle) is homeomorphic to a doughnut.

8.23. Arrange the following items to homeomorphism classes: a cup, a saucer, a glass, a spoon, a fork, a knife, a plate, a coin, a nail, a screw, a bolt, a nut, a wedding ring, a drill, a flower pot (with hole in the bottom), a key.

8.24. In a spherical shell (the space between two concentric spheres) one drilled out a cylindrical hole connecting the boundary spheres. Prove that the rest is homeomorphic to D^3.

8.25. In a spherical shell one made a hole connecting the boundary spheres and having the shape of a knotted tube (see Figure 1.). Prove that the rest of the shell is homotopic to D^3.

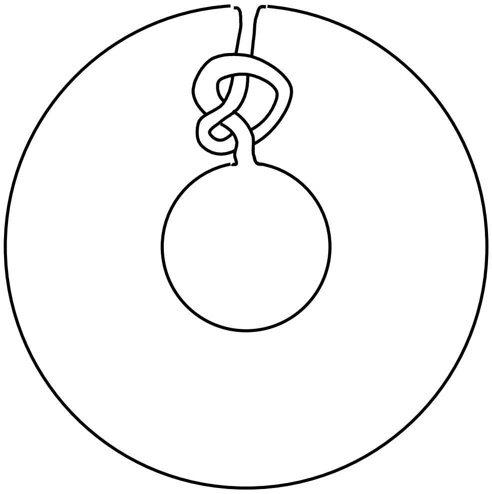


FIGURE 1

8.26. Prove that surfaces shown in Figure 2 are homeomorphic (they are called *handles*).

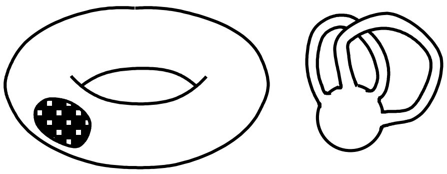


FIGURE 2

8.27. Prove that surfaces shown in the Figure 3 are homeomorphic. (They are homeomorphic to *Klein bottle with two holes.* More details about this is given in Section 18.)

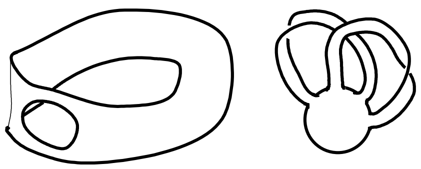


FIGURE 3

8.28*. Prove that $\mathbb{R}^3 \smallsetminus S^1 \cong \mathbb{R}^3 \smallsetminus \big(\mathbb{R}^1 \cup \{(1,1,1)\}\big)$.

8.29. Prove that subset of the sphere S^n defined in standard coordinates in $\mathbb{R}^{n+1}$ by inequality $x_1^2 + x_2^2 + \dots + x_k^2 < x_{k+1}^2 + \dots + x_n^2$ is homeomorphic to $\mathbb{R}^n \smallsetminus \mathbb{R}^{n-k}$.

Examples of Nonhomeomorphic Spaces

8.V. Spaces consisting of different number of points are not homeomorphic.

8.W. A discrete space and an indiscrete space (which have more than one point) are not homeomorphic.

> ***8.30.*** Prove that the spaces $\mathbb{Z}$, $\mathbb{Q}$ (with topology induced from $\mathbb{R}$), $\mathbb{R}$, $\mathbb{R}_{T_1}$ and the arrow are pairwise non-homeomorphic.
>
> ***8.31.*** Find two non-homeomorphic spaces X and Y for which there exist continuous bijections $X \to Y$ and $Y \to X$.

Homeomorphism Problem and Topological Properties

One of the classic problems of topology is the *homeomorphism problem:* to find out whether two given topological spaces are homeomorphic. In each special case the character of solution depends mainly on the answer. To prove that spaces are homeomorphic, it is enough to present a homeomorphism between them. Essentially this is what one usually does in this case. To prove that spaces are **not** homeomorphic, it does not suffice to consider any special mapping, and usually it is impossible to review all the mappings. Therefore for proving non-existence of a homeomorphism one uses indirect arguments. In particular, one finds a property or a characteristic shared by homeomorphic spaces and such that one of the spaces has it, while the other does not. Properties and characteristics which are shared by homeomorphic spaces are called *topological properties* and *invariants.* Obvious examples of them are the cardinality (i.e., the number of elements) of the set of points and the set of open sets (cf. Problems *8.29* and *8.V*). Less obvious examples are the main object of the next chapter.

Information (Without Proof)

Euclidean spaces of different dimensions are not homeomorphic. The balls D^p, D^q with $p \neq q$ are not homeomorphic. The spheres S^p, S^q with $p \neq q$ are not homeomorphic. Euclidean spaces are homeomorphic neither to balls, nor to spheres (of any dimension). Letters A and B are not homeomorphic (if the lines are absolutely thin!). Punctured plane $\mathbb{R}^2 \smallsetminus \{\text{point}\}$ is not homeomorphic to the plane with hole $\mathbb{R}^2 \smallsetminus \{ x^2 + y^2 < 1 \}$.

These statements are of different degrees of difficulty. Some of them will be considered in the next section. However some of them can not be proven by techniques of this course. (See, e.g., D. B. Fuchs, V. A. Rokhlin. Beginner's course in topology: Geometric chapters. Berlin; New York: Springer-Verlag, 1984.)

Embeddings

Continuous mapping $f : X \to Y$ is called a (*topological*) *embedding* if the submapping $\mathrm{ab}(f) : X \to f(X)$ is a homeomorphism.

8.X. The inclusion of a subspace into a space is an embedding.

8.Y. Composition of embeddings is an embedding.

8.Z. Give an example of continuous injective map, which is not a topological embedding. (Find such an example above and create a new one.)

8.32. Find topological spaces X and Y such that X can be embedded into Y, Y can be embedded into X, but $X \not\cong Y$.

8.33. Prove that Q cannot be embedded into $\mathbb{Z}$.

8.34. Can a discrete space be embedded into an indiscrete space? How about vice versa?

8.35. Prove that spaces $\mathbb{R}$, $\mathbb{R}_{T_1}$, and the arrow cannot be embedded into each other.

8.36 Corollary of Inverse Function Theorem. Deduce from the Inverse Function Theorem (see, e.g., any course of advanced calculus) the following statement:

For any differentiable function $f : \mathbb{R}^n \to \mathbb{R}^n$ whose Jacobian $\det(\frac{\partial f_i}{\partial x_j})$ does not vanish at the origin $0 \in \mathbb{R}^n$ there exists a neighborhood U of the origin such that $f|_U : U \to \mathbb{R}^n$ is an embedding and $f(U)$ is open.

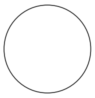
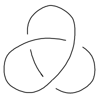
Embeddings $f_1, f_2 : X \to Y$ are said to be *equivalent* if there exist homeomorphisms $h_X : X \to X$ and $h_Y : Y \to Y$ such that $f_2 \circ h_X = h_Y \circ f_1$ (the latter equality mayb stated as follows: the diagram

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "h_X"'] & Y \arrow[d, "h_Y"] \\ X \arrow[r, "f_2"] & Y \end{tikzcd}$$

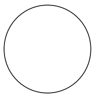
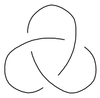
is commutative).

An embedding of the circle S^1 into $\mathbb{R}^3$ is called a *knot*.

8.37. Prove that knots $f_1, f_2 : S^1 \to \mathbb{R}^3$ with $f_1(S^1) = f_2(S^1)$ are equivalent.

8.38. Prove that knots   are equivalent.

Information

There are nonequivalent knots. For instance,  and .

CHAPTER 2

Topological Properties

9. Connectedness

Definitions of Connectedness and First Examples

A topological space X is said to be *connected* if it has only two subsets which are both open and closed: $\varnothing$ and the entire X.

A *partition* of a set is a cover of this set with pairwise disjoint sets. To *partition* a set means to construct such a cover.

9.A. *A topological space is connected, iff it cannot be partitioned into two nonempty open sets, iff it cannot be partitioned into two nonempty closed sets.*

> ***9.1.*** Is an indiscrete space connected? The same for the arrow and $\mathbb{R}_{T_1}$.
>
> ***9.2.*** Describe explicitly all connected discrete spaces.
>
> ***9.3.*** Is the set $\mathbb{Q}$ of rational numbers (with the topology induced from $\mathbb{R}$) connected? The same about the set of irrational numbers.
>
> ***9.4.*** Let Ω_1, Ω_2 be topological structures in a set X, and Ω_2 be finer than Ω_1 (i.e., $\Omega_1 \subset \Omega_2$). If (X, Ω_1) is connected, is (X, Ω_2) connected? If (X, Ω_2) is connected, is (X, Ω_1) connected?

Connected Sets

When one says that a set is connected, it means that this set lies in some topological space (which should be clear from the context), and, with the induced topology, is a connected topological space.

> ***9.5.*** Give a definition of disconnected subset without relying on the induced topology.
>
> ***9.6.*** Is the set $\{0, 1\}$ connected in $\mathbb{R}$, in the arrow, in $\mathbb{R}_{T_1}$?
>
> ***9.7.*** Describe explicitly all connected subsets of the arrow, of $\mathbb{R}_{T_1}$.
>
> ***9.8.*** Show that the set $[0, 1] \cup (2, 3]$ is disconnected in $\mathbb{R}$.
>
> ***9.9.*** Prove that every non-convex subset of the real line is disconnected.
>
> ***9.10.*** Let A be a subset of a topological space X. Prove that A is disconnected, iff there exist non-empty sets B and C such that $A = B \cup C$, $B \cap \operatorname{Cl}_X C = \varnothing$, and $C \cap \operatorname{Cl}_X B = \varnothing$.

9.11. Find a topological space X and disconnected subset $A \subset X$ such that for any disjoint open sets U and V, which form a cover of X, either $U \supset A$, or $V \supset A$.

9.12. Prove that for every disconnected set A in $\mathbb{R}^n$ there exist disjoint open sets U and V such that $A \subset U \cup V$, $U \cap A \neq \varnothing$, and $V \cap A \neq \varnothing$.

Compare *9.10–9.12* with *9.5*.

Properties of Connected Sets

9.B. *The closure of a connected set is connected.*

9.13. Prove that if a set A is connected and $A \subset B \subset \operatorname{Cl} A$, then B is connected.

9.C. *Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a family of connected subsets of a space X. Assume that any two sets of this family intersect. Then $\bigcup_{\lambda\in\Lambda} A_\lambda$ is connected. (In other words: the union of pairwise intersecting connected sets is connected.)*

9.D. Let $\{A_k\}_{k\in\mathbb{Z}}$ be a family of connected sets such that $A_k \cap A_{k+1} \neq \varnothing$ for any $k \in \mathbb{Z}$. Prove that $\bigcup_{k\in\mathbb{Z}} A_k$ is connected.

9.14. Let A, B be connected sets, and $A \cap \operatorname{Cl} B \neq \varnothing$. Prove that $A \cup B$ is connected.

9.15. Let A be a connected subset of a connected space X, and $B \subset X \smallsetminus A$ be an open and closed set in the topology of the subspace $X \smallsetminus A$ of the space X. Prove that $A \cup B$ is connected.

9.16. Does connectedness of $A \cup B$ and $A \cap B$ imply connectedness of A and B?

9.17. Prove that if A and B are either both closed or both open sets, and their union and intersection are connected then A and B are connected, too.

9.18. Let $A_1 \supset A_2 \supset \cdots$ be an infinite descending sequence of connected spaces. Is $\bigcap_{k=1}^{\infty} A_k$ a connected set?

Connected Components

A *connected component* of a space X is its maximal connected subset, that is a connected subset, which is not contained in any other (strictly) larger connected subset of X.

9.E. *Every point belongs to some connected component. Moreover, this component is unique. It is the union of all connected sets containing this point.*

9.F. *Connected components are closed.*

9.G. *Two connected components either are disjoint or coincide.*

A connected component of a space X is called just a *component* of X. Theorems *9.E* and *9.G* mean that connected components comprise a partition of the whole space. The next theorem describes the corresponding equivalence relation.

9.H. *Prove that two points are in the same component, iff they belong to the same connected set.*

> ***9.19.*** Let x and y belong to the same component. Prove that any set, which is closed and open, either contains both x and y or does not contain either of them (cf. *9.29*).
>
> ***9.20.*** Let a space X has a group structure, and the multiplication by an element of the group is a continuous map. Prove that the component of unity is a normal subgroup.

Totally Disconnected Spaces

A topological space is called *totally disconnected* if each of its components consists of a single point.

9.I Obvious Example. Any discrete space is totally disconnected.

9.J. The space $\mathbb{Q}$ (with the topology induced from $\mathbb{R}$) is totally disconnected.

Note that $\mathbb{Q}$ is not discrete.

> ***9.21.*** Give an example of an uncountable closed totally disconnected subset of the line.
>
> ***9.22.*** Prove that Cantor set (see *1:A*) is totally disconnected.

Frontier and Connectedness

> ***9.23.*** Prove that if A is a proper nonempty subset of a connected topological space then $\operatorname{Fr} A \neq \varnothing$.
>
> ***9.24.*** Let F be a connected subset of X. Prove that if $A \subset X$, $F \cap A$, and $F \cap (X \smallsetminus A) \neq \varnothing$ then $F \cap \operatorname{Fr} A \neq \varnothing$.
>
> ***9.25.*** Let A be a subset of connected topological space. Prove that if $\operatorname{Fr} A$ is a connected set then $\operatorname{Cl} A$ is also connected.

Behavior Under Continuous Maps

A *continuous image* of a space is its image under a continuous mapping.

9.K. *A continuous image of a connected space is connected. (In other words if $f : X \to Y$ is a continuous map, and X is connected then $f(X)$ is also connected.)*

9.L Corollary. *Connectedness is a topological property. The number of connected components is a topological invariant.*

9.M. *A space X is not connected, iff there is a continuous surjection $X \to S^0$.*

Connectedness on Line

9.N. *The segment $I = [0,1]$ is connected.*

There are several ways to prove *9.N*. One is suggested by *9.M*, but refers to a famous Intermediate Value Theorem from calculus, see *9.S*. Basically the same proof as a combination of *9.M* with a traditional proof of Intermidiate Value Theorem is sketched in the following two problems. Cf. also *9.26* below.

9.N.1. Let U, V be subsets of I with $V = U \smallsetminus V$. Let $a \in U$, $b \in V$ and $a > b$. Prove that there exists a descending sequence a_n with $a_1 = a$, $a_n \in U$ and an ascending sequence b_n with $b_1 = b$, $b_n \in V$ such that both a_n and b_n have the same limit c.

9.N.2. If under assumptions of *9.N.1* U and V are open, then in which of them can be c?

9.26. Prove that every open subset of the real line is a union of disjoint open intervals (do not use *9.N*). Deduce *9.N* from this.

9.O. Prove that the set of connected components of an open subset of $\mathbb{R}$ is countable.

9.P. Prove that R^1 is connected.

9.Q. Describe explicitly all connected subsets of the line.

9.R. Prove that every convex set in R^n is connected.

9.27. Consider the union of spiral

$$r = exp\left(\frac{1}{1+\varphi^2}\right), \text{ with } \varphi \geq 0$$

(r, φ are the polar coordinates) and circle S^1. Is this set connected? Would the answer change, if the entire circle was replaced by some its subset? (Cf. *9.13*)

9.28. Consider the subset of the plane $\mathbb{R}^2$ consisting of points with both coordinates rational or both coordinates irrational. Is it connected?

9.29. Find a space and two points belonging to its different components such that each simultaneously open and closed set contains either both of the points, or neither of them (cf. *9.19*).

Intermediate Value Theorem and Its Genralizations

The following theorem is usually included in Calculus. You can easily deduce it from the matterial of this section. In fact, in a sense it is equivalent to connectedness of interval.

9.S Intermediate Value Theorem. *A continuous function $f : [a,b] \to \mathbb{R}$ takes every value between $f(a)$ and $f(b)$ for an argument between a and b.*

Many problems which can be solved using Intermediate Value Theorem can be found in Calculus textbooks. Here are few of them.

9.30. Prove that any polynomial of odd degree in one variable with real coefficients has at least one real root.

9.T Generalization. Let X be a connected space and $f : X \to \mathbb{R}$ a continuous function. Then $f(X)$ is a convex subset of $\mathbb{R}$.

Dividing Pancakes

9.31. Any irregularly shaped pancake can be cut in half by one stroke of the knife made in any prescribed direction. In other words, if A is a bounded open set in the plane and l is a line in the plane, then there exists a line L parallel to l which divides A in half by area.

9.32. If, under the conditions of *9.31*, A is connected then L is unique.

9.33. Suppose two irregularly shaped pancakes lie on the same platter; show that it is possible to cut both exactly in half by one stroke of the knife. In other words: if A and B are two bounded regions in the plane, then there exists a line in the plane which divides each region in half by area.

9.34 Dividing Pancake. Prove that a plane pancake of any shape can be divided to four pieces of equal area by two straight cuts orthogonal to each other. In other words, if A is a bounded connected open set in the plane, then there are two perpendicular lines which divide A into four parts having equal areas.

9.35. Riddle. What if the knife is not makes cuts of a shape different from straight line? For which shapes of the blade you can formulate and solve problems similar to *9.31 – 9.34*?

9.36. Riddle. Formulate and solve counter-parts of Problems *9.31 – 9.34* for regions in the three-dimensional space. Can you increase the number of regions in the counter-part of *9.31* and *9.33*?

9.37. Riddle. What about pancakes in $\mathbb{R}^n$?

Induction on Connectedness

A function is said to be *locally constant* if each point of its source space has a neighborhood such that the restriction of the function to this neighborhood is constant.

9.U. A locally constant function on a connected set is constant.

9.38. Riddle. How are *9.24* and *9.U* related?

9.39. Let G be a group equipped with a topology such that for any $g \in G$ the map $G \to G$ defined by $x \mapsto xgx^{-1}$ is continuous, and let G with this topology be connected. Prove that if the topology induced in a normal subgroup H of G is discrete, then H is contained in the center of G (i.e., $hg = gh$ for any $h \in H$ and $g \in G$).

9.40 Induction on Connectedness. Let $\mathcal{E}$ be a property of subsets of a topological space such that the union of sets with nonempty pairwise intersections inherits this property from the sets involved. Prove that if the space is connected and each its point has a neighborhood with property $\mathcal{E}$, then the space has property $\mathcal{E}$.

9.41. Prove *9.U* and solve *9.39* using *9.40.*

For more applications of induction on connectedness see *10.R*, *10.14*, *10.16* and *10.18.*

Applications to Homeomorphism Problem

Connectedness is a topological property, and the number of connected components is a topological invariant (see Section 8).

9.V. $[0,2]$ and $[0,1] \cup [2,3]$ are not homeomorphic.

Simple constructions, which assign homeomorphic spaces to homeomorphic ones (e.g. deleting one or several points), allow one to use connectedness for proving that some *connected* spaces are not homeomorphic.

9.W. I, $\mathbb{R}^1$, S^1 and $[0,\infty)$ are pairwise nonhomeomorphic.

9.42. Prove that a circle is not homeomorphic to any subspace of $\mathbb{R}^1$.

9.43. Give a topological classification of the letters: A, B, C, D, ... , considered as subsets of the plane (the arcs comprising the letters are assumed to have zero thickness).

9.44. Prove that square and segment are not homeomorphic.

Recall that there exist continuous surjections of the segment onto square and these maps are called *Peano curves*, see Section 7.

9.X. $\mathbb{R}^1$ and $\mathbb{R}^n$ are not homeomorphic if $n > 1$.

Information. $\mathbb{R}^p$ and $\mathbb{R}^q$ are not homeomorphic unless $p = q$. It follows, for instance, from the Lebesgue-Brower Theorem on invariance of dimension (see, e.g., W. Hurewicz and H. Wallman, *Dimension Theory* Princeton, NJ, 1941).

9.45. The statement "$\mathbb{R}^p$ is not homeomorphic to $\mathbb{R}^q$ unless $p = q$" implies that S^p is not homeomorphic to S^q unless $p = q$.

10. Path-Connectedness

Paths

A *path* in a topological space X is a continuous mapping of the interval $I = [0,1]$ to X. The point $s(0)$ is called the *initial* point of a path $s : I \to X$, while $s(1)$ is called its *final* point. One says that path s *connects* $s(0)$ with $s(1)$. This terminology is inspired by an image of moving point: at the moment $t \in [0,1]$ it is in $s(t)$. To tell the truth, this is more than what is usually called a path, since besides an information on trajectory of the point it contains a complete account on the movement: the schedule saying when the point goes through each point.

A constant map $s : I \to X$ is called a *stationary* path and denoted by e_a where $a = s(I)$. For a path s the *inverse* path is the path defined by $t \mapsto s(1-t)$. It is denoted by s^{-1}. Although, strictly speaking, this notation is already used (for the inverse mapping), the ambiguity of notations does not lead to confusion: in the context involving paths, inverse mappings, as a rule, do not appear.

Let $u : I \to X$, $v : I \to X$ be paths such that $u(1) = v(0)$. Set

$$uv(t) = \begin{cases} u(2t), & \text{if } t \in [0, \frac{1}{2}] \\ v(2t-1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \tag{20}$$

10.A. Prove that the map $uv : I \to X$ defined by (10) is continuous (i.e., it is a path). Cf. *7.T* and *7.V*.

Path uv is called the *product* of paths u and v. Recall that it is defined only if the final point $u(1)$ of u coincides with the initial point $v(0)$ of v.

Path-Connected Spaces

A topological space is said to be *path-connected* or *pathwise connected*, if any two points can be connected in it by a path.

10.B. Prove that I is pathwise connected.

10.C. Prove that Euclidean space $\mathbb{R}^n$ of any dimension is path-connected.

10.D. Prove that sphere of dimension $n > 0$ is path-connected.

10.E. Prove that the zero-dimensional sphere S^0 is not path-connected.

10.1. Which of the following topological spaces are path-connected:

(a) a discrete space;
(b) an indiscrete space;
(c) the arrow;
(d) $\mathbb{R}_{T_1}$;
(e) $4pT$?

Path-Connected Sets

By a *path-connected set* or *pathwise connected set* one calls a subset of a topological space (which should be clear from the context) path-connected as a space with the topology induced from the ambient space.

> ***10.2.*** Prove that a subset A of a topological space X is path-connected, iff any two points in it can be connected by a path $s : I \to X$ with $s(I) \subset A$.
>
> ***10.3.*** Prove that a convex subset of Euclidean space is path-connected.
>
> ***10.4.*** Prove that the set of plane convex polygons with topology defined by the Hausdorff metric is path-connected.

Path-connectedness is very similar to connectedness. Further, in some important situations it is even equivalent to connectedness. However, some properties of connectedness do not carry over path-connectedness (see *10.O*, *10.P*). For properties, which carry over, proofs are usually easier in the case of path-connectedness.

10.F. *The union of a family of pairwise intersecting path-connected sets is path-connected.*

> ***10.5.*** Prove that if sets A and B are both closed or both open and their union and intersection are path-connected, then A and B are also path-connected.
>
> ***10.6.*** Prove that interior and frontier of a path-connected set may not be path-connected and that connectedness shares this property.
>
> ***10.7.*** Let A be a subset of Euclidean space. Prove that if $\operatorname{Fr} A$ is connected then $\operatorname{Cl} A$ is also connected.
>
> ***10.8.*** Prove that the same holds true for a subset of an arbitrary path-connected space.

Path-Connected Components

A *path-connected component* or *pathwise connected component* of a topological space X is a path-connected subset of X such that no other path-connected subset of X contains it.

10.G. *Every point belongs to a path-connected component.*

10.H. *Two path-connected components either coincide or are disjoint.*

10.I. Prove that two points belong to the same path-connected component, iff they can be connected by a path.

Unlike to the case of connectedness, path-connected components may be non-closed. (See *10.O*, cf. *10.N*, *10.P*.)

10.J. *A continuous image of a pathwise connected space is pathwise connected.*

> ***10.9.*** Let $s : I \to X$ be a path connecting a point of a set A with a point of $X \smallsetminus A$. Prove that $s(I) \cap \operatorname{Fr}(A) \neq \varnothing$.

Path-Connectedness Versus Connectedness

10.K. *Any path-connected space is connected.*

Put
$$A = \left\{(x, y) \in \mathbb{R}^2 \ : \ x > 0, y = \sin\frac{1}{x}\right\}$$
and $X = A \cup \{(0, 0)\}$.

> ***10.10.*** Draw A.

10.L. Prove that A is path-connected and X is connected.

10.M. Prove that deleting any point from A makes A and X disconnected (and hence, not path-connected).

10.N. *X is not path-connected.*

10.O. Find an example of a path-connected set, whose closure is not path-connected.

10.P. Find an example of a path-connected component that is not closed.

10.Q. *If each point of a space has a path-connected neighborhood, then each path-connected component is open.*

10.R. *If each point of a space has a path-connected neighborhood, then the space is path-connected, iff it is connected.*

10.S. *For an open subset of Euclidean space connectedness is equivalent to path-connectedness.*

> ***10.11.*** For subsets of the real line path-connectedness and connectedness are equivalent.
>
> ***10.12.*** Prove that for any $\varepsilon > 0$ an ε-neighborhood of a connected subset of Euclidean space is path-connected.
>
> ***10.13.*** Prove that any neighborhood of a connected subset of Euclidean space contains a path-connected neighborhood of the same set.

Polygon-Connectedness

> A subset A of Euclidean space is said to be *polygon-connected* if any two points of A can be connected by a finite polygonal line contained in A.
>
> ***10.14.*** Prove that for open subsets of Euclidean space connectedness is equivalent to polygon-connectedness.
>
> ***10.15.*** Construct a path-connected subset A of Euclidean space such that A consists of more than one point and no two distinct points can be connected with a polygon in A.
>
> ***10.16.*** Let $X \subset \mathbb{R}^2$ be a countable set. Prove that then $\mathbb{R}^2 \smallsetminus X$ is polygon-connected.

10.17. Let $X \subset \mathbb{R}^n$ be a union of a countable collection of affine subspaces with dimensions not greater than $n-2$. Prove that then $\mathbb{R}^n \setminus X$ is polygon-connected.

10.18. Let $X \subset \mathbb{C}^n$ be a union of a countable collection of algebraic subsets (i.e., subsets defined by systems of algebraic equations in the standard coordinates of $\mathbb{C}^n$) Prove that then $\mathbb{C}^n \setminus X$ is polygon-connected.

Recall, that real $n \times n$-matrices comprise a space, which differs from $\mathbb{R}^{n^2}$ only in the way of enumeration of its natural coordinates (they are numerated by pairs of indices). The same relation holds between the set of complex $n \times n$-matrix and $\mathbb{C}^{n^2}$ (homeomorphic to $\mathbb{R}^{2n^2}$).

10.19. Find connected and path-connected components of the following subspaces of the space of real $n \times n$-matrices:

- $GL(n; \mathbb{R}) = \{A : \det A \neq 0\}$;
- $O(n; \mathbb{R}) = \{A : A \cdot ({}^t A) = \mathbb{1}\}$;
- $Symm(n; \mathbb{R}) = \{A : {}^t A = A\}$;
- $Symm(n; \mathbb{R}) \cap GL(n; \mathbb{R})$;
- $\{A : A^2 = \mathbb{1}\}$.

10.20. Find connected and path-connected components of the following subspaces of the space of complex $n \times n$-matrices:

- $GL(n; \mathbb{C}) = \{A : \det A \neq 0\}$;
- $U(n; \mathbb{C}) = \{A : A \cdot ({}^t \bar{A}) = \mathbb{1}\}$;
- $Herm(n; \mathbb{C}) = \{A : {}^t A = \bar{A}\}$;
- $Herm(n; \mathbb{C}) \cap GL(n; \mathbb{C})$.

11. Separation Axioms

The aim of this section is to consider natural restrictions on topological structure making the structure closer to being metrizable.

Hausdorff Axiom

A lot of separation axioms are known. We restrict ourselves to the most important four of them. They are numerated and denoted by T_1, T_2, T_3, and T_4 respectively. Let us start with the most important second axiom. Besides the notation T_2 it has a name, the *Hausdorff axiom.* A topological space satisfying it is called a *Hausdorff space.* This axiom is stated as follows: *any two distinct points possess disjoint neighborhoods.*

11.A. *Any metric space is Hausdorff.*

> ***11.1.*** Which of the following spaces are Hausdorff:
> (a) a discrete space;
> (b) an indiscrete space;
> (c) the arrow;
> (d) $\mathbb{R}_{T_1}$;
> (e) $4pT$?

If the next problem holds you up even for a minute, we advise you to think over all definitions and solve all simple problems.

11.B. Is the segment $[0, 1]$ with the topology induced from $\mathbb{R}$ a Hausdorff space? Do the points 0 and 1 possess disjoint neighborhoods? Which if any?

Limits of Sequence

Let $\{a_n\}$ be a sequence of points of a topological space X. A point $b \in X$ is called its *limit,* if for any neighborhood U of b there exists a number N such that $a_n \in U$ for any $n > N$. The sequence is said to *converge* or *tend* to b as n tends to infinity.

> ***11.2.*** Explain the meaning of the statement " b is not a limit of sequence a_n" avoiding as much as you can negations (i.e., the words no, not, none, etc..)

11.C. *In a Hausdorff space any sequence has at most one limit.*

11.D. Prove that in the space $\mathbb{R}_{T_1}$ each point is a limit of the sequence $\{a_n = n\}$.

Coincidence Set and Fixed Point Set

> Let $f, g : X \to Y$ be maps. Then the set $\{x \in X : f(x) = g(x)\}$ is called the *coincidence set* of f and g.
>
> ***11.3.*** Prove that the coincidence set for two continuous maps of an arbitrary topological space to a Hausdorff space is closed.

11.4. Construct an example proving that the Hausdorff condition in *11.3* is essential.

A point $x \in X$ is called a *fixed point* of a map $f : X \to X$ if $f(x) = x$. The set of all fixed points of a map f is called the *fixed point set* of f.

11.5. Prove that the fixed point set of a continuous map of a Hausdorff space to itself is closed.

11.6. Construct an example proving that the Hausdorff condition in *11.5* is essential.

11.7. Prove that if $f, g : X \to Y$ are continuous maps, Y is Hausdorff, A is everywhere dense in X, and $f|_A = g|_A$ then $f = g$.

11.8. Riddle. How are problems *11.3*, *11.5*, and *11.7* related?

Hereditary Properties

A topological property is called *hereditary* if it is carried over from a space to its subspaces, i.e. if a space X possesses this property then any subspace of X possesses it.

11.9. Which of the following topological properties are hereditary:

- finiteness of the set of points;
- finiteness of the topological structure;
- infiniteness of the set of points;
- connectedness;
- path-connectedness?

11.E. *The property of being Hausdorff space is hereditary.*

The First Separation Axiom

A topological space is said to satisfy the *first separation axiom* T_1 if each of any two points of the space has a neighborhood which does not contain the other point.

11.F. *A topological space X satisfies the first separation axiom,*

- *iff all one-point sets in X are closed,*
- *iff all finite sets in X are closed.*

11.10. Prove that a space X satisfies the first separation axiom, iff any point of X coincides with the intersection of all its neighborhoods.

11.11. Any Hausdorff space satisfies the first separation axiom.

11.G. In a Hausdorff space any finite set is closed.

11.H. A metric space satisfies the first separation axiom.

11.12. Find an example showing that the first separation axiom does not imply the Hausdorff axiom.

11.I. Show that $\mathbb{R}_{T_1}$ meets the first separation axiom, but is not a Hausdorff space (cf. *11.12*).

11.J. *The first separation axiom is hereditary.*

> ***11.13.*** Prove that if for any two distinct points a and b of a topological space X there exists a continuous map f of X to a space with the first separation axiom such that $f(a) \neq f(b)$ then X possesses the first separation axiom.
>
> ***11.14.*** Prove that a continuous mapping of an indiscrete space to a space satisfying axiom T_1 is constant.
>
> ***11.15.*** Prove that in every set there exists a coarsest topological structure satisfying the first separation axiom. Describe this structure.

The Third Separation Axiom

A topological space X is said to satisfy the *third separation axiom* if any closed set and a point of its complement have disjoint neighborhoods, i.e., for any closed set $F \subset X$ and point $b \in X \setminus F$ there exist open sets $U, V \subset X$ such that $U \cap V = \varnothing$, $F \subset U$, and $b \in V$.

A topological space is called *regular* if it satisfies the first and third separation axioms.

11.K. A regular space is Hausdorff space.

11.L. A space is regular, iff it satisfies the second and third separation axioms.

> ***11.16.*** Find a Hausdorff space which is not regular.
>
> ***11.17.*** Find a space satisfying the third, but not the second separation axiom.
>
> ***11.18.*** Prove that a space satisfies the third separation axiom, iff any neighborhood of any point contains the closure of some neighborhood of the same point.
>
> ***11.19.*** Prove that the third separation axiom is hereditary.

11.M. *Any metric space is regular.*

The Fourth Separation Axiom

A topological space X is said to satisfy the *fourth separation axiom* if any two disjoint closed sets have disjoint neighborhoods, i.e., for any closed sets $A, B \subset X$ such that $A \cap B = \varnothing$ there exist open sets $U, V \subset X$ such that $U \cap V = \varnothing$, $A \subset U$, and $B \subset V$.

A topological space is called *normal* if it satisfies the first and fourth separation axioms.

11.N. A normal space is regular (and hence Hausdorff).

11.O. A space is normal, iff it satisfies the second and fourth separation axioms.

> ***11.20.*** Find a space which satisfies the fourth, but not second separation axiom.

11.21. Prove that a space satisfies the fourth separation axiom, iff in any neighborhood of any closed set contains the closure of some neighborhood of the same set.

11.22. Prove that any closed subspace of a normal space is normal.

11.23. Find closed disjoint subsets A and B of some metric space such that $\inf\{\rho(a,b) \mid a \in A, b \in B\} = 0$.

11.P. *Any metric space is normal.*

11.24. Let $f : X \to Y$ be a continuous surjection such that the image of any closed set is closed. Prove that if X is normal then Y is normal.

Niemytski's Space

Denote by H the open upper half-plane $\{(x,y) \in \mathbb{R}^2 \ : \ y > 0\}$ equipped with the topology induced by the Euclidean metric. Denote by X the union of H and its boundary line $L = \{(x,y) \in \mathbb{R}^2 \ : \ y = 0\}$, but equip it with the topology, which is obtained by adjoining to the Euclidean topology the sets of the form $x \cup D$, where $x \in \mathbb{R}^1$ and D is an open disc in H which is tangent to L at the point x. This is the *Niemytski space.* It can be used to clarify properties of the fourth separation axiom.

11.25. Prove that the Niemytski space is Hausdorff.

11.26. Prove that the Niemytski space is regular.

11.27. What topological structure is induced on L from X?

11.28. Prove that the Niemytski space is not normal.

11.29 Corollary. There exists a regular space, which is not normal.

11.30. Embed the Niemytski space into a normal space in such a way that the complement of the image would be a single point.

11.31 Corollary. Theorem *11.22* does not extend to non-closed subspaces, i.e., the property of being normal is not hereditary?

Urysohn Lemma and Tietze Theorem

11:A*. Let Y be a topological space satisfying the first separation axiom. Let T be a subbase[1] of the topology of Y. Let Σ be an open cover of a space X. Prove that if there exists a bijection $\phi : \Sigma \to T$ which preserves inclusions then there exists a continuous map $f : X \to Y$ such that $f^{-1}(V) = \phi^{-1}(V)$ for any $V \in T$.

11:B. Prove that intervals $[0,r)$ and $(r,1]$ where $r = \frac{n}{2^q}$, $n, q \in \mathbb{N}$ form a subbase for $[0,1]$, i.e., a collection of open sets in $[0,1]$, whose finite intersections form a base of the standard topology in $[0,1]$.

11:C Urysohn Lemma. *Let A and B be disjoint closed subsets of a normal space X. Then there exists a continuous function $f : X \to I$ such that $f(A) = 0$ and $f(B) = 1$.*

[1]Recall that a subbase of the topology of Y is a collection T of open sets of Y such that all finite intersections of sets from T form a base of topology of Y, see Section 2.

11:D. Let A be a closed subset of a normal space X. Let $f : A \to [-1, 1]$ be a continuous function. Prove that there exists a continuous function $g : X \to \left[-\frac{1}{3}, \frac{1}{3}\right]$ such that $|f(x) - g(x)| \leq \frac{2}{3}$ for any $x \in A$.

11:E. Prove that under the conditions of *11:D* for any $\varepsilon > 0$ there exists a continuous function $\phi : X \to [-1, 1]$ such that $|f(x) - \phi(x)| \leq \varepsilon$ for any $x \in A$.

11:F Tietze Extension Theorem. *Prove that under the conditions of 11:D there exists a continuous function* $F : X \to [-1, 1]$ *such that* $F|_A = f$.

11:G. Would the statement of Tietze Theorem remain true if in the hypothesis the segment $[-1, 1]$ was replaced by $\mathbb{R}$, $\mathbb{R}^n$, S^1, or S^2?

12. Countability Axioms

In this section we continue to study topological properties which are imposed additionally on a topological structure to make the abstract situation under consideration closer to special situations and hence richer in contents. Restrictions studied in this section bound a topological structure from above: they require something to be countable.

Set-Theoretic Digression. Countability

Recall that two sets are said to be of equal *cardinality* if there exists a bijection of one of them onto the other. A set of the same cardinality as a subset of the set $\mathbb{N}$ of natural numbers is said to be *countable.* Sometimes this term is used only for infinite countable sets, i.e. for set of the cardinality of the whole set $\mathbb{N}$ of natural numbers, while a set countable in the sense above is called *at most countable.* This is less convenient. In particular, if we adopted this terminology, then this section would have to be called "At Most Countability Axioms". This would lead to other more serious inconveniences as well. Our terminology has the following advantageous properties.

12.A. Any subset of a countable set is countable.

12.B. The image of a countable set under any mapping is countable.

12.C. The union of a countable family of countable sets is countable.

Second Countability and Separability

In this section we study three restrictions on topological structure. Two of them have numbers (one and two), the third one has no number. As in the previous section, we start from the restriction having number two.

A topological space is said to satisfy the *second axiom of countability* or to be *second countable* if it has a countable base. A space is called *separable* if it contains a countable dense set. (This is the countability axiom without a number mentioned above.)

12.D. *The second axiom of countability implies separability.*

12.E. The second axiom of countability is hereditary.

> ***12.1.*** Are the arrow and $\mathbb{R}_{T_1}$ second countable?
>
> ***12.2.*** Are the arrow and $\mathbb{R}_{T_1}$ separable?
>
> ***12.3.*** Construct an example proving that separability is not hereditary.

12.F. *A metric separable space is second countable.*

12.G Corollary. *For metric spaces, separability is equivalent to the second axiom of countability.*

12.H. (Cf. *12.3.*) Prove that for metric spaces separability is hereditary.

12.I. Prove that Euclidean spaces and all their subspaces are separable and second countable.

12.4. Construct a metric space which is not second countable.

12.J. *A continuous image of a separable space is separable.*

12.5. Construct an example proving that a continuous image of a second countable space may be not second countable.

12.K Lindelöf Theorem. *Any open cover of a second countable space contains a countable part, which also covers the space.*

12.6. Prove that any base of a second countable space contains a countable part which is also a base.

12.7. Prove that in a separable space any collection of pairwise disjoint open sets is countable.

12.8. Prove that the set of components of an open set $A \subset \mathbb{R}^n$ is countable.

12.9. Prove that any set of disjoint figure eight curves in the plane is countable.

12.10 Brower Theorem*. Let $\{K_\lambda\}$ be a family of closed sets of a second countable space and let for any descending sequence $K_1 \supset K_2 \supset \dots$ of sets belonging to this family the intersection $\cap_1^\infty K_n$ also belongs to the family. Then the family contains a minimal set, i.e., a set such that no proper its subset belongs to the family.

Embedding and Metrization Theorems

12:A. Prove that the space l_2 is separable and second countable.

12:B. Prove that a regular second countable space is normal.

12:C. Prove that a normal second countable space can be embedded into l_2. (Use Urysohn Lemma *11:C*.)

12:D. Prove that a second countable space is metrizable, iff it is regular.

Bases at a Point

Let X be a topological space, and a its point. A *neighborhood base* at a or just *base of* X *at* a is a collection of neighborhoods of a such that any neighborhood of a contains a neighborhood from this collection.

12.L. If Σ is a base of a space X then $\{U \in \Sigma : a \in U\}$ is a base of X at a.

12.11. In a metric space the following collections of balls are neighborhood bases at a point a:

- the set of all open balls of center a;
- the set of all open balls of center a and rational radii;
- the set of all open balls of center a and radii r_n, where $\{r_n\}$ is any sequence of positive numbers converging to zero.

12.12. What are the minimal bases at a point in the discrete and indiscrete spaces?

First Countability

A topological space X is says to satisfy the *first axiom of countability* or to be a *first countable space* if it has a countable neighborhood base at each point.

12.M. *Any metric space is first countable.*

12.N. *The second axiom of countability implies the first one.*

12.O. Find a first countable space which is not second countable. (Cf. *12.4*.)

> ***12.13.*** Which of the following spaces are first countable:
>
> (a) the arrow;
> (b) $\mathbb{R}_{T_1}$;
> (c) a discrete space;
> (d) an indiscrete space?

Sequential Approach to Topology

Specialists in mathematical analysis love sequences and their limits. Moreover they like to talk about all topological notions relying on the notions of sequence and its limit. This tradition has almost no mathematical justification, except for a long history descending from the XIX century studies on the foundations of analysis. In fact, almost always[2] it is more convenient to avoid sequences, provided you deal with topological notions, except summing of series, where sequences are involved in the underlying definitions. Paying a tribute to this tradition we explain here how and in what situations topological notions can be described in terms of sequences.

Let A be a subset of a topological space X. The set of limits of all sequences a_n with $a_n \in A$ is called a *sequential closure* of A and denoted by $\operatorname{SCl} A$.

12.P. Prove that $\operatorname{SCl} A \subset \operatorname{Cl} A$.

12.Q. *If a space X is first countable then the for any $A \subset X$ the opposite inclusion* $\operatorname{Cl} A \subset \operatorname{SCl} A$ *holds also true, and hence* $\operatorname{SCl} A = \operatorname{Cl} A$.

Therefore, in a second countable space (in particular, any metric spaces) one can recover (hence, define) the closure of a set provided it is known which sequences are convergent and what the limits are. In turn, knowledge of closures allows one to recover which sets are closed. As a consequence, knowledge of closed sets allows one to recover open sets and all other topological notions.

> ***12.14.*** Let X be the set of real numbers equipped with the topology consisting of $\varnothing$ and complements of all countable subsets. Describe convergent sequences, sequential closure and closure in X. Prove that in X there exists a set A with $\operatorname{SCl} A \neq \operatorname{Cl} A$.

[2]The exceptions which one may find in the standard curriculum of a mathematical department can be counted on two hands.

Sequential Continuity

Consider now continuity of maps along the same lines. A map $f : X \to Y$ is said to be *sequentially continuous* if for any $b \in X$ and a sequence $a_n \in X$, which converges to b, the sequence $f(a_n)$ converges to $f(b)$.

12.R. *Any continuous map is sequentially continuous.*

12.S. *The preimage of a sequentially closed set under a sequentially continuous map is sequentially closed.*

12.T. *If X is a first countable space then any sequentially continuous map $f : X \to Y$ is continuous.*

Thus for mappings of a first countable space continuity and sequential continuity are equivalent.

> ***12.15.*** Construct a sequentially continuous, but discontinuous map. (Cf. *12.14*)

13. Compactness

Definition of Compactness

This section is devoted to a topological property, which plays a very special role in topology and its applications. It is sort of topological counter-part for the property of being finite in the context of set theory. (It seems though that this analogy has never been formalized.)

Topological space is said to be *compact* if any of its open covers contains a finite part which covers the space.

If Γ is a cover of X and $\Sigma \subset \Gamma$ is a cover of X then GS is called a *subcover* (or *subcovering*) of Γ. Thus, a topological space is compact if every open cover has a finite subcover.

13.A. Any finite topological space and indiscrete space are compact.

13.B. Which discrete topological spaces are compact?

> ***13.1.*** Let $\Omega_1 \subset \Omega_2$ be topological structures in X. Does compactness of (X, Ω_2) imply compactness of (X, Ω_1)? And vice versa?

13.C. Prove that the line $\mathbb{R}$ is not compact.

13.D. Prove that a topological space X is not compact iff there exists an open covering which contains no finite subcovering.

> ***13.2.*** Is the arrow compact? Is $\mathbb{R}_{T_1}$ compact?

Terminology Remarks

Originally the word compactness was used for the following weaker property: any countable open cover contains a finite subcover.

13.E. Prove that for a second countable space the original definition of compactness is equivalent to the modern one.

The modern notion of compactness was introduced by P. S. Alexandroff (1896–1982) and P. S. Urysohn (1898–1924). They suggested for it the term *bicompactness.* This notion appeared to be so successful that it has displaced the original one and even took its name, i.e. compactness. The term bicompactness is sometimes used (mainly by topologists of Alexandroff school).

Another deviation from the terminology used here comes from Bourbaki: we do not include the Hausdorff property into the definition of compactness, which Bourbaki includes. According to our definition, $\mathbb{R}_{T_1}$ is compact, according to Bourbaki it is not.

Compactness in Terms of Closed Sets

A collection of subsets of a set is said to be *centered* if the intersection of any finite subcollection is not empty.

13.F. A collection Σ of subsets of a set X is centered, iff there exists no finite $\Sigma_1 \subset \Sigma$ such that the complements of the sets belonging to Σ_1 cover X.

13.G. *A topological space is compact, iff any centered collection of its closed sets has nonempty intersection.*

Compact Sets

By a *compact set* one means a subset of a topological space (the latter must be clear from the context) provided it is compact as a space with the topology induced from the ambient space.

13.H. *A subset A of a topological space X is compact, iff any cover which consists of sets open in X contains a finite subcover.*

> ***13.3.*** Is $[1, 2) \subset \mathbb{R}$ compact?
>
> ***13.4.*** Is the same set $[1, 2)$ compact in the arrow?
>
> ***13.5.*** Find a necessary and sufficient condition (formulated not in topological terms) for a subset of the arrow to be compact?
>
> ***13.6.*** Prove that any subset of $\mathbb{R}_{T_1}$ is compact.
>
> ***13.7.*** Let A and B be compact subsets of a topological space X. Does it follow that $A \cup B$ is compact? Does it follow that $A \cap B$ is compact?
>
> ***13.8.*** Prove that the set $A = \{0\} \cup \{\frac{1}{n}\}_{n=1}^{\infty}$ in $\mathbb{R}$ is compact.

Compact Sets Versus Closed Sets

13.I. Is compactness hereditary?

13.J. *Any closed subset of a compact space is compact.*

13.K. *Any compact subset of a Hausdorff space is closed.*

13.L Lemma to 13.K, but not only ... . *Let A be a compact subset of a Hausdorff space X and b a point of X which does not belong to A. Then there exists open sets $U, V \subset X$ such that $b \in V$, $A \subset U$ and $U \cap V = \varnothing$.*

> ***13.9.*** Construct a nonclosed compact subset of some topological space. What is the minimal number of points needed?

Compactness and Separation Axioms

13.M. *A compact Hausdorff space is regular.*

13.N. Prove that a compact Hausdorff space is normal.

> ***13.10.*** Prove that the intersection of any family of compact subsets of a Hausdorff space is compact. (Cf. *13.7.*)

13.11. Let X be a Hausdorff space, let $\{K_\alpha\}_{\alpha\in\Lambda}$ be a family of its compact subsets, and let U be an open set containing $\cap_{\alpha\in\Lambda}K_\alpha$. Prove that $U \supset \cap_{\alpha\in A}K_\alpha$ for some finite $A \subset \Lambda$.

13.12. Let $\{K_n\}$ be a decreasing sequence of compact nonempty connected subset of a Hausdorff space. Prove that the intersection $\cap_{n=1}^{\infty}K_n$ is nonempty and connected.

13.13. Construct a decreasing sequence of connected subsets of the plane with nonconnected intersection.

13.14. Let K be a connected component of a compact Hausdorff space X and let U be an open set containing K. Prove that there exists an open and closed set V such that $K \subset V \subset U$.

Compactness in Euclidean Space

13.O. *The interval I is compact.*

Recall that n-dimensional cube is the set

$$I^n = \{x \in \mathbb{R}^n \,:\, x_i \in [0,1] \text{ for } i = 1, \dots, n\}.$$

13.P. *The cube I^n is compact.*

13.Q. *Any compact subset of a metric space is bounded.*

Therefore, any compact subset of a metric space is closed and bounded, see *13.K* and *13.Q*.

13.R. Construct a closed and bounded, but noncompact set of a metric space.

13.15. Are the metric spaces of Problem *3.A* compact?

13.S. *A subset of Euclidean space is compact, iff it is closed and bounded.*

13.16. Which of the following sets are compact:

(a) $[0,1)$;
(b) ray $\mathbb{R}_+ = \{x \in R \,:\, x \geq 0\}$;
(c) S^1;
(d) S^n;
(e) one-sheeted hyperboloid;
(f) ellipsoid;
(g) $[0,1] \cap \mathbb{Q}$?

Matrix (a_{ij}) with $1 \leq i \leq n$, $1 \leq j \leq k$ with real $a_i j$ can be considered as a point of $\mathbb{R}^{nk}$. For this, one needs to enumerate somehow (e.g, lexicographically) its elements by numbers from 1 till nk. This identifies the set $L(nk)$ of all matrices like that with $\mathbb{R}^{nk}$ and endows it with a topological structure. (Cf. Section 10.)

13.17. Which of the following subsets of $L(n,n)$ are compact:

(a) $GL(n) = \{A \in L(n,n) \,:\, \det A \neq 0\}$;
(b) $SL(n) = \{A \in L(n,n) \,:\, \det A = \mathbb{1}\}$;
(c) $O(n) = \{A \in L(n,n) \,:\, A \text{ is an orthogonal matrix}\}$;
(d) $\{A \in L(n,n) \,:\, A^2 = \mathbb{1}\}$, here $\mathbb{1}$ is the unit matrix?

Compactness and Maps

13.T. *A continuous image of a compact set is compact. (In other words, if X is a compact space and $f : X \to Y$ is a continuous map then $f(X)$ is compact.)*

13.U. *On a compact set any continuous function is bounded and attains its maximal and minimal values. (In other words, if X is a compact space and $f : X \to \mathbb{R}$ is a continuous function, then there exist $a, b \in X$ such that $f(a) \leq f(x) \leq f(b)$ for any $x \in X$.) Cf. 13.T and 13.S.*

13.18. Prove that if $f : I \to \mathbb{R}$ is a continuous function then $f(I)$ is an interval.

13.19. Prove that if F and K are disjoint subsets of a metric space, F is closed and G compact then $\rho(F, K) > 0$.

13.20. Prove that any open set containing a compact set A of a metric space X contains an *ε-neighborhood* of A. (i.e., the set $\{x \in X : \rho(x, A) < \varepsilon\}$ for some $\varepsilon > 0$).

13.21. Let A be a closed connected subset of $\mathbb{R}^n$ and let V be its *closed ε-neighborhood* (i.e., $V = \{x \in \mathbb{R}^n : \rho(x, A) < \varepsilon\}$). Prove that V is path-connected.

13.22. Prove that if in a compact metric space the closure of any open ball is the closed ball with the same center and radius then any ball of this space is connected.

13.23. Let X be a compact metric space and $f : X \to X$ be a map such that $\rho(f(x), f(y)) < \rho(x, y)$ for any $x, y \in X$ with $x \neq y$. Prove that f has a unique fixed point. (Recall that a fixed point of f is a point x such that $f(x) = x$.)

13.24. Prove that for any open cover of a compact metric space there exists a number $r > 0$ such that any open ball of radius r is contained in some element of the cover.

13.V Lebesgue Lemma. *Let $f : X \to Y$ be a continuous map of a compact metric space X to a topological space Y, and let Γ be an open cover of Y. Then there exists a number $\delta > 0$ such that for any set $A \subset X$ with diameter $\operatorname{diam}(A) < \delta$ the image $f(A)$ is contained in some element of Γ.*

Norms in $\mathbb{R}^n$

13.25. Prove that any norm $\mathbb{R}^n \to \mathbb{R}$ (see Section 3) is a continuous function (with respect to the standard topology of $\mathbb{R}^n$).

13.26. Prove that any two norms in $\mathbb{R}^n$ are equivalent (i.e. define the same topological structure). See *3.26*, cf. *3.29*.

13.27. Does the same hold true for metrics in $\mathbb{R}^n$?

Closed Maps

A continuous map is said to be *closed* if the image of any closed set under this map is closed.

13.W. *A continuous map of a compact space to a Hausdorff space is closed.*

Here are two important corollaries of this theorem.

13.X. *A continuous injection of a compact space to a Hausdorff space is a topological embedding.*

13.Y. *A continuous bijection of a compact space to a Hausdorff space is a homeomorphism.*

> ***13.28.*** Show that none of the hypothesis in *13.Y* can be omitted without making the statement false.
>
> ***13.29.*** Does there exist a noncompact subspace of Euclidian space such that any its map to a Hausdorff space is closed? (Cf. *13.U* and *13.W*.)

14. Local Compactness and Paracompactness

Local Compactness

A topological space X is called *locally compact* if each of its points has a neighborhood with compact closure.

14:A. Prove that local compactness is a local property, i.e., a space is locally compact, iff each of its points has a locally compact neighborhood.

14:B. Is local compactness hereditary?

14:C. Prove that a closed subset of a locally compact space is locally compact.

14:D. Prove that an open subset of a locally compact Hausdorff space is locally compact.

14:1. Which of the following spaces are locally compact:

(a) $\mathbb{R}$;
(b) $\mathbb{Q}$;
(c) $\mathbb{R}^n$;
(d) a discrete space?

14:2. Find two locally compact sets on the line such that their union is not locally compact.

One-Point Compactification

Let X be a Hausdorff topological space. Let X^* be the set obtained by adding a point to X (of course, the point does not belong to X). Let Ω^* be the collection of subsets of X^* consisting of

- sets open in X and
- sets of the form $X^* \smallsetminus C$, where $C \subset X$ is a compact set.

14:E. Prove that Ω^* is a topological structure.

14:F. Prove that the space (X^*, Ω^*) is compact.

14:G. Prove that the inclusion $X \hookrightarrow X^*$ is a topological embedding (with respect to the original topology of X and Ω^*).

14:H. Prove that if X is locally compact then the space (X^*, Ω^*) is Hausdorff. (Recall that X is assumed to be Hausdorff.)

A topological embedding of a space X into a compact space Y is called a *compactification* of X if the image of X is dense in Y. In this situation Y is also called a *compactification* of X.

14:I. Prove that if X is a locally compact Hausdorff space and Y is its compactification with $Y \smallsetminus X$ consisting of a single point then there exists a homeomorphism $Y \to X^*$ which is the identity on X.

The space Y of Problem *14:I* is called a *one-point compactification* or *Alexandroff compactification* of X.

14:J. Prove that the one-point compactification of the plane is homeomorphic to S^2.

> ***14:3.*** Prove that the one-point compactification of $\mathbb{R}^n$ is homeomorphic to S^n.
>
> ***14:4.*** Give explicit descriptions of one-point compactifications of the following spaces:
>
> (a) annulus $\{(x,y) \in R^2 : 1 < x^2 + y^2 < 2\}$;
> (b) square without vertices $\{(x,y) \in R^2 : x,y \in [-1,1], |xy| < 1\}$;
> (c) strip $\{(x,y) \in R^2 : x \in [0,1]\}$;
> (d) a compact space.

14:K. Prove that a locally compact Hausdorff space is regular.

Proper Maps

A continuous map $f : X \to Y$ is said to be *proper* if the preimage of any compact subset of Y is compact.

Let X, Y be Hausdorff spaces. Any continuous map $f : X \to Y$ is naturally extended to a map $X^* \to Y^*$ defined by the following formula:

$$f^*(x) = \begin{cases} f(x), & \text{if } x \in X \\ Y^* \smallsetminus Y, & \text{otherwise, i.e., if } x = X^* \smallsetminus X. \end{cases}$$

14:L. Prove that f^* is continuous, iff f is proper.

14:M. Prove that any proper map of a Hausdorff space to a Hausdorff locally compact space is closed.

Problem *14:M* is related to Theorem *13.W*.

14:N. Extend this analogy: formulate and prove statements corresponding to theorems *13.X* and *13.Y*.

Locally Finite Collections of Subsets

A collection Γ of subsets of a space X is said to be *locally finite* if each point $b \in X$ has a neighborhood U such that $A \cap U = \emptyset$ for all but finite number of $A \in \Gamma$.

14:O. Any locally finite cover of a compact space is finite.

> ***14:5.*** If a collection Γ of subsets of a space X is locally finite then so is $\{\operatorname{Cl} A \mid A \in \Gamma\}$.
>
> ***14:6.*** If a collection Γ of subsets of a space X is locally finite and $\operatorname{Cl} A$ is compact for each $A \in \Gamma$ then each $A \in \Gamma$ intersects only finite number of elements of Γ.
>
> ***14:7.*** Any locally finite cover of a sequentially compact space is finite.

14:P. Find an example of an open cover of $\mathbb{R}^n$ which does not possess a locally finite subcover.

Let Γ and Δ be covers of a set X. Then Δ is said to be a *refinement* of Γ if for each $A \in \Gamma$ there exists $B \in \Delta$ such that $B \subset A$.

14:Q. Prove that any open cover of $\mathbb{R}^n$ has a locally finite open refinement.

14:R. Let $\{U_i\}_{i\in\mathbb{N}}$ be a locally finite open cover of $\mathbb{R}^n$. Prove that there exist an open cover $\{V_i\}_{i\in\mathbb{N}}$ such that $\operatorname{Cl} V_i \subset U_i$ for each $i \in \mathbb{N}$.

Paracompact Spaces

A space X is said to be *paracompact* if any its open cover has a locally finite open refinement.

14:S. Any compact space is paracompact.

14:T. $\mathbb{R}^n$ is paracompact.

14:U. Let $X = \cup_{i=1}^{\infty} X_i$ and X_i are compact sets. Then X is paracompact.

14:V. Any closed subspace of a paracompact space is paracompact.

14:8. A disjoint union of paracompact spaces is paracompact.

14:9. If X is a paracompact space and Y compact then $X \times Y$ is paracompact.

Paracompactness and Separation Axioms

14:10. Any Hausdorff paracompact space is regular.

14:11. Any Hausdorff paracompact space is normal.

14:12. Let X be a normal space and Γ its locally finite open cover. Then there exists a locally finite open cover Δ such that $\{\operatorname{Cl} V \mid V \in \Delta\}$ is a refinement of Γ.

Information. *Any metrizable space is paracompact.*

Partitions of Unity

For a function $f : X \to \mathbb{R}$, the set $\operatorname{Cl}\{x \in X \mid f(x) \neq 0\}$ is called the *support* of f and denoted by $\operatorname{supp} f$.

14:W. Let $\{f_\alpha\}_{\alpha\in\Lambda}$ be a family of continuous functions $X \to \mathbb{R}$ such that the sets $\operatorname{supp}(f_\alpha)$ comprise a locally finite cover of the space X. Prove that the relation

$$f(x) = \sum_{\alpha\in\Lambda} f_\alpha(x)$$

defines a continuous function $f : X \to \mathbb{R}$.

A family of nonnegative functions $f_\alpha X \to \mathbb{R}_+$ is called a partition of unity if the sets $\operatorname{supp}(f_\alpha)$ comprise a locally finite cover of the space X and $\sum_{\alpha\in\Lambda} f_\alpha(x) = 1$.

A partition of unity $\{f_\alpha\}$ is said to be *subordinate to a cover* Γ if each $\operatorname{supp}(f_\alpha)$ is contained in an element of Γ.

14:X. For every normal space X there exists a partition of unity which is subordinate to a given locally finite open cover of X.

14:Y. A Hausdorff space is paracompact, iff any its open cover admits a partition of unity which is subordinate to this cover.

Application: Making Embeddings from Pieces

14:Z. Let $h_i U_i \to \mathbb{R}^n$, $i = 1, \dots, k$, be embeddings, where U_i comprise an open cover of a space X. Then X can be embedded in $\mathbb{R}^{k(n+1)}$.

> *14:Z:1.* Show that the map $x \mapsto (f_i(x)\hat{h}_i(x))$, where $f_i X \to \mathbb{R}$ comrise a partition of unity, which is subordinate to the given cover and $\hat{h}_i(x) = (h_i(x), 1) \in \mathbb{R}^{n+1}$, is an embedding.

15. Sequential Compactness

Sequential Compactness Versus Compactness

A topological space is said to be *sequentially compact* if every sequence of its points contains a convergent subsequence.

15.A. *Any compact first countable space is sequentially compact.*

A point b is called an *accumulation point* of a set A if every neighborhood of b contains infinitely many points of A.

15.A.1. Prove that in a first countable space the notions of accumulation point and limit point coincide.

15.A.2. In a compact space any infinite set has an accumulation point.

15.A.3. Deduce Theorem *15.A* from *15.A.2*.

15.B. *A sequentially compact second countable space is compact.*

15.B.1. In a sequentially compact space a decreasing sequence of nonempty closed sets has a nonempty intersection.

15.B.2. Prove that in a topological space any decreasing sequence of nonempty closed sets has nonempty intersection, iff any centered countable collection of closed sets has nonempty intersection.

15.C. *For second countable spaces compactness and sequential compactness are equivalent.*

In Metric Space

A subset A of a metric space X is called an ε-net (where ε is a positive number) if $\rho(x, A) < \varepsilon$ for each point $x \in X$.

15.D. Prove that in any compact metric space for any $\varepsilon > 0$ there exists a finite ε-net.

15.E. Prove that in any sequentially compact metric space for any $\varepsilon > 0$ there exists a finite ε-net.

15.F. Prove that a subset of a metric space is everywhere dense, iff it is an ε-net for any $\varepsilon > 0$.

15.G. *Any sequentially compact metric space is separable.*

15.H. Any sequentially compact metric space is second countable.

15.I. *For metric spaces compactness and sequential compactness are equivalent.*

15.1. Prove that a sequentially compact metric space is bounded. (Cf. *15.E* and *15.I.*)

15.2. Prove that in any metric space for any $\varepsilon > 0$ there exists

(a) a discrete ε-net and even
(b) an ε-net such that the distance between any two of its points is greater than ε.

Completeness and Compactness

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of points of a metric space is called a *Cauchy sequence* if for any $\varepsilon > 0$ there exists a number N such that $\rho(x_n, x_m), \varepsilon$ for any $n, m > N$. A metric space is said to be *complete* if each Cauchy sequence in it is convergent.

15:A. A Cauchy sequence, which contains a convergent subsequence, converges.

15:B. Prove that a metric space is complete, iff any decreasing sequence of its closed balls with radii tending to 0 has nonempty intersection.

15:C. Prove that a compact metric space is complete?

15:D. Is any locally compact, but not compact metric space complete?

15:E. Prove that a complete metric space is compact, iff for any $\varepsilon > 0$ it contains a finite ε-net.

15:F. Prove that a complete metric space is compact iff for any $\varepsilon > 0$ it contains a compact ε-net.

Non-Compact Balls in Infinite Dimension

By l^∞ denote the set of all bounded sequences of real numbers. This is a vector space with respect to the component-wise operations. There is a natural norm in it: $\|x\| = \sup\{|x_n| : n \in \mathbb{N}\}$.

15.3. Are closed balls of l^∞ compact? What about spheres?

15.4. Is the set $\{x \in l^\infty : |x_n| \le 2^{-n}, n \in \mathbb{N}\}$ compact?

15.5. Prove that the set $\{x \in l^\infty : |x_n| = 2^{-n}, n \in \mathbb{N}\}$ is homeomorphic to the Cantor set K introduced in Section 1.

15.6*. Does there exist an infinitely dimensional normed space, in which closed balls are compact?

p-Adic Numbers

Fix a prime integer p. By $\mathbb{Z}_p$ denote the set of series of the form $a_0 + a_1 p + \dots + a_n p^n + \dots$ with $0 \le a_n < p$, $a_n \in \mathbb{N}$. For $x, y \in \mathbb{Z}_p$ put $\rho(x, y) = 0$ if $x = y$ and $\rho(x, y) = p^{-m}$, if m is the smallest number such that the m-th coefficients in the series x and y differ.

15.7. Prove that ρ is a metric in $\mathbb{Z}_p$.

This metric space is called the *space of integer p-adic numbers.* There is an injection $\mathbb{Z} \to \mathbb{Z}_p$ assigning to $a_0 + a_1 p + \dots + a_n p^n \in \mathbb{Z}$ with $0 \le a_k < p$ the series

$$a_0 + a_1 p + \dots + a_n p^n + 0p^{n+1} + 0p^{n+2} + \dots \in \mathbb{Z}_p$$

and to $-(a_0 + a_1 p + \dots + a_n p^n) \in \mathbb{Z}$ with $0 \le a_k < p$ the series

$$b_0 + b_1 p + \dots + b_n p^n + (p-1)p^{n+1} + (p-1)p^{n+2} + \dots,$$

where

$$b_0 + b_1 p + \cdots + b_n p^n = p^{n+1} - (a_0 + a_1 p + \cdots + a_n p^n).$$

Cf. *3.33*.

15.8. Prove that the image of the injection $\mathbb{Z} \to \mathbb{Z}_p$ is dense in $\mathbb{Z}_p$.

15.9. Is $\mathbb{Z}_p$ a complete space?

15.10. Is $\mathbb{Z}_p$ compact?

Induction on Compactness

A function $f : X \to \mathbb{R}$ is *locally bounded* if for any point $a \in X$ there exists a neighborhood U and a number $M > 0$ such that $|f(x)| \leq M$ for $x \in U$ (i.e., each point has a neighborhood such that the restriction of f to this neighborhood is bounded).

15.11. Prove that if a space X is compact and a function $f : X \to \mathbb{R}$ is locally bounded then f is bounded.

This statement is one of the simplest applications of a general principle formulated below in *15.12*. This principle may be called *induction on compactness* (cf. induction on connectedness discussed in Section 9).

Let X be a topological space, $\mathcal{C}$ a property of subsets of X. We say that $\mathcal{C}$ is *additive* if the union of any finite family of sets having $\mathcal{C}$ also has $\mathcal{C}$. The space X is said to *possess $\mathcal{C}$ locally* if each point of X has a neighborhood with property $\mathcal{C}$.

15.12. Prove that a compact space which possesses locally an additive property has this property itself.

15.13. Deduce from this principle the statements of problems *13.Q*, *15:E*, and *15:F*.

Spaces of Convex Figures

Let $D \subset \mathbb{R}^2$ be a closed disc of radius p. Consider the set of all convex polygons P with the following properties:

- the perimeter of P is at most p;
- P is contained in D;
- P has $\leq n$ vertices (the cases of one and two vertices are not excluded).

See *3.39*, cf. *3.41*.

15.14. Equip this set with a natural topological structure. For instance, define a natural metric.

15.15. Prove that this space is compact.

15.16. Prove that there exists a polygon belonging to this set and having the maximal area.

15.17. Prove that this is a regular n-gon.

Consider now the set of all convex polygons of perimeter $\leq p$ contained in D. In other words, consider the union of the sets of $\leq n$-gons considered above.

15.18. Construct a topological structure in this set such that it induces the structures introduced above in the spaces of $\leq n$-polygons.

15.19. Prove that the space provided by the solution of Problem *15.18* is not compact.

Consider now the set of all convex subsets of the plane of perimeter $\leq p$ contained in D.

15.20. Construct a topological structure in this set such that it induces the structure introduced above in the spaces of polygons.

15.21. Prove that the space provided by the solution of Problem *15.20* is compact.

15.22. Prove that there exists a convex plane set with perimeter $\leq p$ having a maximal area.

15.23. Prove that this is a disc of radius $\frac{p}{2\pi}$.

15.24. Consider the set of all bounded subsets of a compact metric space. Prove that this set endowed with the Hausdorff metric (see *3.40*) is a compact space.

Problems for Tests

Test.1. Let X be a topological space. Fill Table 1 with pluses and minuses according to your answers to the corresponding questions.

If X is: Has Y the same property, if:	connected	Hausdorff	non-Hausdorff	separable	compact	non-compact	second countable
$Y \subset X$							
Y is open subset of X							
Y is closed subset of X							
X is dense in Y							
Y is quotient space of X							
$Y = X$ as sets, $\Omega_X \subset \Omega_Y$							
Y is open subset of $\mathbb{R}^n$							
Y is anti-discrete							

TABLE 1

Test.2. Let X be a topological space. Fill Table 2 with pluses and minuses according to your answers to the corresponding questions.

Test.3. Give as many proves as you can for non-existence of a homeomorphism between

(a) S^1 and R^1,
(b) I and I^2,
(c) $\mathbb{R}$ and $\mathbb{R}_{T_1}$
(d) $\mathbb{R}$ and $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$.

If X is: Has Y the same property, if:	connected	Hausdorff	non-Hausdorff	separable	compact	non-compact	second countable
$X = Y \times Z$							
$Y = X \times Z$							
Y is open dense in X							
X is open dense in Y							
X is quotient space of Y							
$Y = X$ as sets, $\Omega_X \supset \Omega_Y$							
Y is closed and bounded subset of $\mathbb{R}^n$							
Y is discrete							

TABLE 2

CHAPTER 3

Topological Constructions

16. Multiplication

Set-Theoretic Digression. Product of Sets

Let X and Y be sets. The set of ordered pairs (x, y) with $x \in X$ and $y \in Y$ is called a *direct product* or *Cartesian product* or just *product* of X and Y and denoted by $X \times Y$. If $A \subset X$ and $B \subset Y$ then $A \times B \subset X \times Y$. Sets $X \times \{b\}$ with $b \in Y$ and $\{a\} \times Y$ with $a \in X$ are called *fibers* of the product $X \times Y$.

16.A. Prove that for any $A_1, A_2 \subset X$ and $B_1, B_2 \subset Y$

$$(A_1 \cup A_2) \times (B_1 \cup B_2) = (A_1 \times B_1) \cup (A_1 \times B_2) \cup (A_2 \times B_1) \cup (A_2 \times B_2),$$

$$(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2).$$

There are natural maps of $X \times Y$ onto X and Y defined by formulas $(x, y) \mapsto x$ and $(x, y) \mapsto y$. They are denoted by pr_X and pr_Y and are called *(natural) projections.*

16.B. Prove that $pr_X^{-1}(A) = A \times Y$ for $A \subset X$. Write down the corresponding formula for $B \subset Y$

To a map $f : X \to Y$ there corresponds a subset Γ_f of $X \times Y$ defined by $\Gamma_f = \{(x, f(x)) : x \in X\}$ and called the *graph* of f.

16.C. A set $\Gamma \subset X \times Y$ is the graph of a map $X \to Y$, iff for each $a \in X$ the intersection $\Gamma \cap (a \times Y)$ contains exactly one point.

16.1. Prove that for any map $f : X \to Y$ and any set $A \subset X$,

$$f(A) = pr_Y(\Gamma_f \cap (A \times Y)) = pr_Y(\Gamma_f \cap pr_X^{-1}(A))$$

and $f^{-1}(B) = pr_X(\Gamma \cap (X \times B))$ for any $B \subset Y$.

16.2. Let A and B be subsets of X and $\Delta = \{(x, y) \in X \times X : x = y\}$. Prove that $(A \times B) \cap \Delta = \varnothing$, iff $A \cap B = \varnothing$

16.3. Prove that the map $pr_X|_{\Gamma_f}$ is bijective.

16.4. Prove that f is injective, iff $pr_X|_{\Gamma_f}$ is injective.

16.5. Let $T : X \times Y \to Y \times X$ be the map defined by $(x, y) \mapsto (y, x)$. Prove that $\Gamma_{f^{-1}} = T(\Gamma_f)$ for any invertible map $f : X \to Y$.

Product of Topologies

Let X and Y be topological spaces. If U is an open set of X and B is an open set of Y, then we say that $U \times V$ is an *elementary* set of $X \times Y$.

16.D. *The set of elementary sets of $X \times Y$ is a base of a topological structure in $X \times Y$.*

The *product* of topological spaces X and Y is the set $X \times Y$ with the topological structure defined by the base consisting of elementary sets.

> ***16.6.*** Prove that for any subspaces A and B of spaces X and Y the topology of the product $A \times B$ coincides with the topology induced from $X \times Y$ via the natural $A \times B \subset X \times Y$.

16.E. The product $Y \times X$ is (canonically) homeomorphic to $X \times Y$. The product $X \times (Y \times Z)$ is canonically homeomorphic to $(X \times Y) \times Z$.

> ***16.7.*** Prove that if A is closed in X and B is closed in Y then $A \times B$ is closed in $X \times Y$.
>
> ***16.8.*** Prove that $\mathrm{Cl}(A \times B) = \mathrm{Cl}\, A \times \mathrm{Cl}\, B$ for any $A \subset X$ and $B \subset Y$.
>
> ***16.9.*** Is it true that $\mathrm{Int}(A \times B) = \mathrm{Int}\, A \times \mathrm{Int}\, B$?
>
> ***16.10.*** Is it true that $\mathrm{Fr}(A \times B) = \mathrm{Fr}\, A \times \mathrm{Fr}\, B$?
>
> ***16.11.*** Prove that $\mathrm{Fr}(A \times B) = (\mathrm{Fr}\, A \times B) \cup (A \times \mathrm{Fr}\, B)$.

Topological Properties of Projections and Fibers

16.F. *The natural projections pr_X and pr_Y are continuous.*

16.G. Prove that the topology of product is the coarsest topology with respect to which pr_X and pr_Y are continuous.

16.H. *A fiber of a product is canonically homeomorphic to the corresponding factor. The canonical homeomorphism is the restriction to the fiber of the natural projection of the product onto the factor.*

16.I. Prove that $\mathbb{R}^1 \times \mathbb{R}^1 = \mathbb{R}^2$, $(\mathbb{R}^1)^n = \mathbb{R}^n$, $(I)^n = I^n$ (recall that I^n is the n-dimensional cube).

> ***16.12.*** Let Σ_X and Σ_Y be bases of topological spaces X and Y. Prove that sets $U \times V$ with $U \in \Sigma_X$ and $V \in \Sigma_Y$ comprise a base for $X \times Y$.
>
> ***16.13.*** Prove that a map $f : X \to Y$ is continuous iff $pr_X|_{\Gamma_f}$ is a homeomorphism.
>
> ***16.14.*** Prove that if W is open in $X \times Y$ then $pr_X(W)$ is open in X.
>
> A map of a topological space X to a topological space Y is said to be *open* if the image of any open set under this map is open. Therefore *16.14* states that $pr_X : X \times Y \to X$ is an open map.
>
> ***16.15.*** Is pr_X a closed map?
>
> ***16.16.*** Prove that for each topological space X and each compact topological space Y the map $\mathrm{pr}_X : X \times Y \to X$ is closed.

Cartesian Products of Maps

Let X, Y, and Z be sets. To a map $f : Z \to X \times Y$ one assigns the compositions $f_1 = pr_X \circ f : Z \to X$ and $f_2 = pr_Y \circ f : Z \to Y$. They are called *factors* of f. Indeed, f can be recovered from them as a sort of product.

16.17. Prove that for any maps $f_1 : Z \to X$ and $f_2 : Z \to Y$ there exists a unique map $f : Z \to X \times Y$ with $pr_X \circ f = f_1$ and $pr_Y \circ f = f_2$

16.18. Let X, Y, and Z be topological spaces. Prove that f is continuous iff f_1 and f_2 are continuous.

For any maps $g_1 : X_1 \to Y_1$ and $g_2 : X_2 \to Y_2$ there is a map $X_1 \times X_2 \to Y_1 \times Y_2$ defined by formula $(x_1, x_2) \mapsto (g_1(x_1), g_2(x_2))$. This map is called a *(Cartesian) product* of g_1 and g_2 and denoted by $g_1 \times g_2$.

16.19. Prove that the Cartesian product of continuous maps is continuous, and the Cartesian product of open maps is open.

16.20. Prove that a metric $\rho : X \times X \to \mathbb{R}$ is continuous with respect to the topology defined by the metric.

Properties of Diagonal and Graph

16.21. Prove that a topological space is Hausdorff iff the set $\Delta = \{(x, x) : x \in X\}$ (which is called the *diagonal* of $X \times X$) is closed.

16.22. Prove that if Y is a Hausdorff space and a map $f : X \to Y$ is continuous then the graph Γ_f is closed in $X \times Y$.

16.23. Let Y be a compact space and Γ_f be closed. Prove that then f is continuous.

16.24. Prove that in *16.23* the hypothesis on compactness is necessary.

16.25. Let $f\ R \to R$ be a continuous function. Prove that its graph is:

(a) closed;
(b) connected;
(c) path connected;
(d) locally connected;
(e) locally compact.

16.26. Does any of properties of the graph of a function mentioned in *16.25* imply its continuity?

16.27. Let Γ_f be closed. Then the following assertions are equivalent:

(a) f is continuous;
(b) f is locally bounded;
(c) the graph Γ_f of f is connected.

16.28. Prove that if Γ_f is connected and locally connected then f is continuous.

16.29. Prove that if Γ_f is connected and locally coompact then f is continuous.

16.30. Are some of assertions in problems *16.27 – 16.29* true for mappings $f : R^2 \to R$?

Topological Properties of Products

16.J. *The product of Hausdorff spaces is Hausdorff.*

16.31. Prove that the product of regular spaces is regular.

16.32. The product of normal spaces is not necessarily normal.

16.32.1. Prove that the set of real numbers with the topology defined by the base which consists of all the rays $[a, \infty)$ is normal.

16.32.2. Prove that in the Cartesian square of the space introduced in *16.32.1* the subspace $\{(x, y) : x = -y\}$ is closed and discrete.

16.32.3. Find two disjoint subsets of $\{(x, y) : x = -y\}$ which have no disjoint neighborhoods in the Cartesian square of the space of *16.32.1*.

16.K. *The product of separable spaces is separable.*

16.L. *First countability of factors implies first countability of the product .*

16.M. *The product of second countable spaces is second countable.*

16.N. *The product of metrizable spaces is metrizable.*

16.O. *The product of connected spaces is connected.*

16.33. Prove that for connected spaces X and Y and any proper subsets $A \subset X$, $B \subset Y$ the set $X \times Y \setminus A \times B$ is connected.

16.P. *The product of path-connected spaces is path-connected.*

16.Q. *The product of compact spaces is compact.*

16.34. Prove that the product of locally compact spaces is locally compact.

16.35. For which of the topological properties studied above, if $X \times Y$ has the property then X also has?

Representation of Special Spaces as Products

16.R. Prove that $\mathbb{R}^2 \smallsetminus \{0\}$ is homeomorphic to $S^1 \times \mathbb{R}$.

16.36. Prove that $\mathbb{R}^n \smallsetminus \mathbb{R}^k$ is homeomorphic to $S^{n-k-1} \times \mathbb{R}^{k+1}$.

16.37. Prove that $S^n \cap \{x \in \mathbb{R}^{n+1} : x_1^2 + \dots + x_k^2 \le x_{k+1}^2 + \dots + x_{n+1}^2\}$ is homeomorphic to $S^{k-1} \times D^{n-k+1}$.

16.38. Prove that $O(n)$ is homeomorphic to $SO(n) \times O(1)$.

16.39. Prove that $GL(n)$ is homeomorphic to $SL(n) \times GL(1)$.

16.40. Prove that $GL_+(n)$ is homeomorphic to $SO(n) \times \mathbb{R}^{\frac{n(n+1)}{2}}$, where

$$GL_+(n) = \{A \in L(n, n) : \det A > 0\}.$$

16.41. Prove that $SO(4)$ is homeomorphic to $S^3 \times SO(3)$.

The space $S^1 \times S^1$ is called a *torus.*

16.S. Construct a topological embedding of the torus to $\mathbb{R}^3$

The product $S^1 \times \cdots \times S^1$ of k factors is called the *k-dimensional torus.*

16.T. Prove that the k-dimensional torus can be topologically embedded into $\mathbb{R}^{k+1}$.

16.U. Find topological embeddings of $S^1 \times D^2$, $S^1 \times S^1 \times I$, and $S^2 \times I$ into $\mathbb{R}^3$.

17. Quotient Spaces

Set-Theoretic Digression. Partitions and Equivalence Relations

Recall that a *partition* of a set is its cover consisting of pairwise disjoint sets. Each partition of a set X gives rise to an *equivalence relation* (i.e., a relation, which is reflexive, symmetric and transitive): two elements of X are said to be equivalent if they belong to the same element of the partition. Vice versa, each equivalence relation in X gives rise to the partition of X to classes of equivalent elements. Thus partitions of a set into nonempty subsets and equivalence relations in the set are essentially the same. More precisely, they are two ways of describing the same phenomenon.

Let X be a set, and S be a partition. The set whose elements are members of the partition S (which are subsets of X) is called the *quotient set* or *factor set* of X by S and denoted by $X/_S$.

> ***17.1. Riddle.*** How is this operation related to division of numbers? Why is there a similarity in terminology and notations?

At first glance, the definition of quotient set contradicts one of the very profound principles of the set theory which states that a set is defined by its elements. Indeed, according to this principle, $X/_S = S$, since S and $X/_S$ have the same elements. Hence, there seems to be no need to introduce $X/_S$.

The real sense of the notion of quotient set is not in its literal set-theoretic meaning, but in our way of thinking of elements of partitions. If we remember that they are subsets of the original set and want to keep track of their internal structure (at least, of their elements), we speak of a partition. If we think of them as atoms, getting rid of their possible internal structure then we speak on the quotient set.

The set $X/_S$ is called also the *set of equivalence classes* for the equivalence relation corresponding to the partition S.

The mapping $X \to X/_S$ that maps $x \in X$ to the element of S containing this point is called a *(canonical) projection* and denoted by pr. A subset of X which is a union of elements of a partition is said to be *saturated.* The smallest saturated set containing a subset A of X is called the *saturation* of A.

> ***17.2.*** Prove that $A \subset X$ is an element of a partition S of X, iff $A = \mathrm{pr}^{-1}(\mathit{point})$ where $\mathrm{pr} : X \to X/_S$ is the natural projection.

17.A. Prove that the saturation of a set A equals $\mathrm{pr}^{-1}\big(\mathrm{pr}(A)\big)$.

17.B. Prove that a set is saturated iff it is equal to its saturation.

Quotient Topology

A quotient set $X/_S$ of a topological space X with respect to a partition S into nonempty subsets is provided with a natural topology: a set $U \subset X/_S$ is said to be open in $X/_S$ if its preimage $\mathrm{pr}^{-1}(U)$ under the canonical projection $\mathrm{pr} : X \to X/_S$ is open.

17.C. *The collection of these sets is a topological structure in the quotient set $X/_S$.*

This topological structure is called the *quotient topology.* The set $X/_S$ with this topology is called the *quotient space* of the space X by partition S.

> ***17.3.*** Give an explicit description of the quotient space of the segment $[0, 1]$ by the partition consisting of $[0, \frac{1}{3}]$, $(\frac{1}{3}, \frac{2}{3}]$, $(\frac{2}{3}, 1]$.
>
> ***17.4.*** What can you say about a partition S of a topological space X if the quotient space $X/_S$ is known to be discrete?

17.D. *A subset of a quotient space $X/_S$ is open iff it is the image of an open saturated set under the canonical projection* pr.

17.E. A subset of a quotient space $X/_S$ is closed, iff its preimage under pr is closed in X, iff it is the image of a closed saturated set.

17.F. *The canonical projection* $\mathrm{pr} : X \to X/_S$ *is continuous.*

17.G. Prove that the quotient topology is the finest topology in $X/_S$ such that the canonical projection pr is continuous with respect to it.

Topological Properties of Quotient Spaces

17.H. *A quotient space of a connected space is connected.*

17.I. *A quotient space of a path-connected space is path-connected.*

17.J. *A quotient space of a separable space is separable.*

17.K. *A quotient space of a compact space is compact.*

17.L. The quotient space of the real line by partition $\mathbb{R}_+$, $\mathbb{R} \setminus \mathbb{R}_+$ is not Hausdorff.

17.M. The quotient space of a topological space X by a partition S is Hausdorff, iff any two elements of S possess disjoint saturated neighborhoods.

> ***17.5.*** Formulate similar necessary and sufficient conditions for a quotient space to satisfy other separation axioms and countability axioms.
>
> ***17.6.*** Give an example showing that second countability may get lost when we go over to a quotient space.

Set-Theoretic Digression. Quotients and Maps

Let S be a partition of a set X into nonempty subsets. Let $f : X \to Y$ be a map which is constant on each element of S. Then there is a map $X/_S \to Y$ which assigns to each element A of S the element $f(A)$. This map is denoted by $f/_S$ and called the *quotient map* or *factor map* of f (by partition S).

17.N. Prove that a map $f : X \to Y$ is constant on each element of a partition S of X iff there exists a map $g : X/_S \to Y$ such that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{pr}}\big\downarrow & \nearrow g & \\ X/_S & & \end{array}$$

Prove that such a map g coincides with $f/_S$.

More generally, if S and T are partitions of sets X and Y then every map $f : X \to Y$, which maps each element of S into an element of T, gives rise to a map $X/_S \to Y/_T$ which assigns to an element A of partition S the element of partition T containing $f(A)$. This map is denoted by $f/_{S,T}$ and called the *quotient map* or *factor map* of f (*with respect to* S and T).

17.O. Formulate and prove for $f/_{S,T}$ a statement which generalizes *17.N.*

A map $f : X \to Y$ defines a partition of the set X into nonempty preimages of the elements of Y. This partition is denoted by $S(f)$.

17.P. The map $f/_{S(f)} : X/_{S(f)} \to Y$ is injective.

This map is called *injective factor* (or *injective quotient*) of the map f.

Continuity of Quotient Maps

17.Q. *Let X, Y be topological spaces, S be a partition of X into nonempty sets, and $f : X \to Y$ be a continuous map, which is constant on each element of S. Then the factor $f/_S$ of f is continuous.*

> ***17.7.*** Let X, Y be topological spaces, S be a partition of X into nonempty sets. Prove that the formula $f \mapsto f/_S$ defines a bijection of the set of all continuous maps $X \to Y$, which are constant on each element of the partition S, onto the set of all continuous maps $X/_S \to Y$.

17.R. Let X, Y be topological spaces, S and T partitions of X and Y, and $f : X \to Y$ a continuous map, which maps each element of S into an element of T. Then the map $f/_{S,T} : X/_S \to Y/_T$ is continuous.

Closed Partitions

A partition S of a topological space X is called *closed,* if the saturation of each closed set is closed.

> ***17:1.*** Prove that a partition is closed iff the canonical projection $X \to X/_S$ is a closed map.
>
> ***17:2.*** Prove that a partition, which contains only one element consisting of more than one point, is closed if this element is a closed set.

17:A. The quotient space of a topological space satisfying the first separation axiom with respect to a closed partition satisfies the first separation axiom.

17:B. *The quotient space of a normal topological space with respect to a closed partition is normal.*

Open Partitions

A partition S of a topological space X is called *open,* if the saturation of each open set is open.

> ***17:3.*** Prove that a partition is open iff the canonical projection $X \to X/_S$ is an open map.
>
> ***17:4.*** Prove that if a set A is saturated with respect to an open partition, then $\operatorname{Int} A$ and $\operatorname{Cl} A$ are also saturated.

17:C. *The quotient space of a second countable space with respect to an open partition is second countable.*

17:D. *The quotient space of a first countable space with respect to an open partition is first countable.*

17:E. *Let S be an open partition of a topological space X and T be an open partition of a topological space Y. Denote by $S \times T$ the partition of $X \times Y$ consisting of $A \times B$ with $A \in S$ and $B \in T$. Then the injective factor $X \times Y/_{S \times T} \to X/_S \times Y/_T$ of $\mathrm{pr} \times \mathrm{pr}\, X \times Y \to X/_S \times Y/_T$ is a homeomorphism.*

18. Zoo of Quotient Spaces

Tool for Identifying a Quotient Space with a Known Space

18.A. *If $f : X \to Y$ is a continuous map of a compact space X onto a Hausdorff space Y then the injective factor $f/_{S(f)} : X/_{S(f)} \to Y$ is a homeomorphism.*

18.B. *The injective factor of a continuous map of a compact space to a Hausdorff one is a topological embedding.*

> ***18.1.*** Describe explicitly partitions of a segment such that the corresponding quotient spaces are all the connected letters of the alphabet.
>
> ***18.2.*** Prove that there exists a partition of a segment I with the quotient space homeomorphic to square $I \times I$.

Tools for Describing Partitions

Usually an accurate literal description of a partition is cumbersome, but can be shortened and made more understandable. Of course, this requires a more flexible vocabulary with lots of words with almost the same meanings. For instance, the words *factorize* and *pass to a quotient* can be replaced by *attach, glue, identify, contract,* and other words accompanying these ones in everyday life.

Some elements of this language are easy to formalize. For instance, factorization of a space X with respect to a partition consisting of a set A and one-point subsets of the complement of A is called a *contraction* (of the subset A to a point), and the result is denoted by $X/_A$.

> ***18.3.*** Let $A, B \subset X$ comprise a fundamental cover of a topological space X. Prove that the quotient map $A/_{A \cap B} \to X/_B$ of the inclusion $A \hookrightarrow X$ is a homeomorphism.

If A and B are disjoint subspaces of a space X, and $f : A \to B$ is a homeomorphism then passing to the quotient of the space X by the partition into one-point subsets of the set $X \smallsetminus (A \cup B)$ and two-point sets $\{x, f(x)\}$, where $x \in A$, is called *gluing* or *identifying* (of sets A and B by homeomorphism f).

Rather convenient and flexible way for describing partitions is to describe the corresponding equivalence relations. The main advantage of this approach is that, due to transitivity, it suffices to specify only some pairs of equivalent elements: if one states that $x \sim y$ and $y \sim z$ then it is not needed to state $x \sim z$, since this follows.

Hence, a partition is represented by a list of statements of the form $x \sim y$, which are sufficient to recover the equivalence relation. By such a list enclosed into square brackets, we denote the corresponding partition. For example, the quotient of a space X obtained by identifying subsets A and B by a homeomorphism $f : A \to B$ is denoted by $X/_{[a \sim f(a) \text{ for any } a \in A]}$ or just $X/_{[a \sim f(a)]}$.

Some partitions are easy to describe by a picture, especially if the original space can be embedded into plane. In such a case, as in the pictures below, one draws arrows on segments to be identified to show directions which are to be identified.

Below we introduce all these kinds of descriptions for partitions and give examples of their usage, providing simultaneously literal descriptions. The latter are not nice, but they may help to keep the reader confident about the meaning of the new words and, on the other hand, appreciating the improvement the new words bring in.

Entrance to the Zoo

18.C. Prove that $I/_{[0 \sim 1]}$ is homeomorphic to S^1.

In other words, the quotient space of segment I by the partition consisting of $\{0, 1\}$ and $\{a\}$ with $a \in (0, 1)$ is homeomorphic to a circle.

18.C.1. Find a surjective continuous map $I \to S^1$ such that the corresponding partition into preimages of points consists of one-point subsets of the interior of the segment and the pair of boundary points of the segment.

18.D. Prove that $D^n/_{S^{n-1}}$ is homeomorphic to S^n.

In *18.D* we deal with the quotient space of ball D^n by the partition into S^{n-1} and one-point subsets of its interior.

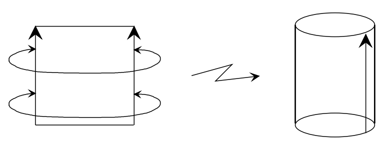
Reformulation of *18.D*: *Contracting* the boundary of an n-dimensional ball to a point gives rise to an n-dimensional sphere.

18.D.1. Find a continuous map of ball D^n to the sphere S^n that maps the boundary of the ball to a single point, and maps the interior of the ball bijectively onto the complement of this point.

18.E. Prove that $I^2/_{[(0,t) \sim (1,t) \text{ for } t \in \mathrm{I}]}$ is homeomorphic to $S^1 \times I$.

Here the partition consisits of pairs of points $\{(0, t), (1, t)\}$ where $t \in I$, and one-point subsets of $(0, 1) \times I$.

Reformulation of *18.E*: If we *glue* the side edges of a square identifying points on the same hight, we get a cylinder.



18.F. *Let X and Y be topological spaces, S a partition of X. Denote by T the partition of $X \times Y$ into sets $A \times y$ with $A \in S$, $y \in Y$. Then the natural bijection $X/_S \times Y \to X \times Y/_T$ is a homeomorphism.*

18.G. Riddle. How are the problems *18.C*, *18.E* and *18.F* related?

18.H. $S^1 \times I/_{[(z,0) \sim (z,1) \text{ for } z \in S^1]}$ is homeomorphic to $S^1 \times S^1$.

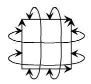
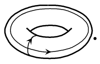
Here the partition consists of one-point subsets of $S^1 \times (0,1)$, and pairs of points of the basis circles lying on the same generatrix of the cylinder.

Reformulation of *18.H*: If we *glue* the basis circles of a cylinder identifying points on the same generatrix, then we get a torus.

18.I. $I^2/_{[(0,t) \sim (1,t),\ (t,0) \sim (t,1)]}$ is homeomorphic to $S^1 \times S^1$.

In *18.I* the partition consists of

- one-point subsets of the interior $(0,1) \times (0,1)$ of the square,
- pairs of points on the vertical sides, which are the same distance from the bottom side (i.e., pairs $\{(0,t),(1,t)\}$ with $t \in (0,1)$),
- pairs of points on the horizontal sides which lie on the same vertical line (i.e., pairs $\{(t,0),(t,1)\}$ with $t \in (0,1)$),
- the four vertices of the square

Reformulation of *18.I*: Identifying the sides of a square as follows  , we get a torus .

Transitivity of Factorization

A solution of Problem *18.I* can be based on Problems *18.E* and *18.H* and the following general theorem.

18.J Transitivity of Factorization. *Let S be a partition of a space X, and let S' be a partition of the space $X/_S$. Then the quotient space $(X/_S)/_{S'}$ is canonically homeomorphic to $X/_T$, where T is the partition of the space X into preimages of elements of the partition S' under projection $X \to X/_S$.*

Möbius Strip

Möbius strip or *Möbius band* is $I^2/_{[(0,t) \sim (1,1-t)]}$. In other words, this is the quotient space of square I^2 by the partition into pairs of points symmetric with respect to the center of the square and lying on the vertical edges and one-point set which do not lie on the vertical edges. Figuratively speaking, the Möbius strip is obtained by identifying the vertical sides of a square in such a way that the directions shown on them by arrows are superimposed.

18.K. Prove that the Möbius strip is homeomorphic to the surface swept in $\mathbb{R}^3$ by an interval, which rotates in a halfplane around the middle point while the halfplane rotates around its boundary line. The ratio of the angular velocities of these rotations is such that rotation of the halfplane by 360° takes the same time as rotation of the interval by 180°. See Figure 1.

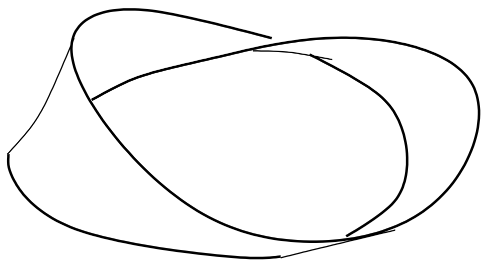


FIGURE 1

Contracting Subsets

18.4. Prove that $[0,1]/[\frac{1}{3},\frac{2}{3}]$ is homeomorphic to $[0,1]$, and $[0,1]/\{\frac{1}{3},1\}$ is homeomorphic to letter P.

18.5. Prove that the following spaces are homeomorphic:

(a) $\mathbb{R}^2$;
(b) $\mathbb{R}^2/I$;
(c) $\mathbb{R}^2/D^2$;
(d) $\mathbb{R}^2/I^2$;
(e) $\mathbb{R}^2/A$ where A is a union of several segments with a common end point;
(f) $\mathbb{R}^2/B$ where B is a simple finite polygonal line, i.e., a union of a finite sequence of segments $I_1, \dots, I_n$ such that the initial point of I_{i+1} coincides with the final point of I_i).

18.6. Prove that if $f : X \to Y$ is a homeomorphism then the quotient spaces X/A and $Y/f(A)$ are homeomorphic.

18.7. Prove that $\mathbb{R}^2/[0,+\infty)$ is homeomorphic to $\operatorname{Int} D^2 \cup \{(0,1)\}$.

Further Examples

18.8. Prove that $S^1/[z \sim e^{2\pi i/3}z]$ is homeomorphic to S^1.

In *18.8* the partition consists of triples of points which are vertices of equilateral inscribed triangles.

18.9. Prove that the following quotient spaces of disk D^2 are homeomorphic to D^2:

(a) $D^2/[(x,y) \sim (-x,-y)]$,
(b) $D^2/[(x,y) \sim (x,-y)]$,
(c) $D^2/[(x,y) \sim (-y,x)]$.

18.10. Find a generalization of *18.9* with D^n substituted for D^2.

18.11. Describe explicitly the quotient space of line $\mathbb{R}^1$ by equivalence relation $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$.

18.12. Present the Möbius strip as a quotient space of cylinder $S^1 \times I$.

Klein Bottle

Klein bottle is $I^2/[(t,0) \sim (t,1),\ (0,t) \sim (1,1-t)]$. In other words, this is the quotient space of square I^2 by the partition into

- one-point subsets of its interior,
- pairs of points $(t,0),(t,1)$ on horizontal edges which lie on the same vertical line,
- pairs of points $(0,t),(1,1-t)$ symmetric with respect to the center of the square which lie on the vertical edges, and
- the quadruple of vertices.

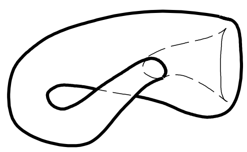
18.13. Present the Klein bottle as a quotient space of

(a) a cylinder;
(b) the Möbius strip.

18.14. Prove that $S^1 \times S^1/[(z,w) \sim (-z,\bar{w})]$ is homeomorphic to the Klein bottle. (Here $\bar{w}$ denotes the complex number conjugate to w.)

18.15. Embed the Klein bottle into $\mathbb{R}^4$ (cf. *18.K* and *16.S*).

18.16. Embed the Klein bottle into $\mathbb{R}^4$ so that the image of this embedding under the orthogonal projection $\mathbb{R}^4 \to \mathbb{R}^3$ would look as follows.



Projective Plane

Let us identify each boundary point of the disk D^2 with the antipodal point, i.e., factorize the disk by the partition consisting of one-point subsets of the interior of the disk and pairs of points on the boundary circle symmetric with respect to the center of the disk. The result is called the *projective plane*. This space cannot be embedded into $\mathbb{R}^3$, too. Thus we are not able to draw it. Instead, we present it in other way.

18.L. A projective plane is the result of gluing of a disk and the Möbius strip by homeomorphism between boundary circle of the disk and boundary circle of the Möbius strip.

You May Have Been Provoked to Perform an Illegal Operation

Solving the previous problem you did something which does not fit into the theory presented above. Indeed, the operation with two spaces called *gluing* in *18.L* has not appeared yet. It is a combination of two operations: first we must make a single space consisting of disjoint copies of the original spaces, and then we factorize this space identifying points of one copy with points of another. Let us consider the first operation in details.

Set-Theoretic Digression. Sums of Sets

A *sum* of a family of sets $\{X_\alpha\}_{\alpha\in A}$ is the set of pairs (x_α, α) such that $x_\alpha \in X_\alpha$. The sum is denoted by $\coprod_{\alpha\in A} X_\alpha$. The map of X_β ($\beta \in A$) to $\coprod_{\alpha\in A} X_\alpha$ defined by formula $x \mapsto (x, \beta)$ is an injection and denoted by in_β. If only sets X and Y are involved and they are distinct, we can avoid indices and define the sum by setting

$$X \amalg Y = \{(x, X) \mid x \in X\} \cup \{(y, Y) \mid y \in Y\}.$$

Sums of Spaces

18.M. If $\{X_\alpha\}_{\alpha\in A}$ is a collection of topological spaces then the collection of subsets of $\coprod_{\alpha\in A} X_\alpha$ whose preimages under all inclusions in_α ($\alpha \in A$) are open, is a topological structure.

The sum $\coprod_{\alpha\in A} X_\alpha$ with this topology is called the *(disjoint) sum of topological spaces* X_α, ($\alpha \in A$).

18.N. Topology described in *18.M* is the finest topology with respect to which all inclusions in_α are continuous.

> ***18.17.*** The maps $\mathrm{in}_\beta : X_\beta \to \coprod_{\alpha\in A} X_\alpha$ are topological embedding, and their images are both open and closed in $\coprod_{\alpha\in A} X_\alpha$.
>
> ***18.18.*** Which topological properties are inherited from summands X_α by the sum $\coprod_{\alpha\in A} X_\alpha$? Which are not?

Attaching Space

Let X, Y be topological spaces, A a subset of Y, and $f : A \to X$ a continuous map. The quotient space $(X \amalg Y)/_{[a \sim f(a) \text{ for } a \in A]}$ is denoted by $X \cup_f Y$, and is said to be the result of *attaching* or *gluing* the space Y to the space X by f. The latter is called the *attaching map.*

Here the partition of $X \amalg Y$ consists of one-point subsets of $\mathrm{in}_2(Y \smallsetminus A)$ and $\mathrm{in}_1(X \smallsetminus f(A))$, and sets $\mathrm{in}_1(x) \cup \mathrm{in}_2\big(f^{-1}(x)\big)$ with $x \in f(A)$.

> ***18.19.*** Prove that the composition of inclusion $X \to X \amalg Y$ and projection $X \amalg Y \to X \cup_f Y$ is a topological embedding.
>
> ***18.20.*** Prove that if X is a point then $X \cup_f Y$ is $Y/_A$.

18.O. Prove that attaching a ball D^n to its copy by the identity map of the boundary sphere S^{n-1} gives rise to a space homeomorphic to S^n.

> ***18.21.*** Prove that the Klein bottle can be obtained as a result of gluing two copies of the Möbius strip by the identity map of the boundary circle.
>
> ***18.22.*** Prove that the result of gluing two copies of a cylinder by the identity map of the boundary circles (of one copy to the boundary circles of the other) is homeomorphic to $S^1 \times S^1$.
>
> ***18.23.*** Prove that the result of gluing two copies of solid torus $S^1 \times D^2$ by the identity map of the boundary torus $S^1 \times S^1$ is homeomorphic to $S^1 \times S^2$.

18.24. Obtain the Klein bottle by gluing two copies of the cylinder $S^1 \times I$ to each other.

18.25. Prove that the result of gluing two copies of solid torus $S^1 \times D^2$ by the map

$$S^1 \times S^1 \to S^1 \times S^1 : (x, y) \mapsto (y, x)$$

of the boundary torus to its copy is homeomorphic to S^3.

18.P. Let X, Y be topological spaces, A a subset of Y, and $f, g : A \to X$ continuous maps. Prove that if there exists a homeomorphism $h : X \to X$ such that $h \circ f = g$ then $X \cup_f Y$ and $X \cup_g Y$ are homeomorphic.

18.Q. Prove that $D^n \cup_h D^n$ is homeomorphic to S^n for any homeomorphism $h : S^{n-1} \to S^{n-1}$.

18.26. Classify up to homeomorphism topological spaces, which can be obtained from a square by identifying a pair of opposite sides by a homeomorphism.

18.27. Classify up to homeomorphism the spaces which can be obtained from two copies of $S^1 \times I$ by identifying of the copies of $S^1 \times \{0, 1\}$ by a homeomorphism.

18.28. Prove that the topological type of the space resulting in gluing two copies of the Möbius strip by a homeomorphism of the boundary circle does not depend on the homeomorphism.

18.29. Classify up to homeomorphism topological spaces, which can be obtained from $S^1 \times I$ by identifying $S^1 \times 0$ with $S^1 \times 1$ by a homeomorphism.

Basic Surfaces

A torus $S^1 \times S^1$ with the interior of an embedded disk deleted is called a *handle*. A two-dimensional sphere with the interior of n disjoint embedded disks deleted is called a *sphere with n holes*.

18.R. A sphere with a hole is homeomorphic to disk D^2.

18.S. A sphere with two holes is homeomorphic to cylinder $S^1 \times I$.

A sphere with three holes has a special name. It is called *pantaloons*.

The result of attaching p copies of a handle to a sphere with p holes by embeddings of the boundary circles of handles onto the boundary circles of the holes (the boundaries of the holes) is called a *sphere with p handles*, or, more ceremonial (and less understandable, for a while), *orientable connected closed surface of genus p*.

18.30. Prove that a sphere with p handles is well-defined up to homeomorphism (i.e., the topological type of the result of gluing does not depend on the attaching embeddings).

18.T. A sphere with one handle is homeomorphic to torus $S^1 \times S^1$.

18.U. A sphere with two handles is homeomorphic to the result of gluing two copies of a handle by the identity map of the boundary circle.

A sphere with two handles is called a *pretzel*. Sometimes this word denotes also a sphere with more handles.

The space obtained from a sphere with q holes by attaching q copies of the Möbius strip by embeddings of the boundary circles of the Möbius strips onto the boundary circles of the holes (the boundaries of the holes) is called a *sphere with q crosscaps*, or *non-orientable connected closed surface of genus q*.

> ***18.31.*** Prove that a sphere with q crosscaps is well-defined up to homeomorphism (i.e., the topological type of the result of gluing does not depend on the attaching embeddings).

18.V. A sphere with one crosscap is homeomorphic to the projective plane.

18.W. A sphere with two crosscaps is homeomorphic to the Klein bottle.

A sphere, spheres with handles, and spheres with crosscaps are called *basic surfaces*.

18.X. Prove that a sphere with p handles and q crosscaps is homeomorphic to a sphere with $2p+q$ crosscaps (here $q>0$).

> ***18.32.*** Classify up to homeomorphisms topological spaces, which can be obtained by attaching to a sphere with $2p$ holes p copies of $S^1 \times I$ by embeddings of the boundary circles of the cylinders onto the boundary circles of the sphere with holes.

19. Projective Spaces

This section can be considered as a continuation of the previous one. The quotient spaces described here are of too great importance to consider them just as examples of quotient spaces.

Real Projective Space of Dimension n

This space is defined as the quotient space of the sphere S^n by the partition into pairs of antipodal points, and denoted by $\mathbb{R}P^n$.

19.A. *The space $\mathbb{R}P^n$ is homeomorphic to the quotient space of the ball D^n by the partition into one-point subsets of the interior of D^n, and pairs of antipodal point of the boundary sphere S^{n-1}.*

19.B. $\mathbb{R}P^0$ is a point.

19.C. The space $\mathbb{R}P^1$ is homeomorphic to the circle S^1.

19.D. The space $\mathbb{R}P^2$ is homeomorphic to the projective plane defined in the previous section.

19.E. *The space $\mathbb{R}P^n$ is canonically homeomorphic to the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ by the partition into one-dimensional vector subspaces of $\mathbb{R}^{n+1}$ punctured at 0.*

A point of the space $\mathbb{R}^{n+1} \setminus \{0\}$ is a sequence of real numbers which are not all zeros. These numbers are called *homogeneous coordinates* of the corresponding point of $\mathbb{R}P^n$. The point with homogeneous coordinates $x_0, x_1, \dots, x_n$ is denoted by $(x_0 : x_1 : \dots : x_n)$. Homogeneous coordinates define a point of $\mathbb{R}P^n$, but are not defined by this point: proportional vectors of coordinates $(x_0, x_1, \dots, x_n)$ and $(\lambda x_0, \lambda x_1, \dots, \lambda x_n)$ define the same point of $\mathbb{R}P^n$.

19.F. *The space $\mathbb{R}P^n$ is canonically homeomorphic to the metric space, whose points are the lines of $\mathbb{R}^{n+1}$ passing through the origin $0 = (0, \dots, 0)$ and the metric is defined to be the angle between lines (which takes values in $[0, \frac{\pi}{2}]$). Prove that this is really a metric.*

Complex Projective Space of Dimension n

This space is defined as the quotient space of unit sphere S^{2n+1} of the space $\mathbb{C}^{n+1}$ by the partition into circles which cut by (complex) lines of $\mathbb{C}^{n+1}$ passing through the point 0. It is denoted by $\mathbb{C}P^n$.

19:A. $\mathbb{C}P^n$ is homeomorphic to the quotient space of the unit ball D^{2n} of the space $\mathbb{C}^n$ by the partition whose elements are one-point subsets of the interior of D^{2n} and circles cut on the boundary sphere S^{2n-1} by (complex) lines of the space $\mathbb{C}^n$ passing through the origin $0 \in \mathbb{C}^n$.

19:B. $\mathbb{C}P^0$ is a point.

19:C. $\mathbb{C}P^1$ is homeomorphic to S^2.

19:D. The space $\mathbb{C}P^n$ is canonically homeomorphic to the quotient space of the space $\mathbb{C}^{n+1} \setminus \{0\}$ by the partition into complex lines of $\mathbb{C}^{n+1}$ punctured at 0.

Hence, $\mathbb{C}P^n$ can be viewed as the space of complex-proportional non-zero complex sequences $(x_0, x_1, \dots, x_n)$. Notation $(x_0 : x_1 : \dots : x_n)$ and term homogeneous coordinates introduced for the real case are used in the same way for the complex case.

19:E. The space $\mathbb{C}P^n$ is canonically homeomorphic to the metric space, whose points are the (complex) lines of the space $\mathbb{C}^{n+1}$ passing through the origin 0 and the metric is defined to be the angle between lines (which takes values in $[0, \frac{\pi}{2}]$).

Quaternion Projective Spaces and Cayley Plane

Must be written

20. Topological Groups

Algebraic Digression. Groups

Recall that a *group* is a set G equipped with a group operation. A *group operation* in set G is a map $\omega : G \times G \to G$ satisfying the following three conditions (known as *group axioms*):

- **Associativity.** $\omega(a, \omega(b, c)) = \omega(\omega(a, b), c)$ for any $a, b, c \in G$,
- **Existence of Neutral Element.** There exists $e \in G$ such that $\omega(e, a) = \omega(a, e) = a$ for every $a \in G$,
- **Existence of Inverse.** For any $a \in G$ there exists $b \in G$ such that $\omega(a, b) = \omega(b, a) = e$.

20:1. In a group a neutral element is unique.

20:2. For any element of a group an inverse element is unique.

The notations above are never used. (The only exception may happen, as here, if the definition of group is discussed.) Instead, one uses either *multiplicative* or *aditive* notations.

Under multiplicative notations the group operations is called *multiplication* and denoted as multiplication: $(a, b) \mapsto ab$. The neutral element is called *unity* and denoted by 1. The element inverse to a is denoted by a^{-1}. These notations are borrowed from the case, say, of group of nonzero rational numbers with the usual multiplication.

Under additive notations the group operations is called *addition* and denoted as addition: $(a, b) \mapsto a + b$. The neutral element is called *zero* and denoted by 0. The element inverse to a is denoted by $-a$. These notations are borrowed from the case, say, of group of integer numbers with the usual addition.

An operation $\omega : G \times G \to G$ is *commutative* provided that $\omega(a, b) = \omega(b, a)$ for all $a, b \in G$. A group with commutative group operation is called *commutative* or *abelian.* Traditionally the additive notations are used only in the case of commutative group, while the multiplicative notations are used both for commutative and non-commutative cases. Below we use mostly the multiplicative notations.

20:3. Check that in each of the following situations we have a group:

(a) the set $\mathbb{S}_n$ of bijections of the set $\{1, 2, \dots, n\}$ of n first natural numbers with composition (*symmetric group of degree n*,)
(b) the set $Homeo(X)$ of all homeomorphisms of a topological space X with composition,
(c) the set of invertible real $n \times n$-matrices $GL(n, \mathbb{R})$ with matrix multiplication,
(d) the set of all real $p \times q$-matrices with addition of matrices,
(e) the set of all subsets of a set X with symmetric difference

$$(A, B) \mapsto (A \cup B) \smallsetminus (A \cap B)$$

Topological Groups

A *topological group* is a set G equipped with both topological and group structures such that the maps $G \times G \to G : (x, y) \mapsto xy$ and $G \to G : x \mapsto x^{-1}$ are continuous.

> ***20:4.*** Prove that if G is a group and a topological space then $G \times G \to G : (x, y) \mapsto xy$ and $G \to G : x \mapsto x^{-1}$ are continuous, iff $G \times G \to G : (x, y) \mapsto x^{-1}y$ is continuous.
>
> ***20:5.*** Prove that for a topological group G the inversion $G \to G : x \mapsto x^{-1}$ is a homeomorphism.
>
> ***20:6.*** Let G be a topological group, X a topological space, and $f, g : X \to G$ be maps continuous at a point $x_0 \in X$. Prove that maps $X \to G : x \mapsto f(x)g(x)$ and $X \to G : x \mapsto (f(x))^{-1}$ are continuous at x_0.

20:A. Any group equipped with the discrete topological structure is a topological group.

> ***20:7.*** Is a group equipped with the indiscrete topological structure a topological group?

20:B. The real line $\mathbb{R}$ with the addition is a topological group.

20:C. The punctured real line $\mathbb{R} \setminus 0$ with the multiplication is a topological group.

20:D. The punctured complex line $\mathbb{C} \setminus 0$ with the multiplication is a topological group.

> ***20:8.*** Check that in each of the following situations we have a topological group:
>
> (a) the set $GL(n, \mathbb{R})$ of invertible real $n \times n$-matrices with the matrix multiplication and the topology induced by the inclusion to the set of all real $n \times n$-matrices considered as $\mathbb{R}^{n^2}$,
> (b) the set $GL(n, \mathbb{C})$ of invertible complex $n \times n$-matrices with the matrix multiplication and the topology induced by the inclusion to the set of all complex $n \times n$-matrices considered as $\mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$.

Self-Homeomorphisms Making a Topological Group Homogeneous

Recall that the maps of a group G to itself defined by formula $x \mapsto xa^{-1}$ and $x \mapsto ax$, respectively, are called (*right* and *left*) *translations* and denoted by R_a and L_a.

20:E. *Any translation of a topological group is a homeomorphism.*

Recall that the *conjugation* of a group G by $a \in G$ is the map $G \to G : x \mapsto a^{-1}xa$.

20:F. *Conjugation of a topological group by any its element is a homeomorphism.*

Given subsets A, B of a group G, the set $\{ab : a \in A, b \in B\}$ is denoted by AB, and $\{a^{-1} : a \in A\}$ is denoted by A^{-1}.

20:G. If U is an open set in a topological group G then for any $x \in G$ the sets xU, Ux and U^{-1} are open.

20:9. Does the same hold true for closed sets?

20:10. Prove that if U and V are subsets of a topological group G and U is open then UV and VU are open.

20:11. Does the same hold true if one replaces all the words open by closed?

20:11:1. Which of the following sugroups of the additive group $\mathbb{R}$ are closed:

(a) $\mathbb{Z}$,
(b) $\sqrt{2}\mathbb{Z}$,
(c) $\mathbb{Z} + \sqrt{2}\mathbb{Z}$?

Neighborhoods

20:H. *If Γ is a neighborhood basis at the unity 1 in a topological group G then $\Sigma = \{aU : a \in G, U \in \Gamma\}$ is a basis for topology of G.*

A subset A of a group G is said to be *symmetric* if $A^{-1} = A$.

20:I. Any neighborhood of unity of a topological group contains a symmetric neighborhood of unity.

20:J. For any neihgborhood U of 1 of a topological group there exists a neighborhood V of 1 such that $VV \subset U$.

20:12. For any neihgborhood U of 1 of a topological group and any natural number n there exists a symmetric neighborhood V of 1 such that $V^n \subset U$.

20:13. Let G be a group and Σ be a collection of its subsets. Prove that there exists a unique topology on G such that G with this topology is a topological group and Σ is its neighborhood basis at the unity, iff Σ satisfies the following five conditions:

(a) each $U \in \Sigma$ contains the unity of G,
(b) for every $x \in U \in \Sigma$ there exists $V \in \Sigma$ such that $xV \subset U$,
(c) for each $U \in \Sigma$ there exists $V \in \Sigma$ such that $V^{-1} \subset U$,
(d) for each $U \in \Sigma$ there exists $V \in \Sigma$ such that $VV \subset U$,
(e) for every $x \in G$ and $U \in \Sigma$ there exists $V \in \Sigma$ such that $V \subset x^{-1}Ux$.

20:K. Riddle. For what reasons *20:J* is similar to the triangle inequality?

Separaion Axioms

20:L. A topological group is Hausdorff, iff it satisfies the first separation axiom, iff the unity is closed.

20:M. A topological group is Hausdorff, iff the unity is equal to the intersection of its neighborhoods.

20:N. If the unity of a topological group G is closed, then G (as a topological space) is regular.

Consequently, for topological groups the first three separation axioms are equivalent.

Countability Axioms

20:O. If Γ is a neighborhood basis at the unity 1 in a topological group G and $S \subset G$ is dense in G, then $\Sigma = \{aU \,:\, a \in S, U \in \Gamma\}$ is a basis for topology of G. Cf. *20:H* and *12.F*.

20:P. A first countable separable topological group is second countable.

Subgroups

Recall that a subset H of a group G such that $HH = H$ and $H^{-1} = H$ is called a *subgroup* of G. It is a group with the operation defined by the group operation of G. If G is a topological group, then H inherits also a topological structure from G.

20:Q. If H is a subgroup of a topological group G, then the topological and group structures induced from G make H a topological group.

> ***20:14.*** Prove that a subgroup of a topological group is open, iff it contains an interior point.
>
> ***20:15.*** Prove that every open subgroup of a topological group is also closed.
>
> ***20:16.*** Find an example of a subgroup of a topological group, which
>
> (a) is closed, but not open,
> (b) is neither closed, nor open.
>
> ***20:17.*** Prove that a subgroup of a topological group is discrete, iff it contains an isolated point.
>
> ***20:18.*** Prove that a subgroup H of a topological group G is closed, iff it is locally closed, i.e., there exists an open set $U \subset G$ such that $U \cap H = U \cap \operatorname{Cl} H \neq \varnothing$.
>
> ***20:19.*** Prove that if H is a non-closed subgroup of a topological group G then $\operatorname{Cl} H \smallsetminus H$ is dense in $\operatorname{Cl} H$.
>
> ***20:20.*** Prove that the closure of a subgroup of a topological group is a subgroup.

20:21. Is it true that the interior of a subgroup of a topological group is a subgroup?

Recall that the smallest subgroup of a group G containing a set S is said to be generated by S.

20:22. The subgroup generated by S is the intersection of all the subgroups which contain S. On the other hand, this is the set of all the elements which can be obtained as products of elements of S and elements inverse to elements of S.

20:R. A connected topological group is generated by any neighborhood of the unity.

Recall that for a subgroup H of a group G *right cosets* are sets $Ha = \{xa : x \in H\}$ with $a \in G$. Analogously, sets aH are *left cosets* of H in G.

20:23. Let H be a subgroup of a group G. Define a relation: $a \sim b$ if $ab^{-1} \in H$. Prove that this is an equivalence relation and the right cosets of H in G are the equivalence classes.

20:24. What is the counter-part of *20:23* for left cosets?

The set of left cosets of H in G is denoted by G/H, the set of right cosets of H in G, by $H \setminus G$. If G is a topological group and H is its subgroup then the sets G/H and $H \setminus G$ are provided with the quotient topology. Equipped with these topologies, they are called *spaces of cosets.*

20:S. For any topological group G and its subgroup H, the natural projections $G \to G/H$ and $G \to H \setminus G$ are open (i.e., the image of every open set is open).

20:25. The space of left (or right) cosets of a closed subgroup in a topological group is regular.

Normal Subgroups

Recall that a subgroup H of a group G is said to be *normal* if $a^{-1}ha \in H$ for all $h \in H$ and $a \in G$. Normal subgroups are called also *normal divisors* or *invariant subgroups.*

20:26. Prove that the closure of a normal subgroup of a topological group is a normal subgroup.

20:27. The connected component of the unity of a topological group is a closed normal subgroup.

20:28. The path-connected component of the unity of a topological group is a normal subgroup.

Recall that for a normal subgroup left cosets coincide with right cosets and the set of cosets is a group with the multiplication defined by formula $(aH)(bH) = abH$. The group of cosets of H in G is called the *quotient group* or *factor group* of G by H and denoted by G/H.

20:T. The quotient group of a topological group is a topological group (provided that it is considered with the quotient topology).

> ***20:29.*** The natural projection of a topological group onto its quotient group is open.
>
> ***20:30.*** A quotient group of a first (or second) countable group is first (respectively, second) countable.
>
> ***20:31.*** The quotient group G/H of a topological group G is regular, iff H is closed.
>
> ***20:32.*** Prove that if a normal subgroup H of a topological group G is open then the quotient group G/H is discrete.
>
> ***20:33.*** Let G be a finite topological group. Prove that there exists a normal subgroup H of G such that a set $U \subset G$ is open, iff it is a union of several cosets of H in G.

Homomorphisms

Recall that a map f of a group G to a group H is called a *(group) homomorphism* if $f(xy) = f(x)f(y)$ for all $x, y \in G$. If G and H are topological groups then by a homomorphism $G \to H$ one means a group homomorphism which is *continuous.*

20:U. *A group homomorphism of a topological group to a topological group is continuous, iff it is continuous at* 1.

Besides similar modifications, which can be summarized by the following principle: *everything is assumed to respect the topological structures*, the terminology of group theory passes over without changes. In particular, the kernel $\operatorname{Ker} f$ of a homomorphism $f : G \to H$ is defined as the preimage of the unity of H. A homomorphism f is a *monomorphism* if it is injective. This is known to be equivalent to $\operatorname{Ker} f = 1$. A homomorphism $f : G \to H$ is an *epimorphism* if it is surjective, i.e, its image $\operatorname{Im} f = f(G)$ is the whole H.

In group theory, an *isomorphism* is an invertible homomorphism. Its inverse is a homomorphism (and hence an isomorphism) automatically. In theory of topological groups this must be included in the definition of isomorphism: an *isomorphism* of topological groups is an invertible homomorphism whose inverse is also a homomorphism. In other words, an isomorphism of topological groups is a map which is both an algebraic homomorphism and a homeomorphism. Cf. Section 8.

> ***20:34.*** An epimorphism $f : G \to H$ is open, iff its injective factor, $f/S(f) : G/\operatorname{Ker} f \to H$, is an isomorphism.

20:35. An epimorphism of a compact topological group onto a topological group with closed unity is open.

20:36. Prove that the quotient group $\mathbb{R}/\mathbb{Z}$ of the additive group of real numbers by the subgroup of integers is isomorphic to the multiplicative group $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ of complex numbers with absolute value 1.

Local Isomorphisms

Let G and H be topological groups. A *local isomorphism* of G to H is a homeomorphism f of a neighborhood U of the unity of G to a neighborhood V of the unity of H such that

- $f(xy) = f(x)f(y)$ for every $x, y \in U$ such that $xy \in U$,
- $f^{-1}(zt) = f^{-1}(z)f^{-1}(t)$ for every $z, t \in V$ such that $zt \in V$.

Topological groups G, H are said to be *locally isomorphic* if there exists a local isomorphism of G to H.

20:V. Isomorphic topological groups are locally isomorphic.

20:W. Additive group $\mathbb{R}$ of real numbers and multiplicative group S^1 of complex numbers with absolute value 1 are locally isomorphic, but not isomorphic.

20:37. Prove that the relation of being locally isomorphic is an equivalence relation on the class of topological groups.

20:38. Find neighborhoods of unities in $\mathbb{R}$ and S^1 and a homeomorphism between them, which satisfies the first condition from the definition of local isomorphism, but does not satisfy the second one.

20:39. Prove that for any homeomorphism between neighborhods of unities of two topological groups, which satisfies the first condition from the definition of local isomorphism, but does not satisfy the second one, there exists a submapping, which is a locall isomorphsm between these topological groups.

Direct Products

Let G and H be topological groups. In group theory, the product $G \times H$ is given a group structure,[1] in topology it is given a topological structure (see Secion 16).

20:X. These two structures are compatible: the group operations in $G \times H$ are continuous with respect to the product topology.

Thus, $G \times H$ is a topological group. It is called the *direct product* of the topological groups G and H. There are canonical homomorphisms related with this: the inclusions $i_G : G \to G \times H : x \mapsto (x, 1)$ and $i_H : H \to G \times H : x \mapsto (1, x)$, which are monomorphisms, and the projections $p_G : G \times H \to G : (x, y) \mapsto x$ and $p_H : G \times H \to H : (x, y) \mapsto y$, which are epimorphisms.

[1] Recall that the multiplication in $G \times H$ is defined by formula $(x, u)(y, v) = (xy, uv)$.

20:40. Prove that the topological groups $G \times H/_{i_H}$ and G are isomorphic.

20:41. The product operation is both commutative and associative: $G \times H$ is (canonically) isomorphic to $H \times G$ and $G \times (H \times K)$ is canonically isomorphic to $(G \times H) \times K$.

A topological group G is said to *decompose into the direct product of its subgroups A and B* if the map $A \times B \to G : (x, y) \mapsto xy$ is an isomorphism of topological groups. If this is the case, the groups G and $A \times B$ are usually identified via this isomorphism.

Recall that a similar definition exists in ordinary group theory. The only difference is that there the isomorphism is just an algebraic isomorphism. Moreover, in that theory, G decomposes into the direct product of its subgroups A and B, iff A and B generate G, are normal subgroups and $A \cap B = 1$. Therefore, if these conditions are satisfied in the case of topological groups, then $(x, y) \mapsto xy : A \times B \to G$ is a group isomorphism.

20:42. Prove that in this situation the map $(x, y) \mapsto xy : A \times B \to G$ is continuous. Find an example where the inverse group isomorphism is not continuous.

20:43. Prove that a compact Hausdorff group which decomposes algebraically into the direct product of two subgroups, decomposes also into the direct product of these subgroups in the category of topological groups.

20:44. Prove that the multiplicative group $\mathbb{R} \setminus 0$ of real numbers is isomorphic (as a topological group) to the direct product of the multiplicative group $S^0 = \{1, -1\}$ and the multiplicative group $\mathbb{R}^*_+ = \{x \in \mathbb{R} : x > 0\}$.

20:45. Prove that the multiplicative group $\mathbb{C} \setminus 0$ of complex numbers is isomorphic (as a topological group) to the direct product of the multiplicative group $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and the multiplicative group $\mathbb{R}^*_+$.

20:46. Prove that the multiplicative group $\mathbb{H} \setminus 0$ of quaternions is isomorphic (as a topological group) to the direct product of the multiplicative group $S^3 = \{z \in \mathbb{H} : |z| = 1\}$ and the multiplicative group $\mathbb{R}^*_+$.

20:47. Prove that the subgroup $S^0 = \{1, -1\}$ of $S^3 = \{z \in \mathbb{H} : |z| = 1\}$ is not a direct factor.

20:48. Find a topological group homeomorphic to $\mathbb{R}P^3$ (the three-dimensional real projective space).

21. Actions of Topological Groups

Actions of Group in Set

Must be written!

Continuous Actions

Must be written!

Orbit Spaces

Must be written!

Homogeneous Spaces

Must be written!

22. Spaces of Continuous Maps

Sets of Continuous Mappings

By $\mathcal{C}(X,Y)$ we denote the set of all continuous mappings of a topologival space X to a topological space Y.

22:1. Prove that $\mathcal{C}(X,Y)$ consists of a single element iff so does Y.

22:2. Prove that there exists an injection $Y \to \mathcal{C}(X,Y)$. In other words, the cardinality $\operatorname{card}\mathcal{C}(X,Y)$ of $\mathcal{C}(X,Y)$ is greater than or equal to $\operatorname{card} Y$.

22:3. Riddle. Find natural conditions implying $\mathcal{C}(X,Y) = Y$.

22:4. Let $Y = \{0,1\}$ equipped with topology $\{\varnothing, \{0\}, Y\}$. Prove that there exists a bijection between $\mathcal{C}(X,Y)$ and the topological structure of X.

22:5. Let X be a set of n points with discrete topology. Prove that $\mathcal{C}(X,Y)$ can be identified with $Y \times \ldots \times Y$ (n times).

22:6. Let Y be a set of k points with discrete topology. Find necessary and sufficient condition for the set $\mathcal{C}(X,Y)$ contain k^2 elements.

Topological Structures on Set of Continuous Mappings

Let X, Y be topological spaces, $A \subset X$, $B \subset Y$. Denote by $W(A,B)$ the set $\{f \in \mathcal{C}(X,Y) \mid f(A) \subset B\}$. Denote by $\Delta^{(pw)}$ the set

$$\{W(a,U) \mid a \in X, \quad U \text{ is open in } Y\}$$

and by $\Delta^{(co)}$ the set

$$\{W(C,U) \mid C \subset X \text{ is compact}, U \text{ is open in } Y\}$$

22:A. $\Delta^{(pw)}$ is a subbase of a topological structure on $\mathcal{C}(X,Y)$.

The topological structure generated by $\Delta^{(pw)}$ is called the *topology of pointwise convergency.* The set $\mathcal{C}(X,Y)$ equipped with this structure is denoted by $\mathcal{C}^{(pw)}(X,Y)$.

22:B. $\Delta^{(co)}$ is a subbase of a topological structures on $\mathcal{C}(X,Y)$.

The topological structure defined by $\Delta^{(co)}$ is called the *compact-open topology.* Hereafter we denote by $\mathcal{C}(X,Y)$ the space of all continuous mappings $X \to Y$ with the compact-open topology, unless the contrary is specified explicitly.

22:C Compact-Open Versus Pointwise. The compact-open topology is finer than the topology of pointwise convergence.

22:7. Prove that $\mathcal{C}(I,I)$ is not homeomorphic to $\mathcal{C}^{(pw)}(I,I)$.

Denote by $Const(X,Y)$ the set of all constant mappings $f : X \to Y$.

22:8. Prove that the topology of pointwise convergence and compact-open topology of $\mathcal{C}(X, Y)$ induce the same topological structure on $Const(X, Y)$, which, with this topology, is homeomorphic Y.

22:9. Let X be a discrete space of n points. Prove that $\mathcal{C}^{(pw)}(X, Y)$ is homeomorphic $Y \times \ldots \times Y$ (n times). Is this true for $\mathcal{C}(X, Y)$?

Topological Properties of Spaces of Continuous Mappings

22:D. Prove that if Y is Hausdorff, then $\mathcal{C}^{(pw)}(X, Y)$ is Hausdorff for any topological space X. Is this true for $\mathcal{C}(X, Y)$?

22:10. Prove that $\mathcal{C}(I, X)$ is path connected iff X is path connected.

22:11. Prove that $\mathcal{C}^{(pw)}(I, I)$ is not compact. Is the space $\mathcal{C}(I, I)$ compact?

Metric Case

22:E. If Y is metrizable and X is compact then $\mathcal{C}(X, Y)$ is metrizable.

Let (Y, ρ) be a metric space and X a compact space. For continuous maps $f, g : X \to Y$ put

$$d(f, g) = \max\{\rho(f(x), g(x)) \mid x \in X\}.$$

22:F This is a Metric. If X is a compact space and Y a metric space, then d is a metric on the set $\mathcal{C}(X, Y)$.

Let X be a topological space and Y a metric space with metric ρ. A sequence f_n of maps $X \to Y$ is said to *uniformly converge* to $f : X \to Y$ if for any $\varepsilon > 0$ there exists a natural N such that $\rho(f_n(x), f(x)) < \varepsilon$ for any $n > N$ and $x \in X$. This is a straightforward generalization of the notion of uniform convergence which is known from Calculus.

22:G Metric of Uniform Convergence. Let X be a compact space and Y a metric space. A sequence f_n of maps $X \to Y$ converges to $f : X \to Y$ in the topology defined by d, iff f_n uniformly converges to f.

22:H Uniform Convergence Versus Compact-Open. Let X be a compact space and Y a metric space. Then the topology defined by d on $\mathcal{C}(X, Y)$ coincides with the compact-open topology.

22:12. Prove that the space $\mathcal{C}(\mathbb{R}, I)$ is metrizable.

22:13. Let Y be a bounded metric space and X a topological space which admits presentation $X = \bigcup_{i=1}^{\infty} X_i$, where X_i is compact and $X_i \subset \operatorname{Int} X_{i+1}$ for $i = 1, 2, \ldots$. Prove that $\mathcal{C}(X, Y)$ is metrizable.

Denote by $\mathcal{C}_b(X, Y)$ the set of all continuous bounded maps from a topological space X to a metric space Y. For maps $f, g \in \mathcal{C}_b(X, Y)$, put

$$d^{\infty}(f, g) = \sup\{\rho(f(x), g(x)) \mid x \in X\}.$$

22:I Metric on Bounded Mappings. This is a metric in $\mathcal{C}_b(X,Y)$.

22:J d^∞ and Uniform Convergence. Let X be a topological space and Y a metric space. A sequence f_n of bounded maps $X \to Y$ converges to $f : X \to Y$ in the topology defined by d^∞, iff f_n uniformly converges to f.

22:K When Uniform Is Not Compact-Open. Find X and Y such that the topology defined by d^∞ on $\mathcal{C}_b(X,Y)$ does not coincide with the compact-open topology.

Interactions With Other Constructions

22:L Continuity of Restricting. Let X, Y be topological spaces and $A \subset X$. Prove that the map $\mathcal{C}(X,Y) \to \mathcal{C}(A,Y) : f \mapsto f|_A$ is continuous.

22:M Continuity of Composing. Let X be a topological space and Y a locally compact Hausdorff space. Prove that the map

$$\mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z) \ : \ (f,g) \mapsto g \circ f$$

is continuous.

> ***22:14.*** Is local compactness of Y necessary in *22:M*?

22:N Extending Target. For any topological spaces X, Y and $B \subset Y$ the map $\mathcal{C}(X,B) \to \mathcal{C}(X,Y) : f \mapsto i_B \circ f$ is a topological embedding.

22:O Maps to Product. For any topological spaces X, Y and Z the space $\mathcal{C}(X, Y \times Z)$ is canonically homeomorphic to $\mathcal{C}(X,Y) \times \mathcal{C}(X,Z)$.

22:P Restricting to Sets Covering Source. Let $\{X_1, \dots, X_n\}$ be a fundumental cover of X. Prove that for any topological space Y,

$$\mathcal{C}(X,Y) \to \prod_{i=1}^{n} \mathcal{C}(X_i,Y) \ : \ f \mapsto (f|_{X_1}, \dots, f|_{X_n})$$

is a topological embedding. What if the cover is not fundamental?

22:Q Factorizing Source. Let S be a closed partition[2] of a Hausdorff compact space X. Prove that for any topological space Y the mapping

$$\mathcal{C}(X/S,Y) \to \mathcal{C}(X,Y)$$

is a topological embedding.

> ***22:15.*** Are the conditions imposed on S and X in *22:Q* necessary?

22:R The Evaluation Map. Let X, Y be topological spaces. Prove that if X is locally compact and Hausdorff then the map

$$\mathcal{C}(X,Y) \times X \to Y \ : \ (f,x) \mapsto f(x)$$

is continuous.

> ***22:16.*** Are the conditions imposed on X in *22:R* necessary?

[2] Recall that a partition is called *closed,* if the saturation of each closed set is closed.

Mappings $X \times Y \to Z$ and $X \to \mathcal{C}(Y, Z)$

22:S. Let X, Y and Z be topological spaces and $f\ X \times Y \to Z$ be a continuous map. Then the map

$$F : X \to \mathcal{C}(Y, Z)\ :\ F(x) : y \mapsto f(x, y),$$

is continuous.

22:T. Let X, Z be topological spaces and Y a Hausdorff locally compact space. Let $F : X \to \mathcal{C}(Y, Z)$ be a continuous mapping. Then the mapping $f : X \times Y \to Z : (x, y) \mapsto F(x)(y)$ is continuous.

22:U. Let X, Y and Z be topological spaces. Let the mapping

$$\Phi : \mathcal{C}(X \times Y, Z) \to \mathcal{C}(X, \mathcal{C}(Y, Z))$$

be defined by the relation

$$\Phi(f)(x) : y \mapsto f(x, y).$$

Then

(a) Φ is continuous;
(b) if Y is locally compact and Hausdorff then Φ is a homeomorphism.

Part 2

Algebraic Topology

CHAPTER 4

Fundamental Group and Covering Spaces

This part of the book can be considered as an introduction to algebraic topology. This is a part of topology, which relates topological and algebraic problems. The relationship is used in both directions, but reduction of topological problems to algebra is at first stages more useful, since algebra is usually easier. The relation is established according to the following scheme. One invents a construction, which assigns to each topological space X under consideration an algebraic object $A(X)$. The latter may be a group, or a ring, or a quadratic form, or algebra, etc. Another construction assigns to a continuous mapping $f : X \to Y$ a homomorphism $A(f) : A(X) \to A(Y)$. The constructions should satisfy natural conditions (in particular, they form a functor), which make it possible to relate topological phenomena with their algebraic images obtained via the constructions.

There are infinitely many useful constructions of this kind. In this part we deal mostly with one of them. This is the first one, first from both the viewpoints of history and its role in mathematics. It was invented by Henri Poincaré in the end of the nineteenth century.

23. Homotopy

Continuous Deformations of Maps

23.A. Is it possible to deform continuously

(a) The identity map $\mathrm{id} : \mathbb{R}^2 \to \mathbb{R}^2$ to the constant map $\mathbb{R}^2 \to \mathbb{R}^2 : x \mapsto 0$,
(b) The identity map $\mathrm{id} : S^1 \to S^1$ to the symmetry $S^1 \to S^1 : x \mapsto -x$ (here x is considered as a complex number, since the circle S^1 is $\{x \in \mathbb{C} : |x| = 1\}$),
(c) The identity map $\mathrm{id} : S^1 \to S^1$ to the constant map $S^1 \to S^1 : x \mapsto 1$,
(d) The identity map $\mathrm{id} : S^1 \to S^1$ to the two-fold wrapping $S^1 \to S^1 : x \mapsto x^2$,
(e) The inclusion $S^1 \to \mathbb{R}^2$ to a constant map,
(f) The inclusion $S^1 \to \mathbb{R}^2 \smallsetminus 0$ to a constant map?

23.B. Riddle. When you (tried to) solve the previous problem, what did you mean by "*deform continuously*"?

This section is devoted to the notion of *homotopy* formalizing the naive idea of the continuous deformation of a map.

Homotopy as Map and Family of Maps

Let f, g be continuous maps of a topological space X to a topological space Y, and $H : X \times I \to Y$ a continuous map such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for any $x \in X$. Then f and g are said to be *homotopic*, and H is called a *homotopy* between f and g.

For $x \in X$, $t \in I$ denote $H(x, t)$ by $h_t(x)$. This change of notation results in a change of the point of view of H. Indeed, for a fixed t the formula $x \mapsto h_t(x)$ defines a map $h_t : X \to Y$ and H appears to be a family of maps h_t enumerated by $t \in I$.

23.C. Prove that each h_t is continuous.

23.D. Does continuity of all h_t imply continuity of H?

The conditions $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ in the definition of homotopy above can be reformulated as $h_0 = f$ and $h_1 = g$. Thus a homotopy between f and g can be considered as a family of continuous maps, which connects f and g. Continuity of a homotopy allows one to say that it is a *continuous family of continuous maps.*

Homotopy as Relation

23.E. *Homotopy of maps is an equivalence relation.*

23.E.1. If $f : X \to Y$ is a continuous map then $H : X \times I \to Y$ defined by $H(x, t) = f(x)$ is a homotopy between f and f.

23.E.2. If H is a homotopy between f and g then H' defined by $H'(x,t) = H(x, 1-t)$ is a homotopy between g and f.

23.E.3. If H is a homotopy between f and f' and H' is a homotopy between f' and f'' then H'' defined by

$$H''(x,t) = \begin{cases} H(x,2t) & \text{for } t \le 1/2, \\ H(x,2t-1) & \text{for } t \ge 1/2 \end{cases}$$

is a homotopy between f and f''.

Homotopy, being an equivalence relation by *23.E*, divides the set $\mathcal{C}(X,Y)$ of all continuous mappings of a space X to a space Y into equivalence classes. The latter are called *homotopy classes.* The set of these classes is denoted by $\pi(X,Y)$.

23.1. Prove that for any X, the set $\pi(X,I)$ has a single element.

23.2. Prove that the number of elements of $\pi(I,Y)$ coincides with the number of path connected components of Y.

Straight-Line Homotopy

23.F. Any two continuous maps of the same space to $\mathbb{R}^n$ are homotopic.

23.G. Solve the preceding problem by proving that for continuous maps $f, g : X \to \mathbb{R}^n$ formula $H(x,t) = (1-t)f(x) + g(x)$ defines a homotopy between f and g.

The homotopy defined in *23.G* is called a *straight-line* homotopy.

23.H. Prove that any two continuous maps of a space to a convex subspace of $\mathbb{R}^n$ are homotopic.

A set $A \subset \mathbb{R}^n$ is said to be *star convex,* if there exists a point $b \in A$ such that for any $x \in A$ the whole segment $[a,x]$ connecting x to a is contained in A.

23.3. Prove that any two continuous maps of a space to a star convex subspace of $\mathbb{R}^n$ are homotopic.

23.4. Prove that any continuous map of a convex set $C \subset \mathbb{R}^n$ to any space is homotopic to a constant map.

23.5. Under what conditions (formulated in terms of known topological properties of a space X) any two continuous maps of any convex set to X are homotopic?

23.6. Prove that any non-surjective map of an arbitrary topological space to S^n is homotopic to a constant map.

23.7. Prove that any two maps of a one-point space to $\mathbb{R}^n \setminus \{0\}$ with $n > 1$ are homotopic.

23.8. Find two non-homotopic maps of a one-point space to $\mathbb{R} \setminus \{0\}$.

23.9. For various m, n, k, calculate the number of homotopy classes of maps $\{1, 2, \dots, m\} \to \mathbb{R}^n \setminus \{x_1, x_2, \dots, x_k\}$, where $\{1, 2, \dots, m\}$ is equipped with discrete topology.

23.10. Let f, g be maps of a topological space X to $\mathbb{C} \smallsetminus 0$. Prove that if $|f(x) - g(x)| < |f(x)|$ for any $x \in X$ then f and g are homotopic.

23.11. Prove that for any polynomials p and q over $\mathbb{C}$ of the same degree in one variable there exists $r > 0$ such that for any $R > r$ formulas $z \mapsto p(z)$ and $z \mapsto q(z)$ define maps of circle $\{z \in \mathbb{C} : |z| = R\}$ to $\mathbb{C} \smallsetminus 0$ and these maps are homotopic.

23.12. Let f, g be maps of an arbitrary topological space X to S^n. Prove that if $|f(a) - g(a)| < 2$ for any $a \in X$ then f is homotopic to g.

23.13. Let $f : S^n \to S^n$ be a continuous map. Prove that if it is fixed point free, i.e., $f(x) \neq x$ for any $x \in S^n$, then f is homotopic to the symmetry $x \mapsto -x$.

Two Natural Properties of Homotopies

23.I. Let $f, f' : X \to Y$, $g : Y \to B$, $h : A \to X$ be continuous maps and $F : X \times I \to Y$ a homotopy between f and f'. Prove that then $g \circ F \circ (h \times \mathrm{id}_I)$ is a homotopy between $g \circ f \circ h$ and $g \circ f' \circ h$.

23.J. Riddle. Under conditions of *23.I* define a natural mapping

$$\pi(X, Y) \to \pi(A, B).$$

How does it depend on g and h? Write down all the nice properties of this construction.

23.K. Prove that maps $f_0, f_1 : X \to Y \times Z$ are homotopic iff $\mathrm{pr}_Y \circ f_0$ is homotopic to $pr_Y \circ f_1$ and $\mathrm{pr}_Z \circ f_0$ is homotopic to $pr_Z \circ f_1$.

Stationary Homotopy

Let A be a subset of X. A homotopy $H : X \times I \to Y$ is said to be *fixed* or *stationary* on A, or, briefly, to be an *A-homotopy,* if $H(x, t) = H(x, 0)$ for all $x \in A$, $t \in I$. Maps which can be connected by an A-homotopy are said to be *A-homotopic.*

Of course, A-homotopic maps coincide on A. If one wants to emphasize that a homotopy is not assumed to be fixed, one says that it is *free.* If one wants to emphasize the opposite (that it is fixed), one says that the homotopy is *relative.*

Warning: there is a similar, but different kind of homotopy, which is also called *relative.* See below.

23.L. Prove that, like free homotopy, A-homotopy is an equivalence relation.

The classes into which A-homotopy divides the set of continuous maps $X \to Y$ that agree on A with a map $f : A \to Y$ are called *A-homotopy classes of continuous extensions of f to X.*

23.M. For what A is a straight-line homotopy fixed on A?

Homotopies and Paths

Recall that by a *path* in a space X we mean a continuous mapping of the interval I into X. (See Section 10.)

23.N. Riddle. In what sense is any path a homotopy?

23.O. Riddle. In what sense does any homotopy consist of paths?

23.P. Riddle. In what sense is any homotopy a path?

23.Q. Riddle. Introduce a topology in the set $\mathcal{C}(X,Y)$ of all continuous mappings $X \to Y$ in such a way that for any homotopy $h_t : X \to Y$ the map $I \to \mathcal{C}(X,Y) : t \mapsto h_t$ would be continuous.

Recall that the *compact-open topology* in $\mathcal{C}(X,Y)$ is the topology generated by the sets $\{\varphi \in \mathcal{C}(X,Y) \mid \varphi(A) \subset B\}$ for compact $A \subset X$ and open $B \subset Y$.

23:A. Prove that any homotopy $h_t : X \to Y$ defines (by the formula presented in *23.Q*) a path in $\mathcal{C}(X,Y)$ with compact-open topology.

23:B. Prove that if X is locally compact and regular then any path in $\mathcal{C}(X,Y)$ with compact-open topology is defined by a homotopy.

Homotopy of Paths

23.R. Prove that any two paths in the same space X are freely homotopic, iff their images belong to the same pathwise connected component of X.

This shows that the notion of free homotopy in the case of paths is not interesting. On the other hand, there is a sort of relative homotopy playing a very important role. This is $(0 \cup 1)$-homotopy. This causes the following commonly accepted deviation from the terminology introduced above: homotopy of paths always means not a free homotopy, but a homotopy fixed on the end points of I (i.e. on $0 \cup 1$).

Notation: a homotopy class of a path s is denoted by $[s]$.

24. Homotopy Properties of Path Multiplication

Multiplication of Homotopy Classes of Paths

Recall (see Section 10) that paths u and v in a space X can be multiplied, provided the initial point $v(0)$ of v coincides with the final point $u(1)$ of u. The product uv is defined by

$$uv(t) = \begin{cases} u(2t), & \text{if } t \le 1/2 \\ v(2t-1), & \text{if } t \ge 1/2. \end{cases}$$

24.A. Prove that if a path u is homotopic to u' and a path v is homotopic to v' and there exists product uv, then $u'v'$ exists and is homotopic to uv.

Define a product of homotopy classes of paths u and v to be the homotopy class of uv. So, $[u][v]$ is defined as $[uv]$, provided uv is defined. This is a definition which demands a proof.

24.B. Prove that the product of homotopy classes of paths is well-defined (of course, when the initial point of paths of the first class coincides with the final point of paths of the second class).

Associativity

24.C. Is multiplication of paths associative?

Of course, this question might be formulated with more details:

24.D. Let u, v, w be paths in the same space such that products uv and vw are defined (i.e., $u(1) = v(0)$ and $v(1) = w(0)$). Is it true that $(uv)w = u(vw)$?

> ***24.1.*** Prove that for paths in a metric space $(uv)w = u(vw)$ implies that u, v, w are constant maps.
>
> ***24.2. Riddle.*** Find non-constant paths u, v, and w in an indiscrete space such that $(uv)w = u(vw)$.

24.E. Find a map $\varphi : I \to I$ such that for any paths u, v, w with $u(1) = v(0)$ and $v(1) = w(0)$

$$((uv)w) \circ \varphi = u(vw).$$

24.F. *Multiplication of homotopy classes of paths is associative.*

If you are troubled by *24.F*, consider the following problem.

24.G. Reformulate Theorem *24.F* in terms of paths and their homotopies.

If you want to understand the essence of *24.F*, you have to realize that paths $(uv)w$ and $u(vw)$ have the same trajectories and differs by time spent in the fragments of the path. Therefore to find a homotopy between them one has to find a continuous way to change one schedule to the other.

If there is still a trouble in a formal prove, recall *24.E* and solve the following problem.

24.H. Prove that any path in I beginning in 0 and finishing in 1 is homotopic to $\mathrm{id} : I \to I$.

Also, it may be useful to take into account *23.I.*

Unit

Let a be a point of a space X. Denote by e_a the path $I \to X : t \mapsto a$.

24.I. Is e_a a unit for multiplication of paths?

The same question in more detailed form:

24.J. For a path u with $u(0) = a$ is $e_a u = u$? For a path v with $v(1) = a$ is $v e_a = v$?

Problems *24.I* and *24.J* are similar to *24.C* and *24.D*, respectively.

24.3. Riddle. Extending this analogy, formulate and solve problems similar to *24.E.*

24.4. Prove that $e_a u = u$ implies $u = e_a$.

24.K. *The homotopy class of e_a is a unit for multiplication of homotopy classes of paths.*

Inverse

Recall that for a path u there is inverse path u^{-1} defined by $u^{-1}(t) = u(1-t)$ (see Section 10).

24.L. Is the inverse path inverse with respect to multiplication of paths?

In other words:

24.M. For a path u beginning in a and finishing in b is $uu^{-1} = e_a$ and $u^{-1}u = e_b$?

24.5. Prove that for a path u with $u(0) = a$ equality $uu^{-1} = e_a$ implies $u = e_a$.

24.6. Find a map $\varphi : I \to I$ such that $(uu^{-1}) = u \circ \varphi$ for any path u.

24.N. *For any path u the homotopy class of path u^{-1} is inverse to the homotopy class of u.*

We see that from the algebraic viewpoint multiplication of paths is terrible, but it defines multiplication of homotopy classes of paths, which has nice algebraic properties. The only unfortunate property is that the multiplication of homotopy classes of paths is not defined for any two classes.

24.O. Riddle. How to select a subset of the set of homotopy classes of paths to obtain a group?

25. Fundamental Group

Definition of Fundamental Group

Let X be a topological space, x_0 its point. A path in X which starts and ends at x_0 is called a *loop* in X at x_0. Denote by $\Omega(X, x_0)$ the set of loops in X at x_0. Denote by $\pi_1(X, x_0)$ the set of homotopy classes of loops in X at x_0.

Both $\Omega(X, x_0)$ and $\pi_1(X, x_0)$ are equipped with multiplication.

25.A. *For any topological space X and a point $x_0 \in X$ the set $\pi_1(X, x_0)$ of homotopy classes of loops at x_0 with multiplication defined above is a group.*

$\pi_1(X, x_0)$ is called the *fundamental group* of the space X with base point x_0. It was introduced by Poincaré and that is why it is called also *Poincaré group*. The letter π in its notation is also due to Poincaré.

Why Index 1?

The index 1 in the notation $\pi_1(X, x_0)$ appeared later than the letter π. It is related to one more name of the fundamental group: the first (or one-dimensional) homotopy group. There is an infinite series of groups $\pi_r(X, x_0)$ with $r = 1, 2, 3, \dots$ and the fundamental group is one of them. The higher-dimensional homotopy groups were defined by Witold Hurewicz in 1935, 30 years after the fundamental group was defined.

There is even a zero-dimensional homotopy group $\pi_0(X, x_0)$, but it is not a group, as a rule. It is the set of path-wise connected components of X. Although there is no natural multiplication in $\pi_0(X, x_0)$, unless X is equipped with some special additional structures, there is a natural 1 in $\pi_0(X, x_0)$. This is the component containing x_0.

Roughly speaking, the general definition of $\pi_r(X, x_0)$ is obtained from the definition of $\pi_1(X, x_0)$ by replacing I with the cube I^r.

25.B. Riddle. How to generalize all the previous problems of this section in such a way that in each of them I would be replaced by I^r?

High Homotopy Groups

Let X be a topological space and x_0 its point. A continuous map $I^r \to X$ which maps the boundary ∂I^r of I^r to x_0 is called a *spheroid of dimension* r of X at x_0. Two r-dimensional spheroids are said to be homotopic, if they are ∂I^r-homotopic. For spheroids u, v of X at x_0 of dimension $r \geq 1$ define their product uv by formula

$$uv(t_1, t_2, \dots, t_r) = \begin{cases} u(2t_1, t_2, \dots, t_r), & \text{if } t_1 \leq 1/2 \\ v(2t_1 - 1, t_2, \dots, t_r), & \text{if } t_1 \geq 1/2. \end{cases}$$

The set of homotopy classes of r-dimensional spheroids of a space X at x_0 is the r-th (or r-dimensional) homotopy group $\pi_r(X, x_0)$ of X at x_0. Thus,

$$\pi_r(X, x_0) = \pi(I^r, \partial I^r;\ X, x_0).$$

Multiplication of spheroids induces multiplication in $\pi_r(X, x_0)$, which makes $\pi_r(X, x_0)$ a group.

25.1. *For any X and x_0 the group $\pi_r(X, x_0)$ with $r \geq 2$ is Abelian.*

25.2. Riddle. For any X, x_0 and $r \geq 2$ present group $\pi_r(X, x_0)$ as the fundamental group of some space.

Circular loops

Let X be a topological space, x_0 its point. A continuous map $l : S^1 \to X$ such that[1] $l(1) = x_0$ is called a *(circular) loop* at x_0. Assign to each circular loop l the composition of l with the exponential map $I \to S^1 : t \mapsto e^{2\pi i t}$. This is a usual loop at the same point.

25.C. Prove that any loop can be obtained in this way from a circular loop.

Circular loops l_1, l_2 are said to be *homotopic* if they are 1-homotopic. Homotopy of a circular loop not fixed at x_0 is called a *free* homotopy.

25.D. Prove that circular loops are homotopic, iff the corresponding loops are homotopic.

25.3. What kind of homotopy of loops corresponds to free homotopy of circular loops?

25.4. Describe the operation with circular loops corresponding to the multiplication of paths.

25.5. Outline a construction of fundamental group based on circular loops.

Similarly, high-dimensional homotopy groups can be constructed not out of homotopy classes of maps $(I^r, \partial I^r) \to (X, x_0)$, but as

$$\pi(S^r, (1, 0, \dots, 0);\ X, x_0).$$

Another, also quite a popular way, is to define $\pi_r(X, x_0)$ as

$$\pi(D^r, \partial D^r;\ X, x_0).$$

25.6. Establish natural bijections

$$\pi(I^r, \partial I^r;\ X, x_0) \to \pi(D^r, \partial D^r;\ X, x_0) \to \pi(S^r, (1, 0, \dots, 0);\ X, x_0)$$

[1] Recall, that S^1 is considered as a subset of the plane R^2, which is identified in a canonical way with $\mathbb{C}$. Hence $1 \in \{z \in \mathbb{C} : |z| = 1\}$.

The Very First Calculations

25.E. Prove that $\pi_1(\mathbb{R}^n, 0)$ is a trivial group (i.e., consists of one element).

25:A. What about $\pi_r(\mathbb{R}^n, 0)$?

25.F. Generalize *25.E* to the cases suggested by *23.H* and *23.3*.

25.7. Calculate the fundamental group of an indiscrete space.

25.8. Calculate the fundamental group of the quotient space of disk D^2 obtained by identification of each $x \in D^2$ with $-x$.

25.G. Prove that $\pi_1(S^n, (1, 0, \dots, 0))$ with $n \geq 2$ is a trivial group.

Whether you have solved *25.G* or not, we would recommend you consider problems *25.G.1*, *25.G.3*, *25.G.4*, *25.G.5* and *25.G.6* designed to give an approach to *25.G*, warn about a natural mistake and prepare an important tool for further calculations of fundamental groups.

25.G.1. Prove that any loop $s : I \to S^n$, which does not fill the whole S^n (i.e., $s(I) \neq S^n$) is homotopic to the constant loop, provided $n \geq 2$. (Cf. Problem *23.6*.)

Warning: for any n there exists a loop filling S^n. See *7:I*

25.G.2. Is a loop filling S^2 homotopic to the constant loop?

25.G.3 ***Corollary of Lebesgue Lemma 13.V.*** *Let $s : I \to X$ be a path, and Γ be an open covering of a topological space X. There exists a sequence of points $a_1, \dots a_N \in I$ with $0 = a_1 < a_2 < \dots < a_{N-1} < a_N = 1$ such that $s([a_i, a_{i+1}])$ is contained in an element of Γ for each i.*

25.G.4. Prove that if $n \geq 2$ then for any path $s : I \to S^n$ there exists a subdivision of I into a finite number of subintervals such that the restriction of s to each of the subintervals is homotopic, via a homotopy fixed on the endpoints of the subinterval, to a map with nowhere dense image.

25.G.5. Prove that if $n \geq 2$ then any loop in S^n is homotopic to a loop which is not surjective.

25.G.6. Deduce *25.G* from *25.G.1* and *25.G.5*. Find all the points of the proof of *25.G* obtained in this way, where the condition $n \geq 2$ is used.

Fundamental Group of Product

25.H. *The fundamental group of the product of topological spaces is canonically isomorphic to the product of the fundamental groups of the factors:*

$$\pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

25.9. Prove that $\pi_1(\mathbb{R}^n \smallsetminus 0, (1, 0, \dots, 0))$ is trivial if $n \geq 3$

25:B. Prove the following generalization of *25.H*:

$$\pi_r(X \times Y, (x_0, y_0)) = \pi_r(X, x_0) \times \pi_r(Y, y_0).$$

Simply-Connectedness

A non-empty topological space X is said to be *simply connected* or *one-connected* if it is path-connected and any loop in it is homotopic to a constant map.

25.I. *For a path-connected topological space X the following statements are equivalent:*

(a) *X is simply connected,*
(b) *any continuous map $f : S^1 \to X$ is (freely) homotopic to a constant map,*
(c) *any continuous map $f : S^1 \to X$ can be extended to a continuous map $D^2 \to X$,*
(d) *any two paths $s_1, s_2 : I \to X$ connecting the same points x_0 and x_1 are homotopic.*

The following theorem implies Theorem *25.I*. However, since it treats a single loop, it can be applied to more situations. Anyway, proving *25.I*, one proves *25.J* in fact.

25.J. *Let X be a topological space and $s : I \to X$ be a loop. Then the following statements are equivalent:*

(a) *s is homotopic to the constant loop,*
(b) *the map $S^1 \to X : t \mapsto s(e^{2\pi i t})$ is* freely *homotopic to a constant map,*
(c) *the map $S^1 \to X : t \mapsto s(e^{2\pi i t})$ can be extended to a continuous map $D^2 \to X$,*
(d) *the paths $s_+, s_- : I \to X$ defined by formula $s_\varepsilon(t) = s(e^{\varepsilon \pi i t})$ are homotopic.*

25.J.1. ***Riddle.*** Proving that 4 statements are equivalent one has to prove at least 4 implications. What implications would you choose for the shortest proof of Theorem *25.J*?

25.10. Which of the following spaces are simply connected:

(a) a discrete space,
(b) an indiscrete space,
(c) $\mathbb{R}^n$,
(d) S^n,
(e) a convex set,
(f) a star convex set,
(g) $\mathbb{R}^n \smallsetminus 0$.

25.11. Prove that a topological space X, which is presented as the union of open simply connected sets U and V with simply connected $U \cap V$, is simply connected.

25.12. Show that the assumption that U and V are open is necessary in *25.11*.

25.13*. Let X be a topological space, U and V its open sets. Prove that if $U \cup V$ and $U \cap V$ are simply connected, then U and V are simply connected, too.

Fundamental Group of a Topological Group

Let G be a topological group. Given loops $u, v : I \to G$ starting at the unity $1 \in G$, let us define a loop $u \odot v : I \to G$ by the formula $u \odot v(t) = u(t) \cdot v(t)$, where $\cdot$ denotes the group operation in G.

25:C. Prove that the set $\Omega(G, 1)$ of all the loops in G starting at 1 equipped with the operation $\odot$ is a group.

25:D. Prove that the operation $\odot$ on $\Omega(G, 1)$ defines a group operation on $\pi_1(G, 1)$ and that this operation coincides with the standard group operation (defined by multiplication of paths).

25:D:1. For loops $u, v \to G$ starting at 1, find $(ue_1) \odot (e_1 v)$.

25:E. The fundamental group of a topological group is abelian.

25:F. Formulate and prove the analogues of Problems *25:C* and *25:D* for high homotopy groups and $\pi_0(G, 1)$.

26. The Role of Base Point

Overview of the Role of Base Point

Roughly, the role of base point may be described as follows:

- While the base point changes within the same path-connected component, the fundamental group remains in the same class of isomorphic groups.
- However, if the group is not commutative, it is impossible to find a natural isomorphism between fundamental groups at different base points even in the same path-connected component.
- Fundamental groups of a space at base points belonging to different path-connected components have no relation to each other.

In this section these will be demonstrated. Of course, with much more details.

Definition of Translation Maps

Let x_0 and x_1 be points of a topological space X, and let s be a path connecting x_0 with x_1. Denote by σ the homotopy class $[s]$ of s. Define a map $T_s : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by formula $T_s(\alpha) = \sigma^{-1}\alpha\sigma$.

> ***26.1.*** Prove that for any loop $a : I \to X$ representing $\alpha \in \pi_1(X, x_0)$ and a path $s : I \to X$ with $s(0) = x_0$ there exists a free homotopy $H : I \times I \to X$ between a and a loop representing $T_s(\alpha)$ such that $H(0,t) = H(1,t) = s(t)$ for $t \in I$.
>
> ***26.2.*** Let $a, b : I \to X$ be loops which are homotopic via a homotopy $H : I \times I \to X$ such that $H(0,t) = H(1,t)$ (i.e., H is a free homotopy of loops: at each moment $t \in I$ it keeps the end points of the path coinciding). Set $s(t) = H(0,t)$ (hence s is the path run over by the initial point of the loop under the homotopy). Prove that the homotopy class of b is the image of the homotopy class of a under $T_s : \pi_1(X, s(0)) \to \pi_1(X, s(1))$.

Properties of T_s

26.A. *T_s is a (group) homomorphism. (Recall that it means that $T_s(\alpha\beta) = T_s(\alpha)T_s(\beta)$.)*

26.B. *If u is a path connecting x_0 to x_1 and v is a path connecting x_1 with x_2 then $T_{uv} = T_v \circ T_u$. In other words the diagram*

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "T_u"] \arrow[dr, "T_{uv}"'] & \pi_1(X, x_1) \arrow[d, "T_v"] \\ & \pi_1(X, x_2) \end{tikzcd}$$

is commutative.

26.C. *If paths u and v are homotopic then $T_u = T_v$.*

26.D. $T_{e_a} = \mathrm{id} : \pi_1(X, a) \to \pi_1(X, a)$

26.E. $T_{s^{-1}} = T_s^{-1}$.

26.F. *T_s is an isomorphism for any path s.*

26.G. *For any points x_0 and x_1 lying in the same path-connected component of X groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic.*

26.H. *Let $f : X \to Y$ be a continuous map, x_0 and x_1 points of X connected by a path $s : I \to X$. Denote $f(x_0)$ by y_0 and $f(x_1)$ by y_1. Then the diagram*

$$\begin{array}{ccc}
\pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, y_0) \\
{\scriptstyle T_s}\downarrow & & \downarrow{\scriptstyle T_{f\circ s}} \\
\pi_1(X, x_1) & \xrightarrow{f_*} & \pi_1(Y, y_1)
\end{array}$$

is commutative, i.e., $T_{f\circ s} \circ f_* = f_* \circ T_s$.

Role of Path

26.I. If s is a loop representing an element σ of fundamental group $\pi_1(X, x_0)$ then T_s is the internal automorphism of $\pi_1(X, x_0)$ defined by $\alpha \mapsto \sigma^{-1}\alpha\sigma$.

26.J. *Let x_0 and x_1 be points of a topological space X belonging to the same path-connected component. Isomorphisms $T_s : \pi_1(X, x_0) \to \pi_1(X, x_1)$ do not depend on s, iff $\pi_1(X, x_0)$ is commutative.*

High Homotopy Groups

26.3. Riddle. Guess how T_s is generalized to $\pi_r(X, x_0)$ with any r.

Here is another form of the same question. We put it since it contains in its statement a greater piece of an answer.

26.4. Riddle. Given a path $s : I \to X$ with $s(0) = x_0$ and a spheroid $f : I^r \to X$ at x_0, how to cook up a spheroid at $x_1 = s(1)$ out of these?

26.5. Prove that for any path $s : I \to X$ and a spheroid $f : I^r \to X$ with $f(\operatorname{Fr} I^r) = s(0)$ there exists a homotopy $H : I^r \times I \to X$ of f such that $H(\operatorname{Fr} I^r \times t) = s(t)$ for any $t \in I$ and that the spheroid obtained by such a homotopy is unique up to homotopy and defines an element of $\pi_r(X, s(1))$ well-defined by the homotopy class of s and the element of $\pi_r(X, s(0))$ represented by f.

Of course, a solution of *26.5* gives an answer to *26.4* and *26.3*. The map $\pi_r(X, s(0)) \to \pi_r(X, s(1))$ defined by *26.5* is denoted by T_s. By *26.2* this T_s generalizes T_s defined in the beginning of the section for the case $r = 1$.

26.6. Prove that the properties of T_s formulated in Problems *26.A* – *26.G* hold true in all dimensions.

In Topological Group

In a topological group G there is another way to relate $\pi_1(G, x_0)$ with $\pi_1(G, x_1)$: there are homeomorphisms $L_g : G \to G : x \mapsto xg$ and $R_g : G \to G : x \mapsto gx$, so that there are the induced isomorphisms $(L_{x_0^{-1}x_1})_* : \pi_1(G, x_0) \to \pi_1(G, x_1)$ and $(R_{x_1 x_0^{-1}})_* : \pi_1(G, x_0) \to \pi_1(G, x_1)$.

26:A. Let G be a topological group, $s\, I \to G$ be a path. Prove that

$$T_s = (L_{s(0)^{-1}s(1)})_* = (R_{s(1)s(0)^{-1}}) : \pi_1(G, s(0)) \to \pi_1(G, s(1)).$$

26:B. Deduce from *26:A* that the fundamental group of a topological group is abelian (cf. *25:E*).

> ***26:1.*** Prove that the fundamental groups of the following spaces are commutative:
>
> (a) the space of non-degenerate real $n \times n$ matrices $GL(n, \mathbb{R}) = \{A \mid \det A \neq 0\}$;
> (b) the space of orthogonal real $n \times n$ matrices $O(n, \mathbb{R}) = \{A \mid A \cdot ({}^tA) = 1\}$;
> (c) the space of special unitary complex $n \times n$ matrices $SU(n) = \{A \mid A \cdot ({}^t\bar{A}) = 1, \det A = 1\}$
> (d) $\mathbb{R}P^n$;
> (e) $V_{k,n} = \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^n)$;

26:C. Generalize *26:A* and *26:B* to a homogeneous space G/H.

26:D. Riddle. What are the counterparts for *26:A* and *26:B* and *26:C* for high homotopy groups?

27. Covering Spaces

Definition

Let X, B topological spaces, $p : X \to B$ a continuous map. Assume that p is surjective and each point of B possesses a neighborhood U such that the preimage $p^{-1}(U)$ of U is presented as a disjoint union of open sets V_α and p maps each V_α homeomorphically onto U. Then $p : X \to B$ is called a *covering,*(of the space B), the space B is called the *base* of this covering, X is called the *covering space* for B and the *total space* of the covering. Neighborhoods like U are said to be *trivially covered.* The map p is called also a *covering map,* or a *covering projection.*

27.A. Let B be a topological space and F be a discrete space. Prove that the projection $pr_B : B \times F \to B$ is a covering.

The following statement shows that in a sense locally any covering is organized as the covering of *27.A*.

27.B. A continuous surjective map $p : X \to B$ is a covering, iff for each point a of B the preimage $p^{-1}(a)$ is discrete and there exist a neighborhood U of a and a homeomorphism $h : p^{-1}(U) \to U \times p^{-1}(a)$ such that $p|_{p^{-1}(U)} = pr_U \circ h$.

However, the coverings of *27.A* are not interesting. They are said to be *trivial.* Here is the first really interesting example.

27.C. Prove that $\mathbb{R} \to S^1 : x \mapsto e^{2\pi i x}$ is a covering.

To distinguish the most interesting examples, a covering with a connected total space is called a covering *in narrow sense.* Of course, the covering of *27.C* is a covering in a narrow sense.

Local Homeomorphisms Versus Coverings

A map $f : X \to Y$ is said to be *locally homeomorphic* if each point of X has a neighborhood U such that the image $f(U)$ is open in Y and the map $U \to f(U)$ defined by f is a homeomorphism.

27.1. Any covering is locally homeomorphic.

27.2. Show that there exists a locally homeomorphic map which is not a covering.

27.3. Prove that a restriction of a locally homeomorphic map to an open set is locally homeomorphic.

27.4. For which subsets of $\mathbb{R}$ is the restriction of the map of Problem *27.C* a covering.

27.5. Find nontrivial coverings $X \to B$ with X homeomorphic to B and prove that they satisfy the definition of covering.

Number of Sheets

Let $p : X \to B$ be a covering. The cardinality (i.e., number of points) of the preimage $p^{-1}(a)$ of a point $a \in B$ is called the *multiplicity* of the covering at a or the *number of sheets of the covering over* a.

27.D. *If the base of a covering is connected then the multiplicity of the covering at a point does not depend on the point.*

In the case of covering with connected base the multiplicity is called the *number of sheets* of the covering. If the number of sheets is n then the covering is said to be *n-sheeted* and we talk about *n-fold* covering. Of course, unless the covering is trivial, it is impossible to distinguish the sheets of it, but this does not prevent us from speaking about the number of sheets.

More Examples

27.E. Prove that $\mathbb{R}^2 \to S^1 \times \mathbb{R} : (x, y) \mapsto (e^{2\pi i x}, y)$ is a covering.

27.F. Prove that $\mathbb{C} \to \mathbb{C} \smallsetminus 0 \, : \, z \mapsto e^z$ is a covering.

> ***27.6. Riddle.*** In what sense the coverings of *27.E* and *27.F* are the same? Define an appropriate equivalence relation for coverings.

27.G. Prove that $\mathbb{R}^2 \to S^1 \times S^1 : (x, y) \mapsto (e^{2\pi i x}, e^{2\pi i y})$ is a covering.

27.H. Prove that for any natural n $S^1 \to S^1 : z \mapsto z^n$ is an n-fold covering.

> ***27.7.*** Prove that for any natural n $\mathbb{C} \smallsetminus 0 \to \mathbb{C} \smallsetminus 0 : z \mapsto z^n$ is an n-fold covering.

27.I. Prove that for any natural p and q $S^1 \times S^1 \to S^1 \times S^1 : (z, w) \mapsto (z^p, w^q)$ is a covering. Find its number of sheets.

> ***27.8.*** Prove that if $p : X \to B$ and $p' : X' \to B'$ are coverings, then $p \times p' : X \times X' \to B \times B'$ is a covering.
>
> ***27.9.*** Let $p : X \to Y$ and $q : Y \to Z$ be coverings. Prove that if q is finitely-fold then $q \circ p : x \to Y$ is a covering.
>
> ***27.10*.*** Show that the assumption about the number of sheets in Problem *27.9* is necessary.
>
> ***27.11.*** Let X be a topological space, which can be presented as a union of open connected sets U and V. Prove that if $U \cap V$ is disconnected then X has a connected infinite-fold covering space

27.J. Prove that the natural projection $S^n \to \mathbb{R}P^n$ is a two-fold covering.

27.K. Is $(0, 3) \to S^1 : x \mapsto e^{2\pi i x}$ a covering?

27.L. Is the projection $\mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto x$ a covering? Indeed, why not take an open interval $(a, b) \subset \mathbb{R}$ as a trivially covered neighborhood: its preimage $(a, b) \times \mathbb{R}$ is the union of open intervals $(a, b) \times \{y\}$ which are projected homeomorphically by the projection $(x, y) \mapsto x$ onto (a, b)?

> ***27.12.*** Find coverings of Möbius strip by cylinder. What numbers can you realize as the number of sheets for such a covering?

27.13. Find non-trivial coverings of Möbius strip by itself. What numbers can you realize as the number of sheets for such a covering?

27.14. Find a two-fold covering of the Klein bottle by torus. Cf. Problem *18.14*.

27.15. Find coverings of the Klein bottle by plane $\mathbb{R}^2$, cylinder $S^1 \times \mathbb{R}$ and a non-trivial covering by itself. What numbers can you realize as the numbers of sheets for such coverings?

27.16. Construct a covering of the Klein bottle by $\mathbb{R}^2$. Describe explicitly the partition of $\mathbb{R}^2$ into preimages of points under this covering.

27.17. Construct a d-fold covering of a sphere with p handles by a sphere with $1 + d(p-1)$ handles.

27.18. Find a covering of a sphere with any number of crosscaps by a sphere with handles.

Universal Coverings

A covering $p : X \to B$ is said to be *universal* if X is simply connected. The appearance of word *universal* in this context will be explained below in Section 30.

27.M. Which coverings of the problems stated above in this section are universal?

Theorems on Path Lifting

Let $p : X \to B$ and $f : A \to B$ be arbitrary maps. A map $g : A \to X$ such that $p \circ g = f$ is said to *cover* f or be a *lifting* of f. A lot of topological problems can be phrased in terms of finding a continuous lifting of some continuous map. Problems of this sort are called *lifting problems.* They may involve additional requirements such as having a lifting already given on some subspace.

27.N. Prove that the identity map $S^1 \to S^1$ does not admit a continuous lifting with respect to the covering $\mathbb{R} \to S^1 : x \mapsto e^{2\pi i x}$. (In other words, there exists no continuous map $g : S^1 \to \mathbb{R}$ such that $e^{2\pi i g(x)} = x$ for $x \in S^1$.)

27.O Path Lifting Theorem. *Let $p : X \to B$ be a covering, $x_0 \in X$, $b_0 \in B$ be points such that $p(x_0) = b_0$. Then for any path $s : I \to B$ starting at b_0 there exists a unique path $\tilde{s} : I \to X$ starting at x_0 and being a lifting of s. (In other words, there exists a unique path $\tilde{s} : I \to X$ with $\tilde{s}(0) = x_0$ and $p \circ \tilde{s} = s$.)*

> *27.O.1* ***Lemma 1.*** *Let $p : X \to B$ be a trivial covering. Then for any continuous map f of any space A to B there exists a continuous lifting $\tilde{f} : A \to X$.*
>
> *27.O.2* ***Lemma 2.*** *Let $p : X \to B$ be a trivial covering and $x_0 \in X$, $b_0 \in B$ be points such that $p(x_0) = b_0$. Then for any continuous map f of a space A to B mapping a point a_0 to b_0 a continuous lifting $\tilde{f} : A \to X$ with $\tilde{f}(a_0) = x_0$ is unique.*

27.O.3 **Lemma 3.** [2] *Let $p : X \to B$ be a covering, A a connected space. If $f, g : A \to X$ are continuous maps coinciding in some point and $p \circ f = p \circ g$, then $f = g$.*

27.19. If in the Problem *27.O.2* one replaces x_0, b_0 and a_0 by pairs of points, then it may happen that the lifting problem has no solution $\tilde{f}$ with $\tilde{f}(a_0) = x_0$. Formulate a condition necessary and sufficient for existence of such a solution.

27.20. What goes wrong with the Path Lifting Theorem *27.O* for the local homeomorphism of Problem *27.K*?

27.21. Consider the covering $\mathbb{C} \to \mathbb{C} \smallsetminus 0 : z \mapsto e^z$. Find liftings of the paths $u(t) = 2 - t$, $v(t) = (1 + t)e^{2\pi i t}$, and their product uv.

27.22. Prove that any covering $p : X \to B$ with simply connected B and path connected X is a homeomorphism.

27.P Homotopy Lifting Theorem. *Let $p : X \to B$ be a covering, $x_0 \in X$, $b_0 \in B$ be points such that $p(x_0) = b_0$. Let $u, v : I \to B$ be paths starting at b_0 and $\tilde{u}, \tilde{v} : I \to X$ be the lifting paths for u, v starting at x_0. If the paths u and v are homotopic then the covering paths $\tilde{u}$ and $\tilde{v}$ are homotopic.*

27.Q Corollary. *Under the assumptions of Theorem 27.P, the covering paths $\tilde{u}$ and $\tilde{v}$ have the same final point (i.e., $\tilde{u}(1) = \tilde{v}(1)$).*

Notice that in *27.P* and *27.Q* paths are assumed to share the initial point x_0. In the statement of *27.Q* we emphasize that then they share also the final point.

27.R Corollary of 27.Q. *Let $p : X \to B$ be a covering and $s : I \to B$ be a loop. If there exists a lifting $\tilde{s} : I \to X$ of s with $\tilde{s}(0) \neq \tilde{s}(1)$ (i.e., there exists a covering path which is not a loop), then s is not homotopic to a constant loop.*

27.23. Prove that if a pathwise connected space B has a non trivial pathwise connected covering space, then the fundamental group of B is not trivial.

27.24. What corollaries can you deduce from *27.23* and the examples of coverings presented above in this Section?

High-Dimensional Homotopy Groups of Covering Space

27:A. *Let $p : X \to B$ be a covering. Then for any continuous map $s : I^n \to B$ and a lifting $u : I^{n-1} \to X$ of the restriction $s|_{I^{n-1}}$ there exists a unique lifting of s extending u.*

27:B. *For any covering $p : X \to B$ and points $x_0 \in X$, $b_0 \in B$ such that $p(x_0) = b_0$ the homotopy groups $\pi_r(X, x_0)$ and $\pi_r(B, b_0)$ with $r > 1$ are canonically isomorphic.*

27:C. Prove that homotopy groups of dimensions greater than 1 of circle, torus, Klein bottle and Möbius strip are trivial.

[2]This is rather a generalization of the uniqueness, than a necessary step of the proof. But a good lemma should emphasize the real contents of the proof, and a generalization is one of the best ways to do this.

28. Calculations of Fundamental Groups Using Universal Coverings

Fundamental Group of Circle

For an integer n denote by s_n the loop in S^1 defined by formula $s_n(t) = e^{2\pi i n t}$. The initial point of this loop is 1. Denote the homotopy class of s_1 by α. Thus $\alpha \in \pi_1(S^1, 1)$. Clearly, s_n represents α^n.

28.A. What are the paths in $\mathbb{R}$ starting at $0 \in \mathbb{R}$ and covering the loops s_n with respect to the universal covering $\mathbb{R} \to S^1$?

28.B. *The homomorphism $\mathbb{Z} \to \pi_1(S^1, 1)$ defined by formula $n \mapsto \alpha^n$ is an isomorphism.*

28.B.1. Rephrase the statement that the homomorphism of Theorem *28.B* is surjective in terms of loops and loop homotopies.

28.B.2. Prove that a loop $s : I \to S^1$ starting at 1 is homotopic to s_n if the path $\tilde{s} : I \to \mathbb{R}$ covering s and starting at $0 \in \mathbb{R}$ finishes at $n \in \mathbb{R}$ (i.e., $\tilde{s}(1) = n$).

28.B.3. Rephrase the statement that the homomorphism of Theorem *28.B* is injective in terms of loops and loop homotopies.

28.B.4. Prove that if loop s_n is homotopic to constant then $n = 0$.

28.1. What is the image under the isomorphism of Theorem *28.B* of the homotopy class of loop $t \mapsto e^{2\pi i t^2}$?

For a loop $s : I \to S^1$ starting at 1 take the covering path $\tilde{s} : I \to \mathbb{R}$ starting at 0. By Theorem *27.O* such a path exists and is unique. Its final point belongs to the preimage of 1 under the universal covering projection $\mathbb{R} \to S^1 : x \mapsto e^{2\pi i x}$. Hence, this final point is an integer n. By *27.Q*, it does not change if s is replaced by a homotopic loop. Therefore, this construction provides a well-defined map $\pi_1(S^1, 1) \to \mathbb{Z}$ assigning n to $[s]$. Denote this map by deg.

28.2. Prove that deg is an isomorphism inverse to the isomorphism of Theorem *28.B*

28.C Corollary of Theorem 28.B. *The fundamental group of $(S^1)^n$ is a free abelian group of rank n (i.e., isomorphic to $\mathbb{Z}^n$).*

28.D. On torus $S^1 \times S^1$ find two loops whose homotopy classes generate the fundamental group of the torus.

28.E Corollary of Theorem 28.B. *The fundamental group of punctured plane $\mathbb{R}^2 \smallsetminus 0$ is an infinite cyclic group.*

28.3. Solve Problems *28.C* – *28.E* without reference to Theorems *28.B* and *25.H*, but using explicit constructions of the corresponding universal coverings.

Fundamental Group of Projective Space

The fundamental group of the projective line is an infinite cyclic group. It is calculated in the previous subsection, since the projective line is a circle. The zero-dimensional projective space is a point, hence its fundamental group is trivial. Here we calculate the fundamental groups of projective spaces of all other dimensions.

Let $n \geq 2$ and $l : I \to \mathbb{R}P^n$ be a loop covered by a path $\tilde{l} : I \to S^n$ which connects two antipodal points, say the poles $P_+ = (1, 0, \dots, 0)$ and $P_- = (-1, 0, \dots, 0)$, of S^n. Denote by λ the homotopy class of l. It is an element of $\pi_1(\mathbb{R}P^n, (1 : 0 : \dots : 0))$.

28.F. *For any $n \geq 2$ group $\pi_1(\mathbb{R}P^n, (1 : 0 : \dots : 0))$ is a cyclic group of order 2. It consists of two elements: λ and 1.*

> *28.F.1* **Lemma.** *Any loop in $\mathbb{R}P^n$ at $(1 : 0 : \dots : 0)$ is homotopic either to l or constant. This depends on whether the covering path of the loop connects the poles P_+ and P_-, or is a loop.*
>
> ***28.4.*** Where in the proofs of Theorem *28.F* and Lemma *28.F.1* the assumption $n \geq 2$ is used?

Fundamental Groups of Bouquet of Circles

Consider a family of topological spaces $\{X_\alpha\}$. In each of the spaces let a point x_α be marked. Take the sum $\amalg_\alpha X_\alpha$ and identify all the marked points. The resulting quotient space is called the *bouquet* of $\{X_\alpha\}$ and denoted by $\vee_\alpha X_\alpha$. Hence *bouquet of q circles* is a space which is a union of q copies of circle. The copies meet in a single common point, and this is the only common point for any two of them. The common point is called the *center* of the bouquet.

Denote the bouquet of q circles by B_q and its center by c. Let $u_1, \dots, u_q$ be loops in B_q starting at c and parametrizing the q copies of circle comprising B_q. Denote the homotopy class of u_i by α_i.

28.G. *$\pi_1(B_q, c)$ is a free group freely generated by $\alpha_1, \dots, \alpha_q$.*

Algebraic Digression. Free Groups

Recall that a group G is a free group freely generated by its elements $a_1, \dots, a_q$ if:

- each its element $x \in G$ can be expressed as a product of powers (with positive or negative integer exponents) of $a_1, \dots, a_q$, i.e.,
$$x = a_{i_1}^{e_1} a_{i_2}^{e_2} \dots a_{i_n}^{e_n}$$
and
- this expression is unique up to the following trivial ambiguity: one may insert or delete factors $\alpha_i \alpha_i^{-1}$ and $\alpha_i^{-1} \alpha_i$ or replace a_i^m by $a_i^p a_i^q$ with $p + q = m$.

28.H. *A free group is defined up to isomorphism by the number of its free generators.*

The number of free generators is called the *rank* of the free group. For a standard representative of the isomorphism class of free groups of rank q one can take the group of words in alphabet of q letters $a_1, \dots, a_q$ and their inverses $a_1^{-1}, \dots, a_q^{-1}$. Two words represent the same element of the group, iff they can be obtained from each other by a sequence of insertions or deletions of fragments $a_i a_i^{-1}$ and $a_i^{-1} a_i$. This group is denoted by $\mathbb{F}(a_1, \dots, a_q)$, or just $\mathbb{F}_q$, when the notations for the generators are not to be emphasized.

28.I. *Each element of $\mathbb{F}(a_1, \dots, a_q)$ has a unique shortest representative. This is a word without fragments that could have been deleted.*

The number of letters in the shortest representative of $x \in \mathbb{F}(a_1, \dots, a_q)$ is called the *length* of x and denoted by $l(x)$. Of course, this number is not well defined, unless the generators are fixed.

> ***28.5.*** Show that an automorphism of $\mathbb{F}_q$ can map $x \in \mathbb{F}_q$ to an element with different length. For what value of q does such an example not exist? Is it possible to change the length in this way arbitrarily?

28.J. *A group G is a free group freely generated by its elements $a_1, \dots, a_q$ if and only if every map of the set $\{a_1, \dots, a_q\}$ to any group X can be extended to a unique homomorphism $G \to X$.*

Sometimes Theorem *28.J* is taken as a definition of free group. (A definition of this sort emphasizes relations among different groups, rather than the internal structure of a single group. Of course, relations among groups can tell everything about internal affairs of each group.)

Now we can reformulate Theorem *28.G* as follows:

28.K. *The homomorphism*

$$\mathbb{F}(a_1, \dots, a_q) \to \pi_1(B_q, c)$$

taking a_i to α_i for $i = 1, \dots, q$ is an isomorphism.

First, for the sake of simplicity let us agree to restrict ourselves to the case of $q = 2$. It would allow us to avoid superfluous complications in notations and pictures. This is the simplest case, which really represents the general situation. The case $q = 1$ is too special.

To take advantages of this, let us change notations. Put $B = B_2$, $u = u_1$, $v = u_2$, $\alpha = \alpha_1$, $\beta = \alpha_2$.

Now Theorem *28.K* looks as follows:

The homomorphism $\mathbb{F}(a, b) \to \pi(B, c)$ taking a to α and b to β is an isomorphism.

This theorem can be proved like Theorems *28.B* and *28.F*, provided the universal covering of B is known.

Universal Covering for Bouquet of Circles

Denote by U and V the points antipodal to c on the circles of B. Cut B at these points, removing U and V and putting instead each of them two new points. Whatever this operation is, its result is a cross K, which is the union of four closed segments with a common end point c. There appears a natural map $P : K \to B$, which takes the center c of the cross to the center c of B and maps homeomorphically the rays of the cross onto half-circles of B. Since the circles of B are parametrized by loops u and v, the halves of each of the circles are ordered: the corresponding loop passes first one of the halves and then the other one. Denote by U^+ the point of $P^{-1}(U)$, which belongs to the ray mapped by P onto the second half of the circle, and by U^- the other point of $P^{-1}(U)$. Similarly denote points of $P^{-1}(V)$ by V^+ and V^-.

The restriction of P to $K \smallsetminus \{U^+, U^-, V^+, V^-\}$ maps this set homeomorphically onto $B \smallsetminus \{U, V\}$. Therefore P provides a covering of $B \smallsetminus \{U, V\}$. But it fails to be a covering at U and V: each of this points has no trivially covered neighborhood. Moreover, the preimage of each of these points consists of 2 points (the end points of the cross), where P is not even a local homeomorphism. To recover, we may attach a copy of K at each of the 4 end points of K and extend P in a natural way to the result. But then new 12 end points, where the map is not a local homeomorphism, appear. Well, we repeat the trick and recover the property of being a local homeomorphism at each of the new 12 end points. Then we have to do this at each of the new 36 points, etc. But if we repeat this infinitely many times, all the bad points are turned to nice ones.[3]

28.L. Formalize the construction of a covering for B described above.

Consider $\mathbb{F}(a, b)$ as a discrete topological space. Take $K \times \mathbb{F}(a, b)$. It can be thought of as a collection of copies of K enumerated by elements of $\mathbb{F}(a, b)$. Topologically this is a disjoint sum of the copies, since $\mathbb{F}(a, b)$ is equipped with discrete topology. In $K \times \mathbb{F}(a, b)$ identify points (U^-, g) with (U^+, ga) and (V^-, g) with (V^+, gb) for each $g \in \mathbb{F}(a, b)$. Denote the resulting quotient space by X.

---

[3]This sounds like a story about a battle with a dragon, but the happy ending demonstrates that modern mathematicians have a magic power of the sort that the heros of tales could not dream of. Indeed, we meet a dragon K with 4 heads, cut off all the heads, but, according to the old tradition of the genre, 3 new heads appear in place of each of the original heads. We cut off them, and the story repeats. We do not even try to prevent this multiplication of heads. We just fight. But contrary to the real heros of tales, we act outside of Time and hence have no time restrictions. Thus after infinite repetitions of the exercise with an exponentially growing number of heads we succeed! No heads left! This is a typical story about an infinite construction in mathematics. Sometimes, as in our case, such a construction can be replaced by a finite one, but which deals with infinite objects. However, there are important constructions, in which an infinite fragment is unavoidable.

28.M. The composition of the natural projection $K \times \mathbb{F}(a,b) \to K$ and $P : K \to B$ defines a continuous quotient map $p : X \to B$.

28.N. $p : X \to B$ is a covering.

28.O. X is path-connected. For any $g \in \mathbb{F}(a,b)$ there exists a path connecting $(c,1)$ with (c,g) and covering loop obtained from g by substituting a by u and b by v.

28.P. X is simply connected.

29. Fundamental Group and Continuous Maps

Induced Homomorphisms

Let $f : X \to Y$ be a continuous map of a topological space X to a topological space Y. Let $x_0 \in X$ and $y_0 \in Y$ be points such that $f(x_0) = y_0$. The latter property of f is expressed by saying that f *maps pair* (X, x_0) *to pair* (Y, y_0) and writing $f : (X, x_0) \to (Y, y_0)$.

Denote by $f_\sharp$ the map $\Omega(X, x_0) \to \Omega(Y, y_0)$ defined by formula $f_\sharp(s) = f \circ s$. This map assigns to a loop its composition with f.

29.A. $f_\sharp$ maps homotopic loops to homotopic loops.

Therefore $f_\sharp$ induces a map $\pi_1(X, x_0) \to \pi_1(Y, y_0)$. The latter is denoted by f_*.

29.B. *$f_* : \pi(X, x_0) \to \pi_1(Y, y_0)$ is a homomorphism for any continuous map $f : (X, x_0) \to (Y, y_0)$.*

$f_* : \pi(X, x_0) \to \pi_1(Y, y_0)$ is called a *homomorphism induced* by f.

29.C. *Let $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$ be (continuous) maps. Then*

$$(g \circ f)_* = g_* \circ f_* : \pi_1(X, x_0) \to \pi_1(Z, z_0).$$

29.D. *Let $f, g : (X, x_0) \to (Y, y_0)$ be continuous maps homotopic via a homotopy fixed at x_0. Then $f_* = g_*$.*

29.E. Riddle. How to generalize Theorem *29.D* to the case of freely homotopic f and g?

> ***29.1.*** Prove that the map $\mathbb{C} \smallsetminus 0 \to \mathbb{C} \smallsetminus 0 : z \mapsto z^3$ is not homotopic to the identity map $\mathbb{C} \smallsetminus 0 \to \mathbb{C} \smallsetminus 0 : z \mapsto z$.
>
> ***29.2.*** Let X be a subset of $\mathbb{R}^n$. Prove that a if a continuous map $f : X \to Y$ is extentable to a continuous map $\mathbb{R}^n \to Y$ then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is the trivial homomorphism (i.e., maps everything to 1) for any $x_0 \in X$.
>
> ***29.3.*** Prove that a Hausdorff space, which contains an open set homeomorphic to $S^1 \times S^1 \smallsetminus (1, 1)$, has an infinite non-cyclic fundamental group.
>
> > *29.3.1.* Prove that a space X satisfying the conditions of *29.3* can be continuously mapped to a space with infinite non-cyclic fundamental group in such a way that the map would induce an epimorphism of $\pi_1(X)$ onto this infinite group.
>
> ***29.4.*** Prove that the fundamental group of the space $GL(n, \mathbb{C})$ of complex $n \times n$-matrices with non-zero determinant is infinite.
>
> > *29.4.1.* Construct continuous maps $S^1 \to GL(n, \mathbb{C}) \to S^1$, whose composition is the identity.

Fundamental Theorem of High Algebra

Here our goal is to prove the following theorem, which at first glance has no relation to fundamental group.

29.F Fundamental Theorem of High Algebra. *Any polynomial of positive degree in one variable with complex coefficients has a complex root.*

With more details:

Let $p(z) = z^n + a_1 z^{n-1} + \dots + a_n$ be a polynomial of degree $n > 0$ in z with complex coefficients. Then there exists a complex number w such that $p(w) = 0$.

Although it is formulated in an algebraic way and called "The Fundamental Theorem of High Algebra," it has no purely algebraic proof. Its proofs are based either on topological arguments or use complex analysis. This is because the field $\mathbb{C}$ of complex numbers cannot be described in purely algebraic terms: all its descriptions involve a sort of completion construction, cf. Section 15.

29.F.1 ***Reduction to Problem on a Map.*** Deduce Theorem *29.F* from the following statement:

For any complex polynomial $p(z)$ of a positive degree the zero does not belong to the image of the map $\mathbb{C} \to \mathbb{C} : z \mapsto p(z)$. In other words, the formula $z \mapsto p(z)$ defines a map $\mathbb{C} \to \mathbb{C} \smallsetminus 0$.

29.F.2 ***Estimate of Reminder.*** Let $p(z) = z^n + a_1 z^{n-1} + \dots + a_n$ be a complex polynomial, $q(z) = z^n$ and $r(z) = p(z) - q(z)$. Then there exists a positive number R such that $|r(z)| < |q(z)| = R^n$ for any z with $|z| = R$

29.F.3 ***Lemma on Lady with Doggy.*** A lady $q(z)$ and her dog $p(z)$ walk on punctured plane $\mathbb{C} \smallsetminus 0$ periodically (i.e., say, with $z \in S^1$). Prove that if the lady does not let the dog to run further than by $|q(z)|$ from her then the doggy loop $S^1 \to \mathbb{C} \smallsetminus 0 : z \mapsto p(z)$ is homotopic to the lady loop $S^1 \to \mathbb{C} \smallsetminus 0 : z \mapsto q(z)$.

29.F.4 ***Lemma for Dummy.*** If $f : X \to Y$ is a continuous map and $s : S^1 \to X$ is a loop homotopic to the trivial one then $f \circ s : S^1 \to Y$ is also homotopic to trivial.

Generalization of Intermediate Value Theorem

29.G. Riddle. How to generalize Intermediate Value Theorem *9.S* to the case of maps $f : D^2 \to \mathbb{R}^2$?

29.5. Let $f : D^2 \to \mathbb{R}^2$ be a continuous map which leaves fixed each point of the bounding circle S^1. Then $f(D^2) \supset D^2$.

29.H. Let $f : D^2 \to \mathbb{R}^2$ be a continuous map. If $f(S^1)$ does not contain $a \in \mathbb{R}^2$ and the circular loop $f| : S^1 \to \mathbb{R}^2 \smallsetminus a$ defines a nontrivial element of $\pi_1(\mathbb{R}^2 \smallsetminus a)$ then there exists $x \in D^2$ such that $f(x) = a$.

29.6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous map such that $|f(x) - x| \le 1$. Prove that f is a surjection.

29.7. Let $u, v\ I \to I \times I$ be two paths such that $u(0) = (0,0)$, $u(1) = (1,1)$ and $v(0) = (0,1)$, $v(1) = (1,0)$. Prove that $u(I) \cap v(I) \neq \emptyset$.

29.7.1. Let u, v be as in *29.7.* Denote by w the map $I^2 \to \mathbb{R}^2$ defined by $w(x,y) = u(x) - v(y)$. Prove that $0 \in \mathbb{R}^2$ is a value of w.

29.8. Let C be a smooth simple closed curve on the plane with two inflection points. Prove that there is a line intersecting C in four points a, b, c, d with segments $[a,b]$, $[b,c]$ and $[c,d]$ of the same length.

Winding Number

As we know (see *28.E*), the fundamental group of the punctured plane $\mathbb{R}^2 \smallsetminus 0$ is $\mathbb{Z}$. There are two isomorphisms which differ by multiplication by -1. We choose the one which maps the homotopy class of the loop $t \mapsto (\cos 2\pi t, \sin 2\pi t)$ to $1 \in \mathbb{Z}$. In terms of circular loops, the isomorphism means that to any loop $f : S^1 \to \mathbb{R}^2 \smallsetminus 0$ we associate an integer. It is the number of times the loop goes arround 0 in the counter-clockwise direction.

Now we change the viewpoint in this consideration, and fix the loop, but vary the point. Let $f : S^1 \to \mathbb{R}^2$ be a circular loop and $x \in \mathbb{R}^2 \smallsetminus f(S^1)$. Then f defines an element of $\pi_1(\mathbb{R}^2 \smallsetminus x) = \mathbb{Z}$ (we choose basically the same identification of $\pi_1(\mathbb{R}^2 \smallsetminus x)$ with $\mathbb{Z}$ assigning 1 to the homotopy class of $t \mapsto x + (\cos 2\pi t, \sin 2\pi t)$). This number is denoted by $\operatorname{ind}(f,x)$ and called *winding number* or *index* of x with respect to f.

29:A. Let $f : S^1 \to \mathbb{R}^2$ be a loop and $x, y \in \mathbb{R}^2 \smallsetminus f(S^1)$. Prove that if $\operatorname{ind}(f,x) \neq \operatorname{ind}(f,y)$ then any path connecting x and y in $\mathbb{R}^2$ meets $f(S^1)$.

29:B. Find a loop $f : S^1 \to \mathbb{R}^2$ such that there exist $x, y \in \mathbb{R}^2 \smallsetminus f(S^1)$ with $\operatorname{ind}(f,x) = \operatorname{ind}(f,y)$, but lying in different connected components of $\mathbb{R}^2 \smallsetminus f(S^1)$.

29:C. Prove that for any ray R radiating from x the number of points in $f^{-1}(R)$ is not less than $|\operatorname{ind}(f,x)|$.

Borsuk-Ulam Theorem

29:D One-Dimensional Borsuk-Ulam. For each continuous map $f : S^1 \to \mathbb{R}^1$ there exists $x \in S^1$ such that $f(x) = f(-x)$.

29:E Two-Dimensional Borsuk-Ulam. For each continuous map $f : S^2 \to \mathbb{R}^2$ there exists $x \in S^2$ such that $f(x) = f(-x)$.

29:E:1 ***Lemma.*** If there exists a continuous map $f : S^2 \to \mathbb{R}^2$ with $f(x) \neq f(-x)$ for any $x \in S^2$ then there exists a continuous map $\varphi : \mathbb{R}P^2 \to \mathbb{R}P^1$ inducing a non-zero homomorphism $\pi_1(\mathbb{R}P^1) \to \pi_1(\mathbb{R}P^2)$.

29:1. Prove that at each instant of time, there is a pair of antipodal points on the earth's surface where the pressures and also the temperatures are equal.

Theorems *29:D* and *29:E* are special cases of the following general theorem. We do not assume the reader to be ready to prove Theorem *29:F* in the full generality, but is there another easy special case?

29:F Borsuk-Ulam Theorem. For each continuous map $f : S^n \to \mathbb{R}^n$ there exists $x \in S^n$ such that $f(x) = f(-x)$.

30. Covering Spaces via Fundamental Groups

Homomorphisms Induced by Covering Projections

30.A. Let $p : X \to B$ be a covering, $x_0 \in X$, $b_0 = p(x_0)$. Then $p_* : \pi_1(X, x_0) \to \pi_1(B, b_0)$ is a monomorphism. Cf. *27.L*.

The image of the monomorphism $p_* : \pi_1(X, x_0) \to \pi_1(B, b_0)$ induced by a covering projection $p : X \to B$ is called the *group of covering p with base point* x_0.

30.B. Riddle Lifting Loops. Describe loops in the base space of a covering, whose homotopy classes belong to the group of the covering, in terms provided by Path Lifting Theorem *27.J*.

30.C. Let $p : X \to B$ be a covering, let $x_0, x_1 \in X$ belong to the same path-component of X, and $b_0 = p(x_0) = p(x_1)$. Then $p_*(\pi_1(X, x_0))$ and $p_*(\pi_1(X, x_1))$ are conjugate subgroups of $\pi_1(B, b_0)$ (i.e. there exists an element α of $\pi_1(B, b_0)$ such that $p_*(\pi_1(X, x_1)) = \alpha^{-1} p_*(\pi_1(X, x_0))\alpha$).

30.D. Let $p : X \to B$ be a covering, $x_0 \in X$, $b_0 = p(x_0)$. Let $\alpha \in \pi_1(B, b_0)$. Then there exists $x_1 \in p^{-1}(b_0)$ such that $p_*(\pi_1(X, x_1)) = \alpha^{-1} p_*(\pi_1(X, x_0))\alpha$.

30.E. Let $p : X \to B$ be a covering in a narrow sense and $G \subset \pi_1(B, b_0)$ be the group of this covering with base point x_0. A subgroup $H \subset \pi_1(B, b_0)$ is a group of the same covering, iff it is conjugate to G.

Number of Sheets

30.F Number of Sheets and Index of Subgroup. Let $p : X \to B$ be a covering in narrow sense with finite number of sheets. Then the number of sheets is equal to the index of the group of this covering.

30.G Sheets and Right Cosets. Let $p : X \to B$ be a covering in narrow sense, $b_0 \in B$, $x_0 \in p^{-1}(b_0)$. Construct a natural bijection of $p^{-1}(b_0)$ and the set $p * (\pi_1(X, x_0)) \backslash \pi_1(B, b_0)$ of right cosets of the group of the covering in the fundamental group of the base space.

> ***30.1 Number of Sheets in Universal Covering.*** The number of sheets of a universal covering equals the order of the fundamental group of the base space.
>
> ***30.2 Covering Means Non-Trivial*** π_1***.*** Any topological space, which has a nontrivial path-connected covering space, has a nontrivial fundamental group.

30:A Action of π_1 ***in Fiber.*** Let $p : X \to B$ be a covering, $b_0 \in B$. Construct a natural right action of $\pi_1(B, b_0)$ in $p^{-1}(b_0)$.

30:B. When the action in *30:A* is transitive?

Hierarchy of Coverings

Let $p : X \to B$ and $q : Y \to B$ be coverings, $x_0 \in X$, $y_0 \in Y$ and $p(x_0) = q(y_0) = b_0$. One says that q with base point y_0 is *subordinate* to p with base point x_0 if there exists a map $\varphi : X \to Y$ such that $q \circ \varphi = p$ and $\varphi(x_0) = y_0$. In this case the map φ is called a *subordination.*

30.H. A subordination is a covering map.

30.I. If a subordination exists, then it is unique. Cf. *27.J.*

Coverings $p : X \to B$ and $q : Y \to B$ are said to be *equivalent* if there exists a homeomorphism $h : X \to Y$ such that $p = q \circ h$. In this case h and h^{-1} are called *equivalencies*

30.J. If two coverings are mutually subordinate, then the corresponding subordinations are equivalencies.

30.K. Let $p : X \to B$ and $q : Y \to B$ be coverings, $x_0 \in X$, $y_0 \in Y$ and $p(x_0) = q(y_0) = b_0$. If q with base point y_0 is subordinate to p with base point x_0 then the group of covering p is contained in the group of covering q, i.e. $p_*(\pi_1(X, x_0)) \subset q_*(\pi_1(Y, y_0))$.

A topological space X is said to be *locally path-connected* if for each point $a \in X$ and each neighborhood U of a there is a neighborhood $V \subset U$ which is path-connected.

30.L. Let B be a locally path-connected space, $p : X \to B$ and $q : Y \to B$ be coverings in narrow sense, $x_0 \in X$, $y_0 \in Y$ and $p(x_0) = q(y_0) = b_0$. If $p_*(\pi_1(X, x_0)) \subset q_*(\pi_1(Y, y_0))$ then q is subordinate to p.

30.L.1. Under the conditions of *30.L*, if paths $u, v : I \to X$ have the same initial point x_0 and a common final point, then the paths which cover $p \circ u$ and $p \circ v$ and have the same initial point y_0 also have the same final point.

30.L.2. Under the conditions of *30.L*, the map $X \to Y$ defined by *30.L.1* (guess, what is this map!) is continuous.

30.M. Two coverings, $p : X \to B$ and $q : Y \to B$, with a common locally path-connected base are equivalent, iff for some $x_0 \in X$ and $y_0 \in Y$ with $p(x_0) = q(y_0) = b_0$ the groups $p_*(\pi_1(X, x_0))$ and $q_*(\pi_1(Y, y_0))$ are conjugate in $\pi_1(B, b_0)$.

To be finished

Automorphisms of Covering

Regular Coverings

Existence of Coverings

Lifting Maps

CHAPTER 5

More Applications and Calculations

31. Retractions and Fixed Points

Retractions and Retracts

A continuous map of a topological space onto a subspace is called a *retraction* if the restriction of the map to the subspace is the identity mapping. In other words, if X is a topological space, $A \subset X$ then $\rho : X \to A$ is a retraction if it is continuous and $\rho|A = \mathrm{id}_A$.

31.A. Let ρ be a continuous map of a space X onto its subspace A. Then the following statements are equivalent:

(a) ρ is a retraction,
(b) $\rho(a) = a$ for any $a \in A$,
(c) $\rho \circ \mathrm{in} = \mathrm{id}_A$,
(d) $\rho : X \to A$ is an extension of the identity mapping $A \to A$.

A subspace A of a space X is said to be a *retract* of X if there exists a retraction $X \to A$.

31.1. Any one-point subset is a retract.

Two-point set may be a non-retract.

31.2. Any subset of $\mathbb{R}$ consisting of two points is not a retract of $\mathbb{R}$.

31.3. If A is a retract of X and B is a retract of A then B is a retract of X.

31.4. If A is a retract of X and B is a retract of Y then $A \times B$ is a retract of $X \times Y$.

31.5. A closed interval $[a, b]$ is a retract of $\mathbb{R}$.

31.6. An open interval (a, b) is not a retract of $\mathbb{R}$.

31.7. What topological properties of ambient space are inherited by a retract?

31.8. Prove that a retract of a Hausdorff space is closed.

31.9. Prove that the union of Y-axis and the set $\{(x, y) \in \mathbb{R}^2 : x > 0, y = \sin \frac{1}{x}\}$ is not a retract of $\mathbb{R}^2$ and moreover is not a retract of any of its neighborhoods.

The role of the notion of retract is clarified by the following theorem.

31.B. *A subset A of a topological space X is a retract of X, iff any continuous map $A \to Y$ to any space Y can be extended to a continuous map $X \to Y$.*

Fundamental Group and Retractions

31.C. *If $\rho : X \to A$ is a retraction, $i : A \to X$ is the inclusion and $x_0 \in A$, then $\rho_* : \pi_1(X, x_0) \to \pi_1(A, x_0)$ is an epimorphism and $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ is a monomorphism.*

31.D. Riddle. Which of the two statements of Theorem *31.C* (about ρ_* or i_*) is easier to use for proving that a set $A \subset X$ is not a retract of X?

31.E Borsuk Theorem in Dimension 2. *S^1 is not a retract of D^2.*

31.10. Is the projective line a retract of the projective plane?

The following problem is more difficult than *31.E* in the sense that its solution is not a straightforward consequence of Theorem *31.C*, but rather demands to reexamine the arguments used in proof of *31.C*.

31.11. Prove that the boundary circle of Möbius band is not a retract of Möbius band.

31.12. Prove that the boundary circle of a handle is not a retract of the handle.

The Borsuk Theorem in its whole generality cannot be deduced like Theorem *31.E* from Theorem *31.C*. However, it can be proven using a generalization of *31.C* to higher homotopy groups. Although we do not assume that you can successfully prove it now relying only on the tools provided above, we formulate it here.

31.F Borsuk Theorem. *Sphere S^{n-1} is not a retract of ball D^n.*

At first glance this theorem seems to be useless. Why could it be interesting to know that a map with a very special property of being retraction does not exists in this situation? However in mathematics non-existence theorems may be closely related to theorems, which may seem to be more attractive. For instance, Borsuk Theorem implies Brower Theorem discussed below. But prior to this we have to introduce an important notion related to Brower Theorem.

Fixed-Point Property.

Let $f : X \to X$ be a continuous map. A point $a \in X$ is called a *fixed point* of f if $f(x) = x$. A space X is said to have the *fixed-point property* if any continuous map $X \to X$ has a fixed point. Fixed point property means solvability of a wide class of equations.

31.13. Prove that the fixed point property is a topological property.

31.14. A closed interval $[a, b]$ has the fixed point property.

31.15. Prove that if a topological space has fixed point property then each its retract also has the fixed-point property.

31.16. Prove that if topological spaces X and Y have fixed point property, $x_0 \in X$ and $y_0 \in Y$, then $X \amalg Y/_{x_0 \sim y_0}$ also has the fixed point property.

31.17. Prove that $\mathbb{R}^n$ with $n > 0$ does not have the fixed point property.

31.18. Prove that S^n does not have the fixed point property.

31.19. Prove that $\mathbb{R}P^n$ with odd n does not have the fixed point property.

31.20*. Prove that $\mathbb{C}P^n$ with odd n does not have the fixed point property.

Information. $\mathbb{R}P^n$ and $\mathbb{C}P^n$ with any even n have the fixed point property.

31.G Brower Theorem. *D^n has the fixed point property.*

31.H. Deduce from Borsuk Theorem in dimension n (i.e., from the statement that S^{n-1} is not a retract of D^n) Brower Theorem in dimension n (i.e., the statement that any continuous map $D^n \to D^n$ has a fixed point).

32. Homotopy Equivalences

Homotopy Equivalence as Map

Let X and Y be topological spaces, $f : X \to Y$ and $g : Y \to X$ continuous maps. Consider compositions $f \circ g : Y \to Y$ and $g \circ f : X \to X$. They would be equal to the corresponding identity maps, if f and g were homeomorphisms inverse to each other. If $f \circ g$ and $g \circ f$ are only homotopic to the identity maps then f and g are said to be *homotopy inverse* to each other. If a continuous map possesses a homotopy inverse map then it is called *homotopy invertible* or a *homotopy equivalence.*

32.A. Prove the following properties of homotopy equivalences:

(a) any homeomorphism is a homotopy equivalence,
(b) a map homotopy inverse to a homotopy equivalence is a homotopy equivalence,
(c) the composition of homotopy equivalences is a homotopy equivalence.

> ***32.1.*** Find a homotopy equivalence that is not a homeomorphism.

Homotopy Equivalence as Relation

Topological spaces X and Y are said to be *homotopy equivalent* if there exists a homotopy equivalence $X \to Y$.

32.B. Homotopy equivalence of topological spaces is an equivalence relation.

The classes of homotopy equivalent spaces are called *homotopy types.* Thus homotopy equivalent spaces are said to be of the same homotopy type.

> ***32.2.*** Prove that homotopy equivalent spaces have the same number of path-connected components.
>
> ***32.3.*** Prove that homotopy equivalent spaces have the same number of connected components.
>
> ***32.4.*** Find infinite series of topological spaces belonging to the same homotopy type, but pairwise non-homeomorphic.

Deformation Retraction

A retraction ρ, which is homotopy inverse to the inclusion, is called a *deformation retraction.* Since ρ is a retraction, one of the two conditions from the definition of homotopy inverse maps is satisfied automatically: its composition with the inclusion $\rho \circ \mathrm{in}$ is equal to the identity id_A. The other condition says that $\mathrm{in} \circ \rho$ is homotopic to the identity id_X.

If X admits a deformation retraction onto A, then A is called a *deformation retract* of X.

Examples

32.C. Circle S^1 is a deformation retract of $\mathbb{R}^2 \smallsetminus 0$

32.5. Prove that Möbius strip is homotopy equivalent to circle.

32.6. Prove that a handle is homotopy equivalent to a union of two circles intersecting in a single point.

32.7. Prove that a handle is homotopy equivalent to a union of three arcs with common end points (i.e., letter θ).

32.8. Classify letters of Latin alphabet up to homotopy equivalence.

32.D. Prove that a plane with s points deleted is homotopy equivalent to a union of s circles intersecting in a single point.

32.E. Prove that the union of a diagonal of a square and the contour of the same square is homotopy equivalent to a union of two circles intersecting in a single point.

32.9. Prove that the space obtained from S^2 by identification of a two (distinct) points is homotopy equivalent to the union of a two-dimensional sphere and a circle intersecting in a single point.

32.10. Prove that the space $\{(p, q) \in \mathbb{C} : z^2 + pz + q \text{ has two distinct roots}\}$ of quadratic complex polynomials with distinct roots is homotopy equivalent to the circle.

32.11. Prove that the space $GL(n, \mathbb{R})$ of invertible $n \times n$ real matrices is homotopy equivalent to the subspace $O(n)$ consisting of orthogonal matrices.

Deformation Retraction Versus Homotopy Equivalence

32.F. Spaces of Problem *32.E* cannot be embedded one to another. On the other hand, they can be embedded as strong homotopy retracts to plane with two points removed.

Deformation retractions a special type of homotopy equivalences. They are easier to visualize. However, as follows from *32.F*, homotopy equivalent spaces may be such that none of them can be embedded to the other one, and hence none of them is homeomorphic to a deformation retract of the other one. Therefore deformation retractions seem to be not sufficient for establishing homotopy equivalences.

Though it is not the case:

32.12*. Prove that any two homotopy equivalent spaces can be embedded as deformation retracts to the same topological space.

Contractible Spaces

A topological space X is said to be *contractible* if the identity map $\mathrm{id} : X \to X$ is homotopic to a constant map.

32.13. Show that $\mathbb{R}$ and I are contractible.

32.14. Prove that any contractible space is path-connected.

32.15. Prove that the following three statements about a topological space X are equivalent:

(a) X is contractible,
(b) X is homotopy equivalent to a point,
(c) there exists a deformation retraction of X onto a point,
(d) any point a of X is a deformation retract of X,
(e) any continuous map of any topological space Y to X is homotopic to a constant map,
(f) any continuous map of X to any topological space Y is homotopic to a constant map.

32.16. Is it right that if X is a contractible space then for any topological space Y

(a) any two continuous maps $X \to Y$ are homotopic?
(b) any two continuous maps $Y \to X$ are homotopic?

32.17. Check if spaces of the following list are contractible:

(a) $\mathbb{R}^n$,
(b) a convex subset of $\mathbb{R}^n$,
(c) a star convex subset of $\mathbb{R}^n$,
(d) $\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 \leq 1\}$,
(e) a finite tree (i.e., a connected space obtained from a finite collection of closed intervals by some identifying of their end points such that deleting of an internal point of each of the segments makes the space disconnected.)

32.18. Prove that $X \times Y$ is contractible, iff both X and Y are contractible.

Fundamental Group and Homotopy Equivalences

32.G. *Let $f : X \to Y$ and $g : Y \to X$ be homotopy inverse maps, $x_0 \in X$ and $y_0 \in Y$ be points such that $f(x_0) = y_0$ and $g(y_0) = x_0$ and, moreover, the homotopies relating $f \circ g$ to id_Y and $g \circ f$ to id_X are fixed at y_0 and x_0, respectively. Then f_* and g_* are inverse to each other isomorphisms between groups $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$.*

32.H Corollary. *If $\rho : X \to A$ is a strong deformation retraction, $x_0 \in A$, then $\rho_* : \pi_1(X, x_0) \to \pi_1(A, x_0)$ and $\mathrm{in}_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ are isomorphisms inverse to each other.*

32.19. Calculate the fundamental group of the following spaces:

(a) Möbius strip,
(b) $\mathbb{R}^3 \setminus \mathbb{R}^1$,
(c) $\mathbb{R}^N \setminus \mathbb{R}^n$,
(d) $\mathbb{R}^3 \setminus S^1$,
(e) $\mathbb{R}^N \setminus S^n$,
(f) $S^3 \setminus S^1$,
(g) $S^N \setminus S^k$,
(h) $\mathbb{R}P^3 \setminus \mathbb{R}P^1$,
(i) handle,
(j) sphere with s holes,

(k) Klein bottle with a point removed,
(l) Möbius strip with s holes.

32.20. Prove that the boundary of the Möbius band standardly embedded in $\mathbb{R}^3$ (see *18.18*) could not be the boundary of a disk embedded in $\mathbb{R}^3$ in such a way that its interior does not intersect the band.

32.21. Calculate the fundamental group of the space of all the complex polynomials $ax^2 + bx + c$ with distinct roots. Calculate the fundamental group of the subspace of this space consisting of polynomials with $a = 1$.

32.22. Riddle. Can you solve *32.21* along deriving of the formular for roots of quadratic trinomial?

32.I. What if the hypothesis of Theorem *32.G* were weakened as follows: $g(y_0) \neq x_0$ and/or the homotopies relating $f \circ g$ to id_Y and $g \circ f$ to id_X are *not* fixed at y_0 and x_0, respectively? How would f_* and g_* be related? Would $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$ be isomorphic?

33. Cellular Spaces

Definition of Cellular Spaces

In this section we study a class of topological spaces, which play an important role in algebraic topology. Their role in the context of this book is more restricted: this is the class of spaces for which we learn how to calculate the fundamental group.

This class of spaces was introduced by J.H.C.Whitehead. He called these spaces *CW-complexes,* and they are known under this name. However, for many reasons it is not a good name. For very rare exceptions (one of which is CW-complex, other is simplicial complex), the word *complex* is used nowadays for various algebraic notions, but not for spaces.

We have decided to use instead the term *cellular spaces* following D. B. Fuchs and V. A. Rokhlin, Beginner's Course in Topology: Geometric Chapters. Berlin; New York: Springer-Verlag, 1984.

A *zero-dimensional cellular space* is just a discrete space. Points of a 0-dimensional cellular space are also called *(zero-dimensional) cells* or 0*-cells.*

A *one-dimensional cellular space* is a space, which can be obtained as follows. Take any 0-dimensional cellular space X_0. Take a family of maps $\varphi_\alpha : S^0 \to X_0$. Attach to X_0 by φ_α the sum of a family of copies of D^1 (indexed by the same indices α as the maps φ_α):

$$X_0 \cup_{\varphi_\alpha} (\amalg_\alpha D^1).$$

The images of the interior parts of copies of D^1 are called *(open)* 1*-dimensional cells,* or 1*-cells,* or *edges*. The subsets obtained out of D^1 are called *closed* 1*-cells.* The cells of X_0 (i.e., points of X_0) are also called *vertices.* Open 1-cells and 0-cells comprise a partition of a one-dimensional cellular space. This partition is included in the notion of cellular space, i.e., a one-dimensional cellular space is a topological space equipped with a partition, which can be obtained in this way.

One-dimensional cellular spaces are associated also with the term *graph.* However, rather often this term is used for one-dimensional cellular spaces either equipped with additional structures (like orientations on edges), or satisfying to additional restrictions (such as injectivity of φ_α).

A *two-dimensional cellular space* is a space, which can be obtained as follows. Take any cellular space X_1 of dimension ≤ 1. Take a family of continuous[1] maps $\varphi_\alpha : S^1 \to X_1$. Attach to X_1 by φ_α the sum of a family of copies of D^2:

$$X_1 \cup_{\varphi_\alpha} (\amalg_\alpha D^2).$$

The images of the interior parts of copies of D^2 are called *open* 2*-dimensional cells,* or 2*-cells,* or *faces*. The cells of X_1 are also considered as cells of the

[1] Above, in the definition of 1-dimensional cellular space, the restriction of continuity for φ_α also could be stated, but it would be empty, since any map of S^0 to any space is continuous.

2-dimensional cellular space. A set obtained out of a copy of D^2 is called a *closed* 2*-cell.* Open cells of both kinds comprise a partition of a 2-dimensional cellular space. This partition is included in the notion of cellular space, i.e., a two-dimensional cellular space is a topological space equipped with a partition, which can be obtained in the way described above.

A *cellular space of dimension* n is defined in a similar way: This is a space equipped with a partition. It can be obtained from a cellular space X_{n-1} of dimension $< n$ by attaching a family of copies of ball D^n by a family of continuous maps of their boundary spheres:

$$X_{n-1} \cup_{\varphi_\alpha} (\amalg_\alpha D^n).$$

The images of interior parts of the attached n-dimensional balls are called *(open)* n*-dimensional cells,* or n*-cells*. The images of the whole n-dimensional balls are called *closed* n*-cells.* Cells of X_{n-1} are also considered as cells of the n-dimensional cellular space.

A *cellular space* is obtained as a union of increasing sequence of cellular spaces $X_0 \subset X_1 \subset \dots \subset X_n \subset \dots$ obtained in this way from each other. The sequence may be finite or infinite. In the latter case topological structure is introduced by saying that the cover of the union by X_n's is fundamental, i.e., that a set $U \subset \cup_{n=0}^{\infty} X_n$ is open, iff its intersection $U \cap X_n$ with each X_n is open in X_n.

The union of all cells of dimension $\leq n$ of a cellular space X is called the n*-dimensional skeleton* of X. This term may be misleading, since n-dimensional skeleton may be without cells of dimension n, hence it may coincide with $(n-1)$-dimensional skeleton. Thus n-dimensional skeleton may have dimension $< n$. Therefore it is better to speak about n*-th skeleton* or n*-skeleton.* Cells of dimension n are called also n*-cells.* A cellular space is said to be *finite* if it contains a finite number of cells. A cellular space is said to be *countable* if it contains a countable number of cells. Let X be a cellular space. A subspace $A \subset X$, which can be presented both as a union of closed cells and a union of open cells, is called a *cellular subspace* of X. Of course, it is provided with a partition into the open cells of X contained in A. Obviously, the k-skeleton of a cellular space X is a cellular subspace of X.

33.A. Prove that a cellular subspace of a cellular space is a cellular space.

First Examples

33.B. A cellular space consisting of two cells, one 0-dimensional and one n-dimensional, is homeomorphic to S^n.

33.C. Present D^n with $n > 0$ as a cellular space made of three cells.

33.D. A cellular space consisting of a single zero-dimensional cell and q one-dimensional cells is a bouquet of q circles.

33.E. Present torus $S^1 \times S^1$ as a cellular space with one 0-cell, two 1-cells, and one 2-cell.

33.F. How to obtain a presentation of torus $S^1 \times S^1$ as a cellular space with 4 cells from a presentation of S^1 as a cellular space with 2 cells?

> ***33.1.*** Prove that if X and Y are finite cellular spaces then $X \times Y$ can be equipped in a natural way with a structure of finite cellular space.

> ***33.2*.*** Does the statement of *33.1* remain true if one skips the finiteness condition in it? If yes, prove; if no, find an example when the product is not a cellular space.

33.G. Present sphere S^n as a cellular space such that spheres $S^0 \subset S^1 \subset S^2 \subset \dots \subset S^{n-1}$ are its skeletons.

33.H. Present $\mathbb{R}P^n$ as a cellular space with $n+1$ cells. Describe the attaching maps of its cells.

> ***33.3.*** Present $\mathbb{C}P^n$ as a cellular space with $n+1$ cells. Describe the attaching maps of its cells.

> ***33.4.*** Present the following topological spaces as cellular ones
>
> (a) handle,
> (b) Möbius strip,
> (c) $S^1 \times I$,
> (d) sphere with p handles,
> (e) sphere with p crosscaps.

> ***33.5.*** What is the minimal number of cells in a cellular space homeomorphic to
>
> (a) Möbius strip,
> (b) sphere with p handles,
> (c) sphere with p crosscaps?

> ***33.6.*** Find a cellular space, in which a closure of a cell is not equal to a union of other cells. What is the minimal number of cells in a space containing a cell of this sort?

> ***33.7.*** Consider a disjoint sum of a countable collection of copies of closed interval I and identify the copies of 0 in all of them. Present the result (which is the bouquet of the countable family of intervals) as a countable cellular space. Prove that this space is not first countable.

33.I. Present $\mathbb{R}^1$ as a cellular space.

> ***33.8.*** Prove that for any two cellular spaces homeomorphic to $\mathbb{R}^1$ there exists a homeomorphism between them mapping each cell of one of them homeomorphically onto a cell of the other one.

33.J. Present $\mathbb{R}^n$ as a cellular space.

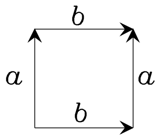
Denote by $\mathbb{R}^\infty$ the union of the sequence of Euclidean spaces $\mathbb{R}^0 \subset \mathbb{R}^1 \subset \dots \subset \mathbb{R}^n \subset$ canonically included to each other: $\mathbb{R}^n = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$. Equip $\mathbb{R}^\infty$ with the topological structure, for which the spaces $\mathbb{R}^n$ comprise a fundamental cover.

33.K. Present $\mathbb{R}^\infty$ as a cellular space.

More Two-Dimensional Examples

Let us consider a class of 2-dimensional cellular spaces, which admit a simple combinatorial description. Each space of this class can be presented as a quotient space of a finite family of convex polygons by identification of sides via affine homeomorphisms. The identification of vertices is defined by the identification of the sides. The quotient space is naturally equipped with decomposition into 0-cells, which are the images of vertices, 1-cells, which are the images of sides, and faces, the images of the interior parts of the polygons.

To describe such a space, one needs, first, to show, what sides are to be identified. Usually this is indicated by writing the same letters at the sides that are to be identified. There are only two affine homeomorphisms between two closed intervals. To specify one of them, it is enough to show orientations of the intervals which are identified by the homeomorphism. Usually this is done by drawing arrows on the sides. Here is a description of this sort for the standard presentation of torus $S^1 \times S^1$ as the quotient space of square:



It is possible to avoid a picture by a description. To do this, go around the polygons counter-clockwise writing down the letters, which stay at the sides of polygon along the contour. The letters corresponding to the sides, whose orientation is opposite to the counter-clockwise direction, put with exponent -1. This gives rise to a collection of words, which contains a sufficient information about the family of polygons and the partition. For instance, the presentation of torus shown above is encoded by the word $ab^{-1}a^{-1}b$.

33.9. Prove that:

(a) word $a^{-1}a$ describes a cellular space homeomorphic to S^2,
(b) word aa describes a cellular space homeomorphic to $\mathbb{R}P^2$,
(c) word $aba^{-1}b^{-1}c$ describes a handle,
(d) word $abcb^{-1}$ describes cylinder $S^1 \times I$,
(e) each of the words aab and $abac$ describe Möbius strip,
(f) word $abab$ describes a cellular space homeomorphic to $\mathbb{R}P^2$,
(g) each of the words $aabb$ and $ab^{-1}ab$ describe Klein bottle,
(h) word
$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\dots a_gb_ga_g^{-1}b_g^{-1}.$$
describes sphere with g handles,
(i) word $a_1a_1a_2a_2\dots a_ga_g$ describes sphere with g crosscaps.

Topological Properties of Cellular Spaces

33:A. Closed cells comprise a fundamental cover of a cellular space.

33:B. If A is cellular subspace of a cellular space X then A is closed in X.

33:C. Prove that any compact subset of a cellular space intersects a finite number of cells.

33:D Corollary. *A cellular space is compact, iff it is finite.*

33:E. Any cell of a cellular space is contained in a finite cellular subspace of this space.

33:F. Any compact subset of a cellular space is contained in a finite cellular subspace.

33:G. A cellular space is separable, iff it is countable.

33:H. Any path-connected component of a cellular space is a cellular subspace.

33:I. Any path-connected component of a cellular space is both open and closed. It is a connected component. In particular, a cellular space is path-connected, iff it is connected.

33:J. Any connected cellular space is countable.

33:K. *A cellular space is connected, iff its* 1*-skeleton is connected.*

33:L. Any cellular space is normal.

Embedding to Euclidean Space

33.L. Any countable 0-dimensional cellular space can be embedded into $\mathbb{R}$.

33.M. Any countable 1-dimensional cellular space can be embedded into $\mathbb{R}^3$.

> ***33.10.*** Find a 1-dimensional cellular space, which you cannot embed into $\mathbb{R}^2$. (We do not ask to prove that it is impossible to embed.)

33.N. *Any countable finite dimensional cellular space can be embedded into Euclidean space of sufficiently high dimension.*

33.N.1. Let X and Y be topological spaces such that X can be embedded into $\mathbb{R}^p$ and Y can be embedded into $\mathbb{R}^q$. Let A be a closed subset of Y. Assume that A has a neighborhood U in Y such that there exists a homeomorphism $h : \operatorname{Cl} U \to A \times I$ mapping A to $A \times 0$. Let $\varphi : A \to X$ be any continuous map. Then there exists an embedding of $X \cup_\varphi Y$ into R^{p+q+1}.

33.N.2. Let X be a countable k-dimensional cellular space and A be its $(k-1)$-skeleton. Prove that if A can be embedded to $\mathbb{R}^p$ then X can be embedded into R^{p+k+1}.

33.O. Any countable cellular space can be embedded into $\mathbb{R}^\infty$.

33.P. Any countable cellular space is metrizable.

One-Dimensional Cellular Spaces

33.Q. *Any connected finite* 1*-dimensional cellular space is homotopy equivalent to a bouquet of circles.*

33.Q.1 ***Lemma.*** Let X be a 1-dimensional cellular space, and e its 1-cell, which is attached by an injective map $S^0 \to X_0$ (i.e., it has two distinct end points). Prove that the natural projection $X \to X/_e$ is a homotopy equivalence. Describe the homotopy inverse map explicitly.

A 1-dimensional cellular space is called a *tree* if it is connected and the complement of any its 1-cell is not connected.

33.R. A cellular space X is a tree, iff there is no an embedding $S^1 \to X$.

33.S. Prove that any point of a tree is a deformation retract of the tree.

33.11. Prove that any finite tree has fixed point property.

Cf. *31.14*, *31.15* and *31.16*.

33.12. Does the same hold true for any tree, for a finite graph?

A cellular subspace A of a cellular space X is called a *maximal tree* of X if A is a tree and is not contained in any other cellular subspace $B \subset X$, which is a tree.

33.T. Prove that any finite connected 1-dimensional cellular space contains a maximal tree.

33.U. Prove that a cellular subspace A of a cellular space X is a maximal tree, iff it is a tree and the quotient space $X/_A$ is a bouquet of circles.

33.V. Let X be a 1-dimensional cellular space and A its cellular subspace. Prove that if A is a tree then the natural projection $X \to X/_A$ is a homotopy equivalence.

Problems *33.T*, *33.V* and *33.U* provide a proof of Theorem *33.Q*.

33:M. Prove that any 1-dimensional connected cellular space has a maximal tree.

33:N. *Any connected one-dimensional cellular space is homotopy equivalent to a bouquet of circles.*

33:O. Prove that if T is a tree and a cellular subspace of a cellular space X then the natural projection $X \to X/_T$ is a homotopy equivalence.

33:P. *Any connected cellular space is homotopy equivalent to a cellular space with 0-skeleton consisting of one point.*

Euler Characteristic

Let X be a finite cellular space. Let $c_i(X)$ denote the number of its cells of dimension i. *Euler characteristic* of X is the alternating sum of $c_i(X)$:

$$\chi(X) = c_0(X) - c_1(X) + c_2(X) - \dots + (-1)^i c_i(X) + \dots$$

.

33:Q. Prove that Euler characteristic is additive in the following sense: for any cellular space X and its finite cellular subspaces A and B

$$\chi(A) \cup (B) = \chi(A) + \chi(B) - \chi(A \cap B).$$

33:R. Prove that Euler characteristic is multiplicative in the following sense: for any finite cellular spaces X and Y the Euler characteristic of their product $X \times Y$ is $\chi(X)\chi(Y)$.

33.W. *A finite connected cellular space X of dimension one is homotopy equivalent to the bouquet of $1 - \chi(X)$ circles.*

34. Fundamental Group of a Cellular Space

One-Dimensional Cellular Spaces

34.A. *If X is a finite 1-dimensional cellular space, then $\pi_1(X)$ is a free group of rank $1 - \chi(X)$.*

34.B. *Homotopy Classification of Finite One-Dimensional Cellular Spaces. Two finite 1-dimensional cellular spaces are homotopy equivalent, iff their Euler characteristics are equal.*

> ***34.1.*** Prove that the fundamental group of 2-dimensional sphere with n points removed is a free group of rank $n - 1$.
>
> ***34.2 Euler Theorem.*** For any bounded convex polyhedron in $\mathbb{R}^3$ the number of edges plus 2 is equal to the sum of the numbers of vertices and faces.

34.C. Let X be a finite 1-dimensional cellular space, T a maximal tree of X and $x_0 \in T$. For each cell $e \subset X \smallsetminus T$ choose a loop s_e, which starts at x_0, goes inside T to e, then goes once along e and then comes back to x_0 in T. Prove that $\pi_1(X, x_0)$ is freely generated by homotopy classes of s_e.

Generators

34.D. Let A be a topological space, $x_0 \in A$. Let $\varphi : S^{k-1} \to A$ be a continuous map, $X = A \cup_\varphi D^k$. Prove that if $k > 1$ then the inclusion homomorphism $\pi_1(A, x_0) \to \pi_1(X, x_0)$ is surjective. Cf. *25.G.5*, *25.G.4*.

34.E. *Let X be a cellular space, x_0 its 0-cell and X_1 the 1-skeleton of X. Then the inclusion homomorphism*

$$\pi_1(X_1, x_0) \to \pi_1(X, x_0)$$

is surjective.

34.F. Let X be a finite cellular space, T a maximal tree of X_1 and $x_0 \in T$. For each cell $e \subset X_1 \smallsetminus T$ choose a loop s_e, which starts at x_0, goes inside T to e, then goes once along e and then comes back to x_0 in T. Prove that $\pi_1(X, x_0)$ is generated by homotopy classes of s_e.

> ***34.3.*** Deduce Theorem *25.G* from Theorem *34.E*.
>
> ***34.4.*** Find $\pi_1(\mathbb{C}P^n)$.

Relators

Let X be a cellular space, x_0 its 0-cell. Denote by X_n the n-skeleton of X. Recall that X_2 is obtained from X_1 by attaching copies of disk D^2 by continuous maps $\varphi_\alpha : S^1 \to X_1$. The attaching maps are circular loops in X_1. For each α choose a path $s_\alpha : I \to X_1$ connecting $\varphi_\alpha(1)$ with x_0. Denote by

N the normal subgroup of $\pi_1(X, x_0)$ generated (as a normal subgroup[2].) by elements

$$T_{s_\alpha}[\varphi_\alpha] \in \pi_1(X_1, x_0).$$

34.G. Prove that N does not depend on the choice of paths s_α.

34.H. *N coincides with the kernel of the inclusion homomorphism*

$$i_* : \pi_1(X_1, x_0) \to \pi_1(X, x_0).$$

34.H.1 ***Lemma 1.*** $N \subset \operatorname{Ker} i_*$, cf. *25.J* (c).

34.H.2 ***Lemma 2.*** Let $p_1 : Y_1 \to X_1$ be a covering with covering group N. Then for any α and a point $y \in p_1^{-1}(\varphi_\alpha(1))$ there exists a lifting $\widetilde{\varphi}_\alpha : S^1 \to Y_1$ of φ_α with $\widetilde{\varphi}_\alpha(1) = y$.

34.H.3 ***Lemma 3.*** Let Y_2 be a cellular space obtained by attaching copies of disk to Y_1 by all liftings of attaching maps φ_α. Then there exists a map $p_2 : Y_2 \to X_2$ extending p_1 and this is a covering.

34.H.4 ***Lemma 4.*** Any loop $s : I \to X_1$ realizing an element of the kernel of the inclusion homomorphism $\pi_1(X_1, x_0) \to \pi_1(X_2, x_0)$ (i.e., homotopic to constant in X_2) is covered by a loop of Y_2. The covering loop is contained in Y_1.

34.H.5 ***Lemma 5.*** N coincides with the kernel of the inclusion homomorphism $\pi_1(X_1, x_0) \to \pi_1(X_2, x_0)$.

34.H.6 ***Lemma 6.*** Attaching maps of n-cells with $n > 3$ are lifted to any covering space. Cf. *27:A*, *27:B*.

34.H.7 ***Lemma 7.*** Covering $p_2 : Y_2 \to X_2$ can be extended to a covering of the whole X.

34.H.8 ***Lemma 8.*** Any loop $s : I \to X_1$ realizing an element of $\operatorname{Ker} i_*$ (i.e., homotopic to constant in X) is covered by a loop of Y. The covering loop is contained in Y_1.

Writing Down Generators and Relators

Theorems *34.F* and *34.H* imply the following prescription for writing down presentation for the fundamental group of a finite dimensional cellular space by generators and relators:

Let X be a finite cellular space, x_0 its 0-cell. Let T a maximal tree of 1-skeleton of X. For each 1-cell $e \not\subset T$ of X choose a loop s_e, which starts at x_0, goes inside T to e, then goes once along e and then comes back to x_0 in T. Let g_1, ... , g_m be the homotopy classes of these loops. Let $\varphi_1, \dots, \varphi_n : S^1 \to X_1$ be attaching maps of 2-cells of X. For each φ_i choose a path s_i connecting $\varphi_i(1)$ with x_0 in 1-skeleton of X. Express the homotopy class of the loop $s_i^{-1}\varphi_i s_i$ as a product of powers of generators g_j. Let r_1, ... , r_n are the words in letters g_1, ... , g_m obtained in this way. The fundamental group of X is generated by g_1, ... , g_m, which are subject to defining relators $r_1 = 1$, ... , $r_n = 1$.

[2]Recall that a subgroup is said to be *normal* if it coincides with conjugate subgroups. The normal subgroup generated by a set A is the minimal normal subgroup containing A. As a subgroup, it is generated by elements of A and elements conjugate to them. This means that each element of this normal subgroup is a product of elements conjugate to elements of A

34.I. Check that this rule gives correct answers in the cases of $\mathbb{R}P^n$ and $S^1 \times S^1$ for the cellular presentations of these spaces provided in Problems *33.H* and *33.E*.

Fundamental Groups of Basic Surfaces

34.J. *The fundamental group of a sphere with g handles admits presentation*

$$\{a_1, b_1, a_2, b_2, \dots a_g, b_g \; : \; a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = 1\}.$$

34.K. *The fundamental group of a sphere with g crosscaps admits presentation*

$$\left\{a_1, a_2, \dots a_g \; : \; a_1^2 a_2^2 \dots a_g^2 = 1\right\}.$$

34.L. Prove that fundamental groups of spheres with different number of handles are not isomorphic.

When one needs to prove that two finitely presented groups are not isomorphic, one of the first natural moves is to abelianize the groups. Recall that to abelianize a group G means to quotient it out by the commutator subgroup. The commutator subgroup $[G, G]$ is the normal subgroup generated by commutators $a^{-1}b^{-1}ab$ for all $a, b \in G$. Abelianization means adding relations that $ab = ba$ for any $a, b \in G$.

Abelian finitely generated groups are well known. Any finitely generated abelian group is isomorphic to a product of a finite number of cyclic groups. If the abelianized groups are not isomorphic then the original groups are not isomorphic as well.

34.L.1. Abelianized fundamental group of a sphere with g handles is a free abelian group of rank $2g$ (i.e., is isomorphic to $\mathbb{Z}^{2g}$).

34.L.2. Prove that fundamental groups of spheres with different number of crosscaps are not isomorphic.

34.L.3. Abelianized fundamental group of a sphere with g crosscaps is isomorphic to $\mathbb{Z}^{g-1} \times \mathbb{Z}_2$.

34.M. *Spheres with different numbers of handles are not homotopy equivalent.*

34.N. *Spheres with different numbers of crosscaps are not homotopy equivalent.*

34.O. *A sphere with handles is not homotopy equivalent to a sphere with crosscaps.*

If X is a path-connected space then the abelianized fundamental group of X is called the 1-*dimensional (or first) homology group* of X and denoted by $H_1(X)$. If X is not path-connected then $H_1(X)$ is the direct sum of the first homology groups of all path-connected components of X. Thus *34.L.1* can be rephrased as follows: if F_g is a sphere with g handles then $H_1(F_g) = \mathbb{Z}^{2g}$.

Seifert - van Kampen Theorem

Let X be a connected cellular space, A and B its cellular subspaces which cover X. Denote $A \cap B$ by C.

34:A. How fundamental groups of X, A, B and C are related?

34:B Seifert - van Kampen Theorem. *Suppose A, B, and C are connected. Let $x_0 \in C$,*

$$\pi_1(A, x_0) = \{\alpha_1, \dots, \alpha_p \,:\, \rho_1 = 1, \dots, \rho_r = 1\},$$
$$\pi_1(B, x_0) = \{\beta_1, \dots, \beta_q \,:\, \sigma_1 = 1, \dots, \sigma_s = 1\},$$

and $\pi_1(C, x_0)$ be generated by $\gamma_1, \dots \gamma_t$. Let the images of γ_i under the inclusion homomorphisms $\pi_1(C, x_0) \to \pi_1(A, x_0)$ and $\pi_1(C, x_0) \to \pi_1(B, x_0)$ be expressed as $\xi_i(\alpha_1, \dots, \alpha_p)$ and $\eta_i(\beta_1, \dots, \beta_q)$, respectively. Then

$$\pi_1(X) = \{\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q \,:\\ \rho_1 = 1, \dots, \rho_r = 1, \sigma_1 = 1, \dots, \sigma_s = 1,\\ \xi_1 = \eta_1, \dots, \xi_t = \eta_t\}.$$

34:C. Let X, A, B and C be as above. Suppose A, B are simply connected and C consists of two path connected components. Prove that $\pi_1(X)$ is isomorphic to $\mathbb{Z}$.

To write details: van Kampen published much more general theorem!

35. One-Dimensional Homology and Cohomology

Sometimes the fundamental group contains too much information to deal with, and it is more convinient to ignore a part of this information. A regular way to do his is to use some of the natural quotient groups of the fundamental group. One of the quotients, the abelianized fundamental group, was introduced and used in Section 34 to prove, in particular, that spheres with different numbers of handles are not homotopy equivalent, see Problems *34.L*, *34.L.1*-*34.L.3* and *34.M*.

Recall that for a path-connected space X the abelianized fundamental group of X is called its one-dimensional homology group and denoted by $H_1(X)$. If X is an arbitrary topological space then $H_1(X)$ is the direct sum of the one-dimensional homology groups of all the connected components of X.

In this Section we will study the one-dimensional homology and its closest relatives. Usually they are studied in the framework of homology theory together with high-dimensional generalizations. This general theory requires much more algebra and takes more time and efforts. On the other hand, one-dimensional case is useful on its own, involves a lot of specific details and provides a geometric intuition, which is useful, in particular, for studying high-dimensional homology.

First, few new words. Elements of a homology group is called *homology classes.* They really admit several interpretations as equivalence classes of objects of various nature. For example, according to the definition we start with, a homology class is a coset consisting of elements of the fundamental group. In turn, each element of the fundamental group consists of loops. Thus, we can think of a homology class as of a set of loops. A loop which belongs to the zero homology class is said to be *zero-homologous.* Loops, which belong to the same homology class, are said to be *homologous* to each other.

35:A Zero-Homologous Loop. Let X be a topological space. A circular loop $s : S^1 \to X$ is zero-homologous, iff there exist a continuous map f of a disk D with handles (i.e., a sphere with a hole and handles) to X and a homeomorphism h of S^1 onto the boundary circle of D such that $f \circ h = s$.

35:A:1. In the fundamental group of a disk with handles, a loop, whose homotopy class generates the fundamental group of the boundary circle, is homotopic to a product of commutators of meridian and longitude loops of the handles.

A homotopy between a loop and a product of commutators of loops can be thought of as an extension of the loop to a continuous map of a sphere with handles and a hole.

Description of $H_1(X)$ in Terms of Free Circular Loops

Factorization by the commutator subgroup kills the difference between translation maps defined by different paths. Therefore the abelianized fundamental groups of a path-connected space can be naturally identified. Hence each free loop defines a homology class. This suggests that $H_1(X)$ can be defined starting with free loops, rather than loops at a base point.

35:B. On the sphere with two handles and three holes shown in Figure 1 the sum of the homology classes of the three loops, which go counter-clockwise arround the three holes, is zero.

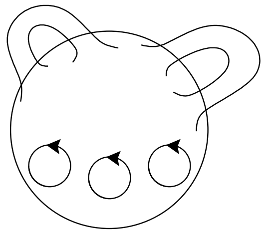


FIGURE 1. Sphere with two handles and three holes. The boundary circles of the holes are equipped with arrows showing the counter-clockwise orientation.

35:C Zero-Homologous Collections of Loops. Let X be a pathwise connected space and $s_1, \dots, s_n : S^1 \to X$ be a collection of n free loops. Prove that the sum of homology classes of $s_1, \dots, s_n$ is equal to zero, iff there exist a continuous map $f : F \to X$, where F is a sphere with handles and n holes, and embeddings $i_1, \dots, i_n : S^1 \to F$ parametrizing the boundary circles of the holes in the counter-clockwise direction (as in Figure 1) such that $s_k = f \circ i_k$ for $k = 1, \dots, n$.

35:D Homologous Collections of Loops. In a topological space X any class $\xi \in H_1(X)$ can be represented by a finite collection of free circular loops. Collections $\{u_1, \dots, u_p\}$ and $\{v_1, \dots, v_q\}$ of free circular loops in X define the same homology class, iff there exist a continuous map $f : F \to X$, where F is a disjoint sum of several spheres with handles and holes with the total number of holes equal $p + q$, and embeddings $i_1, \dots, i_{p+q} : S^1 \to F$ parametrizing the boundary circles of all the holes of F in the counter-clockwise direction such that $u_k = f \circ i_k$ for $k = 1, \dots, p$ and $v_k^{-1} = f \circ i_{k+p}$ for $k = 1, \dots, q$.

35:1. Find $H_1(X)$ for the following spaces

(a) Möbius strip,
(b) handle,
(c) sphere with p handles and r holes,
(d) sphere with p crosscaps and r holes,
(e) the complement in $\mathbb{R}^3$ of the circles $\{(x, y, z) \in \mathbb{R}^3 \mid z = 0, x^2 + y^2 = 1\}$ and $\{(x, y, z) \in \mathbb{R}^3 \mid x = 0, z^2 + (y - 1)^2 = 1\}$,
(f) the complement in $\mathbb{R}^3$ of the circles $\{(x, y, z) \in \mathbb{R}^3 \mid z = 0, x^2 + y^2 = 1\}$ and $\{(x, y, z) \in \mathbb{R}^3 \mid z = 1, x^2 + y^2 = 1\}$,

One-Dimensional Cohomology

Let X be a path-connected topological space and G a commutative group.

35:E. The homomorphisms $\pi_1(X, x_0) \to G$ comprise a commutative group in which the group operation is the pointwise addition.

The group $\mathrm{Hom}(\pi_1(X, x_0), G)$ of all the homomorphisms $\pi_1(X, x_0) \to G$ is called *one-dimensional cohomology group of X with coefficients in G* and denoted by $H^1(X; G)$.

For an arbitrary topological space X, the one-dimensional cohomology group of X with coefficients in G is defined as the direct product of one-dimensional cohomology group with coefficients in G of all the path-connected components of X.

35:F Cohomology via Homology. $H^1(X; G) = \mathrm{Hom}(H_1(X), G)$.

The following subsection is to be rewritten when the section on classification of coverings will be done!

Cohomology and Classification of Regular Coverings

Recall that a covering $p : X \to B$ is a regular G-covering if X is a G-space, in which the orbits of the action of G are the fibers of p and G acts effectively on each of them. Regular G-covering may be with disconnected total space. For example, $X \times G \to X$ is a regular G-covering.

For any loop $s : I \to B$ in the base B of a regular G-covering $p : X \to B$ there is a map $M_s : p^{-1}(s(0)) \to p^{-1}(s(0))$ assigning to $x \in p^{-1}(s(0))$ the final point of the path covering s^{-1} and beginning at x. This map is called the *monodromy transformation* of $p^{-1}(s(0))$ defined by s. It coincides with action of one of the elements of G. In this way a homomorphism $\pi_1(B) \to G$ is defined. It is called the *monodromy representaion of the fundamental group.* Thus any regular G-covering of X defines a cohomology class belonging to $H^1(X; G)$.

35:G Cohomology and Regular Coverings. This map is a bijection of the set of all the regular G-coverings of X onto $H^1(X; G)$.

> ***35:2 Addition of G-Coverings.*** What operation on the set of regular G-coverings corresponds to addition of cohomology classes?

Integer Cohomology and Maps to S^1

Let X be a topological space and $f : X \to S^1$ a continuous map. It induces a homomorphism $f_* : H_1(X) \to H_1(S^1) = \mathbb{Z}$. Therefore it defines an element of $H^1(X; \mathbb{Z})$.

35:H. This construction defines a bijection of the set of all the homotopy classes of maps $X \to S^1$ onto $H^1(X;\mathbb{Z})$.

35:I Addition of Maps to Circle. What operation on the set of homotopy classes of maps to S^1 corresponds to the addition in $H^1(X;\mathbb{Z})$?

35:J. What regular $\mathbb{Z}$-covering of X corresponds to a homotopy class of mappings $X \to S^1$ under the compositions of the bijections described in *35:H* and *35:G*

One-Dimensional Homology Modulo 2

Here we define yet another natural quotient group of the fundamental group. It is even simpler than $H_1(X)$.

For a path-connected X, consider the quotient group of $\pi_1(X)$ by the normal subgroup generated by squares of all the elements of $\pi(X)$. It is denoted by $H_1(X;\mathbb{Z}_2)$ and called *one-dimensional homology group of X with coefficients in $\mathbb{Z}_2$.* For an arbitrary X, the group $H_1(X;\mathbb{Z}_2)$ is defined as the sum of one-dimensional homology group with coefficients in $\mathbb{Z}_2$ of all the path-connected components of X.

Elements of $H_1(X;\mathbb{Z}_2)$ are called *one-dimensional homology classes modulo 2* or *one-dimensional homology classes with coefficients in $\mathbb{Z}_2$.* They can be thought of as classes of elements of the fundamental groups or classes of loops. A loop defining the zero homology class modulo 2 is said to be *zero-homologous modulo 2.*

35:K. In a disk with crosscaps the boundary loop is zero-homologous modulo 2.

35:L Loops Zero-Homologous Modulo 2. Prove that a circular loop $s : S^1 \to X$ is zero-homologous modulo 2, iff there exist a continuous map f of a disk with crosscaps D to X and a homeomorphism h of S^1 onto the boundary circle of D such that $f \circ h = s$.

35:M. If a loop is zero-homologous then it is zero-homologous modulo 2.

35:N Homology and Mod 2 Homology. $H_1(X;\mathbb{Z}_2)$ is commutative for any X, and can be obtained as the quotient group of $H_1(X)$ by the subgroup of all even homology classes, i.e. elements of $H_1(X)$ of the form 2ξ with $\xi \in H_1(X)$. Each element of is of order 2 and $H_1(X;\mathbb{Z}_2)$ is a vector space over the field of two elements $\mathbb{Z}_2$.

35:3. Find $H_1(X;\mathbb{Z}_2)$ for the following spaces

(a) Möbius strip,
(b) handle,
(c) sphere with p handles,
(d) sphere with p crosscaps,
(e) sphere with p handles and r holes,
(f) sphere with p crosscaps and r holes,

(g) the complement in $\mathbb{R}^3$ of the circles $\{(x,y,z) \in \mathbb{R}^3 \mid z = 0, x^2 + y^2 = 1\}$ and $\{(x,y,z) \in \mathbb{R}^3 \mid x = 0, z^2 + (y-1)^2 = 1\}$,

(h) the complement in $\mathbb{R}^3$ of the circles $\{(x,y,z) \in \mathbb{R}^3 \mid z = 0, x^2 + y^2 = 1\}$ and $\{(x,y,z) \in \mathbb{R}^3 \mid z = 1, x^2 + y^2 = 1\}$,

35:4 ***$\mathbb{Z}_2$-Homology of Cellular Space.*** Deduce from the calculation of the fundamental group of a cellular space (see Section 34) an algorithm for calculation of the one-dimensional homology group with $\mathbb{Z}_2$ coefficients of a cellular space.

35:O Collections of Loops Homologous Mod 2. Let X be a topological space. Any class $\xi \in H_1(X;\mathbb{Z}_2)$ can be represented by a finite collection of free circular loops in X. Collections $\{u_1,\dots,u_p\}$ and $\{v_1,\dots,v_q\}$ of free circular loops in X define the same homology class modulo 2, iff there exist a continuous map $f : F \to X$, where F is a disjoint sum of several spheres with crosscaps and holes with the total number of holes equal $p+q$, and embeddings $i_1,\dots,i_{p+q} : S^1 \to F$ parametrizing the boundary circles of all the holes of F such that $u_k = f \circ i_k$ for $k = 1,\dots,p$ and $v_k = f \circ i_{k+p}$ for $k = 1,\dots,q$.

35:5. Compare *35:O* with *35:D*. Why in *35:O* the counter-clockwise direction does not appear? In what other aspects *35:O* is simpler than *35:D* and why?

35:P Duality Between Mod 2 Homology and Cohomology.

$$H^1(X;\mathbb{Z}_2) = \mathrm{Hom}(H_1(X;\mathbb{Z}_2),\mathbb{Z}_2) = \mathrm{Hom}_{\mathbb{Z}_2}(H_1(X;\mathbb{Z}_2),\mathbb{Z}_2)$$

for any space X. If $H_1(X;\mathbb{Z}_2)$ is finite then $H_1(X;\mathbb{Z}_2)$ and $H^1(X;\mathbb{Z}_2)$ are finite-dimensional vector spaces over $\mathbb{Z}_2$ dual to each other.

35:6. A loop is zero-homologous modulo 2 in X, iff it is covered by a loop in any two-fold covering space of X.

35:Q. Riddle. *Homology Modulo n?* Generalize all the theory above about $\mathbb{Z}_2$-homology to define and study $\mathbb{Z}_n$-homology for any natural n.

Part 3

Manifolds

This part is devoted to study of the most important topological spaces. These spaces provide a scene for most of geometric branches of mathematics.

CHAPTER 6

Bare Manifolds

36. Locally Euclidean Spaces

Definition of Locally Euclidean Space

Let n be a non-negative integer. A topological space X is called a *locally Euclidean space of dimension* n if each point of X has a neighborhood homeomorphic either to $\mathbb{R}^n$ or $\mathbb{R}^n_+$. Recall that $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_1 \geq 0\}$, it is defined for $n \geq 1$.

36.A. The notion of 0-dimensional locally Euclidean space coincides with the notion of discrete topological space.

36.B. Prove that the following spaces are locally Euclidean:

(a) $\mathbb{R}^n$,
(b) any open subset of $\mathbb{R}^n$,
(c) S^n,
(d) $\mathbb{R}P^n$,
(e) $\mathbb{C}P^n$,
(f) $\mathbb{R}^n_+$,
(g) any open subset of $\mathbb{R}^n_+$,
(h) D^n,
(i) torus $S^1 \times S^1$,
(j) handle,
(k) sphere with handles,
(l) sphere with holes,
(m) Klein bottle,
(n) sphere with crosscaps.

> ***36.1.*** Prove that an open subspace of a locally Euclidean space of dimension n is a locally Euclidean space of dimension n.
>
> ***36.2.*** Prove that a bouquet of two circles is not locally Euclidean.

36.C. If X is a locally Euclidean space of dimension p and Y is a locally Euclidean space of dimension q then $X \times Y$ is a locally Euclidean space of dimension $p + q$.

Dimension

36.D. Can a topological space be simultaneously a locally Euclidean space of dimension both 0 and $n > 0$?

36.E. Can a topological space be simultaneously a locally Euclidean space of dimension both 1 and $n > 1$?

> ***36.3.*** Prove that any nonempty connected subset of a locally Euclidean space of dimension 1 can be made disconnected by removing two points.
>
> ***36.4.*** Prove that any nonempty locally Euclidean space of dimension $n > 1$ contains a nonempty open set, which cannot be made disconnected by removing any two points.

36.F. Can a topological space be simultaneously a locally Euclidean space of dimension both 2 and $n > 2$?

36.G. Let U be an open subset of $\mathbb{R}^2$ and a $p \in U$. Prove that $\pi_1(U \smallsetminus \{p\})$ admits an epimorphism onto $\mathbb{Z}$.

36.H. Deduce from *36.G* that a topological space cannot be simultaneously a locally Euclidean space of dimension both 2 and $n > 2$.

We see that dimension of locally Euclidean topological space is a topological invariant at least for the cases when it is not greater than 2. It is corrected without this restriction. However, one needs some technique to prove this. One possibility is provided by dimension theory, see, e.g., W. Hurewicz and H. Wallman, *Dimension Theory* Princeton, NJ, 1941. Other possibility is to generalize the arguments used in *36.H* to higher dimensions. However, this demands a knowledge of high-dimensional homotopy groups.

> ***36.5.*** Deduce that a topological space cannot be simultaneously a locally Euclidean space of dimension both n and $p > n$ from the fact that $\pi_{n-1}(S^{n-1}) = \mathbb{Z}$. Cf. *36.H*

Interior and Boundary

A point a of a locally Euclidean space X is said to be an *interior* point of X if a has a neighborhood (in X) homeomorphic to $\mathbb{R}^n$. A point $a \in X$, which is not interior, is called a *boundary* point of X.

> ***36.6.*** Which points of $\mathbb{R}^n_+$ have a neighborhood homeomorphic to $\mathbb{R}^n_+$?

36.I. Formulate a definition of boundary point independent of a definition for interior point.

Let X be a locally Euclidean space of dimension n. The set of all interior points of X is called the *interior* of X and denoted by $\operatorname{int} X$. The set of all boundary points of X is called the *boundary* of X and denoted by ∂X.

These terms (interior and boundary) are used also with different meaning. The notions of boundary and interior points of a set in a topological space and the interior part and boundary of a set in a topological space are introduced in general topology, see Section 5. They have almost nothing to do with the notions discussed here. In both senses the terminology is classical, which is impossible to change. This does not create usually a danger of confusion.

Notations are not as commonly accepted as words. We take an easy opportunity to select unambiguous notations: we denote the interior part of a set A in a topological space X by $\operatorname{Int}_X A$ or $\operatorname{Int} A$, while the interior of a locally Euclidean space X is denoted by $\operatorname{int} X$; the boundary of a set in a topological space is denoted by symbol Fr, while the boundary of locally Euclidean space is denoted by symbol ∂.

36.J. For a locally Euclidean space X the interior $\operatorname{int} X$ is an open dense subset of X, the boundary ∂X is a closed nowhere dense subset of X.

36.K. The interior of a locally Euclidean space of dimension n is a locally Euclidean space of dimension n without boundary (i.e., with empty boundary; in symbols: $\partial(\operatorname{int} X) = \varnothing$).

36.L. The boundary of a locally Euclidean space of dimension n is a locally Euclidean space of dimension $n-1$ without boundary (i.e., with empty boundary; in symbols: $\partial(\partial X) = \varnothing$).

36.M. $\operatorname{int} \mathbb{R}^n_+ \supset \{x \in \mathbb{R}^n \,:\, x_1 > 0\}$ and

$$\partial \mathbb{R}^n_+ \subset \{x \in \mathbb{R}^n \,:\, x_1 = 0\}.$$

> ***36.7.*** For any $x, y \in \{x \in \mathbb{R}^n \,:\, x_1 = 0\}$, there exists a homeomorphism $f : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ with $f(x) = y$.

36.N. Either $\partial \mathbb{R}^n_+ = \varnothing$ (and then $\partial X = \varnothing$ for any locally Euclidean space X of dimension n), or $\partial \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \,:\, x_1 = 0\}$.

In fact, the second alternative holds true. However, this is not easy to prove for any dimension.

36.O. Prove that $\partial \mathbb{R}^1_+ = \{0\}$.

36.P. Prove that $\partial \mathbb{R}^2_+ = \{x \in \mathbb{R}^2 \,:\, x_1 = 0\}$. (Cf. *36.G*.)

> ***36.8.*** Deduce that a $\partial \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \,:\, x_1 = 0\}$ from $\pi_{n-1}(S^{n-1}) = \mathbb{Z}$. (Cf. *36.P*, *36.5*)

36.Q. Deduce from $\partial \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \,:\, x_1 = 0\}$ for all $n \geq 1$ that

$$\operatorname{int}(X \times Y) = \operatorname{int} X \times \operatorname{int} Y$$

and

$$\partial(X \times Y) = (\partial(X) \times Y) \cup (X \times \partial Y).$$

The last formula resembles Leibniz formula for derivative of a product.

36.R. Riddle. Can this be a matter of chance?

36.S. Prove that

(a) $\partial(I \times I) = (\partial I \times I) \cup (I \times \partial I)$,
(b) $\partial D^n = S^{n-1}$,
(c) $\partial(S^1 \times I) = S^1 \times \partial I = S^1 \amalg S^1$,
(d) the boundary of Möbius strip is homeomorphic to circle.

36.T Corollary. Möbius strip is not homeomorphic to cylinder $S^1 \times I$.

37. Manifolds

Definition of Manifold

A topological space is called a *manifold* of dimension n if it is

- locally Euclidean of dimension n,
- second countable,
- Hausdorff.

37.A. Prove that the three conditions of the definition are independent (i.e., there exist spaces not satisfying any one of the three conditions and satisfying the other two.)

37.A.1. Prove that $\mathbb{R} \cup_i \mathbb{R}$, where $i : \{x \in \mathbb{R} : x < 0\} \to \mathbb{R}$ is the inclusion, is a non-Hausdorff locally Euclidean space of dimension one.

37.B. Check whether the spaces listed in Problem *36.B* are manifolds.

A compact manifold without boundary is said to be *closed.* As in the case of interior and boundary, this term coincides with one of the basic terms of general topology. Of course, the image of a closed manifold under embedding into a Hausdorff space is a closed subset of this Hausdorff space (as any compact subset of a Hausdorff space). However absence of boundary does not work here, and even non-compact manifolds may be closed subsets. They are closed in themselves, as any space. Here we meet again an ambiguity of classical terminology. In the context of manifolds the term closed relates rather to the idea of a closed surface.

Components of Manifold

37.C. *A connected component of a manifold is a manifold.*

37.D. *A connected component of a manifold is path-connected.*

37.E. *A connected component of a manifold is open in the manifold.*

37.F. *A manifold is the sum of its connected components.*

37.G. The set of connected components of any manifold is countable. If the manifold is compact, then the number of the components is finite.

37.1. Prove that a manifold is connected, iff its interior is connected.

Making New Manifolds out of Old Ones

37.H. Prove that an open subspace of a manifold of dimension n is a manifold of dimension n.

37.I. The interior of a manifold of dimension n is a manifold of dimension n without boundary.

37.J. The boundary of a manifold of dimension n is a manifold of dimension $n - 1$ without boundary.

37.2. The boundary of a compact manifold of dimension n is a closed manifold of dimension $n-1$.

37.K. If X is a manifold of dimension p and Y is a manifold of dimension q then $X \times Y$ is a manifold of dimension $p+q$.

37.L. Prove that a covering space (in narrow sense) of a manifold is a manifold of the same dimension.

37.M. Prove that if the total space of a covering is a manifold then the base is a manifold of the same dimension.

37.N. Let X and Y be manifolds of dimension n, A and B components of ∂X and ∂Y respectively. Then for any homeomorphism $h : B \to A$ the space $X \cup_h Y$ is a manifold of dimension n.

37.O. Let X and Y be manifolds of dimension n, A and B closed subsets of ∂X and ∂Y respectively. If A and B are manifolds of dimension $n-1$ then for any homeomorphism $h : B \to A$ the space $X \cup_h Y$ is a manifold of dimension n.

37.O.1. Prove that the result of gluing of two copy of $\mathbb{R}^n_+$ by the identity map of the boundary hyperplane is homeomorphic to $\mathbb{R}^n$.

Double

37.P. Can a manifold be embedded into a manifold of the same dimension without boundary?

Let X be a manifold. Denote by DX the space $X \cup_{\mathrm{id}_{\partial X}} X$ obtained by gluing of two copies of X by the identity mapping $\mathrm{id}_{\partial X} : \partial X \to \partial X$ of the boundary.

37.Q. Prove that DX is a manifold without boundary of the same dimension as X.

DX is called the *double* of X.

37.R. Prove that a double of a manifold is compact, iff the original manifold is compact.

Collars and Bites

Let X be a manifold. An embedding $c : \partial X \times I \to X$ such that $c(x, 0) = x$ for each $x \in \partial X$ is called a *collar* of X. A collar can be thought of as a neighborhood of the boundary presented as a cylinder over boundary.

37:A. *Every manifold has a collar.*

Let U be an open set in the boundary of a manifold X. For a continuous function $\varphi : \partial X \to \mathbb{R}_+$ with $\varphi^{-1}(0, \infty) = U$ set

$$B_\varphi = \{(x, t) \in \partial X \times \mathbb{R}_+ \, : \, t \le \varphi(x)\}.$$

A *bite* on X at U is an embedding $b : B_\varphi \to X$ with some $\varphi : \partial X \to \mathbb{R}_+$ such that $b(x, 0) = x$ for each $x \in \partial X$.

This is a generalization of collar. Indeed, a collar is a bite at $U = \partial X$ with $\varphi = 1$.

37:A:1. Prove that if $U \subset \partial X$ is contained in an open subset of X homeomorphic to $\mathbb{R}^n_+$, then there exists a bite of X at U.

37:A:2. Prove that for any bite $b : B \to X$ of a manifold X the closure of $X \smallsetminus b(B)$ is a manifold.

37:A:3. Let $b_1 : B_1 \to X$ be a bite of X and $b_2 : B_2 \to \operatorname{Cl}(X \smallsetminus b_1(B_1))$ be a bite of $\operatorname{Cl}(X \smallsetminus b_1(B_1))$. Construct a bite $b : B \to X$ of X with $b(B) = b_1(B_1) \cup b_2(B_2)$.

37:A:4. Prove that if there exists a bite of X at ∂X then there exists a collar of X.

37:B. *For any two collars $c_1, c_2 : \partial X \times I \to X$ there exists a homeomorphism $h : X \to X$ with $h(x) = x$ for $x \in \partial X$ such that $h \circ c_1 = c_2$.*

This means that a collar is unique up to homeomorphism.

37:B:1. For any collar $c : \partial X \times I \to X$ there exists a collar $c' : \partial X \times I \to X$ such that $c(x,t) = c'(x, t/2)$.

37:B:2. For any collar $c : \partial X \times I \to X$ there exists a homeomorphism

$$h : X \to X \cup_{x \mapsto (x,1)} \partial X \times I$$

with $h(c(x,t)) = (x,t)$.

38. Isotopy

Isotopy of Homeomorphisms

Let X and Y be topological spaces, $h, h' : X \to Y$ homeomorphisms. A homotopy $h_t : X \to Y$, $t \in [0,1]$ connecting h and h' (i.e., with $h_0 = h$, $h_1 = h'$) is called an *isotopy* between h and h' if h_t is a homeomorphism for each $t \in [0,1]$. Homeomorphisms h, h' are said to be *isotopic* if there exists an isotopy between h and h'.

38.A. Being isotopic is an equivalence relation on the set of homeomorphisms $X \to Y$.

38.B. Find a topological space X such that homotopy between homeomorphisms $X \to X$ does not imply isotopy.

This means that isotopy classification of homeomorphisms can be more refined than homotopy classification of them.

> ***38.1.*** Classify homeomorphisms of circle S^1 to itself up to isotopy.
>
> ***38.2.*** Classify homeomorphisms of line $\mathbb{R}^1$ to itself up to isotopy.

The set of isotopy classes of homeomorphisms $X \to X$ (i.e. the quotient of the set of self-homeomorphisms of X by isotopy relation) is called the *mapping class group* or *homeotopy group* of X.

38.C. For any topological space X, the mapping class group of X is a group under the operation induced by composition of homeomorphisms.

> ***38.3.*** Find the mapping class group of the union of the coordinate lines in the plane.
>
> ***38.4.*** Find the mapping class group of the union of bouquet of two circles.

Isotopy of Embeddings and Sets

Homeomorphisms are topological embeddings of special kind. The notion of isotopy of homeomorphism is extended in an obvious way to the case of embeddings. Let X and Y be topological spaces, $h, h' : X \to Y$ topological embeddings. A homotopy $h_t : X \to Y$, $t \in [0,1]$ connecting h and h' (i.e., with $h_0 = h$, $h_1 = h'$) is called an *(embedding) isotopy* between h and h' if h_t is an embedding for each $t \in [0,1]$. Embeddings h, h' are said to be *isotopic* if there exists an isotopy between h and h'.

38.D. Being isotopic is an equivalence relation on the set of embeddings $X \to Y$.

A family A_t, $t \in I$ of subsets of a topological space X is called an *isotopy of the set* $A = A_0$, if the graph $\Gamma = \{(x,t) \in X \times I \mid x \in A_t\}$ of the family is fibrewise homeomorphic to the cylinder $A \times I$, i. e. there exists a homeomorphism $A \times I \to \Gamma$ mapping $A \times \{t\}$ to $\Gamma \cap X \times \{t\}$ for any $t \in I$. Such a homeomorphism gives rise to an isotopy of embeddings $\Phi_t : A \to X$, $t \in I$ with $\Phi_0 = \mathrm{in}$, $\Phi_t(A) = A_t$. An isotopy of a subset is also called a *subset isotopy.* Subsets A and A' of the same topological space X are said to be *isotopic in* X, if there exists a subset isotopy A_t of A with $A' = A_1$.

38.E. It is easy to see that this is an equivalence relation on the set of subsets of X.

As it follows immediately from the definitions, any embedding isotopy determines an isotopy of the image of the initial embedding and any subset isotopy is accompanied with an embedding isotopy. However the relation between the notions of subset isotopy and embedding isotopy is not too close because of the following two reasons:

(a) an isotopy Φ_t accompanying a subset isotopy A_t starts with the inclusion of A_0 (while arbitrary isotopy may start with any embedding);
(b) an isotopy accompanying a subset isotopy is determined by the subset isotopy only up to composition with an isotopy of the identity homeomorphism $A \to A$ (an isotopy of a homeomorphism is a special case of embedding isotopies, since homeomorphisms can be considered as a sort of embeddings).

An isotopy of a subset A in X is said to be *ambient,* if it may be accompanied with an embedding isotopy $\Phi_t : A \to X$ extendible to an isotopy $\tilde{\Phi}_t : X \to X$ of the identity homeomorphism of the space X. The isotopy $\tilde{\Phi}_t$ is said to be *ambient* for Φ_t. This gives rise to obvious refinements of the equivalence relations for subsets and embeddings introduced above.

38.F. Find isotopic, but not ambiently isotopic sets in $[0, 1]$.

38.G. If sets $A_1, A_2 \subset X$ are ambiently isotopic then the complements $X \smallsetminus A_1$ and $X \smallsetminus A_2$ are homeomorphic and hence homotopy equivalent.

38.5. Find isotopic, but not ambiently isotopic sets in $\mathbb{R}$.

38.6. Prove that any isotopic compact subsets of $\mathbb{R}$ are ambiently isotopic.

38.7. Find isotopic, but not ambiently isotopic compact sets in R^3.

38.8. Prove that any two embeddings $S^1 \to \mathbb{R}^3$ are isotopic. Find embeddings $S^1 \to \mathbb{R}^3$ that are not ambiently isotopic.

Isotopies and Attaching

38:A. Any isotopy $h_t : \partial X \to \partial X$ extends to an isotopy $H_t : X \to X$.

38:B. Let X and Y be manifolds of dimension n, A and B components of ∂X and ∂Y respectively. Then for any isotopic homeomorphisms $f, g : B \to A$ the manifolds $X \cup_f Y$ and $X \cup_g Y$ are homeomorphic.

38:C. Let X and Y be manifolds of dimension n, let B be a compact subset of ∂Y. If B is a manifold of dimension $n-1$ then for any embeddings $f, g : B \to \partial X$ ambiently isotopic in ∂X the manifolds $X \cup_f Y$ and $X \cup_g Y$ are homeomorphic.

Connected Sums

38.H. Let X and Y be manifolds of dimension n, and $\varphi : \mathbb{R}^n \to X$, $\psi : \mathbb{R}^n \to Y$ be embeddings. Then

$$X \cup_{\psi(D^n)\to X:\psi(a)\mapsto\varphi(a)} Y$$

is a manifold of dimension n.

This manifold is called a *connected sum* of X and Y.

38.I. Show that the topological type of the connected sum of X and Y depends not only on the topological types of X and Y.

38.J. Let X and Y be manifolds of dimension n, and $\varphi : \mathbb{R}^n \to X$, $\psi : \mathbb{R}^n \to Y$ be embeddings. Let $h : X \to X$ be a homeomorphism. Then the connected sums of X and Y defined via ψ and φ, on one hand, and via ψ and $h \circ \varphi$, on the other hand, are homeomorphic.

38.9. Find pairs of manifolds connected sums of which are homeomorphic to

(a) S^1,
(b) Klein bottle,
(c) sphere with three crosscaps.

38.10. Find a disconnected connected sum of connected manifolds. Describe, under what circumstances this can happen.

39. One-Dimensional Manifolds

Zero-Dimensional Manifolds

This section is devoted to topological classification of manifolds of dimension one. We skip the case of 0-dimensional manifolds due to triviality of the problem. Indeed, any 0-dimensional manifold is just a countable discrete topological space, and the only topological invariant needed for topological classification of 0-manifolds is the number of points: two 0-dimensional manifolds are homeomorphic, iff they have the same number of points.

The case of 1-dimensional manifolds is also simple, but it requires more detailed consideration.

Reduction to Connected Manifolds

Since each manifold is the sum of its connected components, two manifolds are homeomorphic if and only if there exists a one-to-one correspondence between their components such that the corresponding components are homeomorphic. Therefore for topological classification of n-manifolds it suffices to classify only *connected* n-manifolds.

Examples

39.A. What connected 1-manifolds do you know?

(a) Do you know any *closed* connected 1-manifold?
(b) Do you know a connected *compact* 1-manifold, which is not closed?
(c) What *non-compact* connected 1-manifolds do you know?
(d) Is there a *non-compact* connected 1-manifolds with boundary?

39.B. Fill the following table with pluses and minuses.

Manifold X	Is X compact?	Is ∂X empty?
S^1		
$\mathbb{R}^1$		
I		
$\mathbb{R}^1_+$		

Statements of Main Theorems

39.C. *Any connected manifold of dimension 1 is homeomorphic either to one of the following for manifolds:*

- *circle S^1,*
- *line $\mathbb{R}^1$,*
- *interval I,*
- *half-line $\mathbb{R}^1_+$.*

This theorem may be splitted into the following four theorems:

39.D. *Any closed connected manifold of dimension 1 is homeomorphic to circle S^1.*

39.E. *Any non-compact connected manifold of dimension 1 without boundary is homeomorphic to line $\mathbb{R}^1$.*

39.F. *Any compact connected manifold of dimension 1 with nonempty boundary is homeomorphic to interval I.*

39.G. *Any non-compact connected manifold of dimension one with nonempty boundary is homeomorphic to half-line $\mathbb{R}^1_+$.*

Lemma on 1-Manifold Covered with Two Lines

39.H Lemma. *Any connected manifold of dimension 1 covered with two open sets homeomorphic to $\mathbb{R}^1$ is homeomorphic either to $\mathbb{R}^1$, or S^1.*

Let X be a connected manifold of dimension 1 and $U, V \subset X$ be its open subsets homeomorphic to $\mathbb{R}$. Denote by W the intersection $U \cap V$. Let $\varphi : U \to \mathbb{R}$ and $\psi : V \to \mathbb{R}$ be homeomorphisms.

39.H.1. Prove that each connected component of $\varphi(W)$ is either an open interval, or an open ray, or the whole $\mathbb{R}$.

39.H.2. Prove that a homeomorphism between two open connected subsets of $\mathbb{R}$ is a (strictly) monotone continuous function.

39.H.3. Prove that if a sequence x_n of points of W converges to a point $a \in U \smallsetminus W$ then it does not converge in V.

39.H.4. Prove that if there exists a bounded connected component C of $\varphi(W)$ then $C = \varphi(W)$, $V = W$, $X = U$ and hence X is homeomorphic to $\mathbb{R}$.

39.H.5. In the case of connected W and $U \neq V$, construct a homeomorphism $X \to \mathbb{R}$ which takes:

- W to $(0, 1)$,
- U to $(0, +\infty)$, and
- V to $(-\infty, 1)$.

39.H.6. In the case of W consisting of two connected components, construct a homeomorphism $X \to S^1$, which takes:

- W to $\{z \in S^1 : -1/\sqrt{2} < \operatorname{Im}(z) < 1/\sqrt{2}\}$,
- U to $\{z \in S^1 : -1/\sqrt{2} < \operatorname{Im}(z)\}$, and
- V to $\{z \in S^1 : \operatorname{Im}(z) < 1/\sqrt{2}\}$.

Without Boundary

39.D.1. Deduce Theorem *39.D* from Lemma *39.G*.

39.E.1. Deduce from Lemma *39.G* that for any connected non-compact one-dimensional manifold X without a boundary there exists an embedding $X \to \mathbb{R}$ with open image.

39.E.2. Deduce Theorem *39.E* from *39.E.1*.

With Boundary

39.F.1. Prove that any compact connected manifold of dimension 1 can be embedded into S^1.

39.F.2. List all connected subsets of S^1.

39.F.3. Deduce Theorem *39.F* from *39.F.2*, and *39.F.1*.

39.G.1. Prove that any non-compact connected manifold of dimension 1 can be embedded into $\mathbb{R}^1$.

39.G.2. Deduce Theorem *39.G* from *39.G.1*.

Consequences of Classification

39.I. Prove that connected sum of closed 1-manifolds is defined up homeomorphism by topological types of summands.

39.J. Which 0-manifolds bound a compact 1-manifold?

Mapping Class Groups

39.K. Find the mapping class groups of

(a) S^1,
(b) $\mathbb{R}^1$,
(c) $\mathbb{R}^1_+$,
(d) $[0,1]$.

39.1. Find the mapping class group of an arbitrary 1-manifold with finite number of components.

40. Two-Dimensional Manifolds

Examples

40.A. What connected 2-manifolds do you know?

(a) List *closed* connected 2-manifold that you know.
(b) Do you know a connected *compact* 2-manifold, which is not closed?
(c) What *non-compact* connected 2-manifolds do you know?
(d) Is there a *non-compact* connected 2-manifolds with boundary?

40.1. Construct non-homeomorphic non-compact connected manifolds of dimension two without boundary and with isomorphic infinitely generated fundamental group.

Ends and Odds

Let X be a non-compact Hausdorff topological space, which is a union of an increasing sequence of its compact subspaces

$$C_1 \subset C_2 \subset \dots \subset C_n \subset \dots \subset X.$$

Each connected component U of $X \smallsetminus C_n$ is contained in some connected component of $X \smallsetminus C_{n-1}$. A decreasing sequence $U_1 \supset U_2 \supset \dots \supset U_n \supset \dots$ of connected components of

$$(X \smallsetminus C_1) \supset (X \smallsetminus C_2) \supset \dots \supset (X \smallsetminus C_n) \supset \dots$$

respectively is called an *end of X with respect to $C_1 \subset \dots \subset C_n \subset \dots$*.

40:A. Let X and C_n be as above, D be a compact set in X and V a connected component of $X \smallsetminus D$. Prove that there exists n such that $D \subset C_n$.

40:B. Let X and C_n be as above, D_n be an increasing sequence of compact sets of X with $X = \cup_{n=1}^{\infty} D_n$. Prove that for any end $U_1 \subset \dots \subset U_n \subset \dots$ of X with respect to C_n there exists a unique end $V_1 \subset \dots \subset V_n \subset \dots$ of X with respect to D_n such that for any p there exists q such that $V_q \subset U_p$.

40:C. Let X, C_n and D_n be as above. Then the map of the set of ends of X with respect to C_n to the set of ends of X with respect to D_n defined by the statement of *40:B* is a bijection.

Theorem *40:C* allows one to speak about *ends* of X without specifying a system of compact sets

$$C_1 \subset C_2 \subset \dots \subset C_n \subset \dots \subset X$$

with $X = \cup_{n=1}^{\infty} C_n$. Indeed, *40:B* and *40:C* establish a canonical one-to-one correspondence between ends of X with respect to any two systems of this kind.

40:D. Prove that $\mathbb{R}^1$ has two ends, $\mathbb{R}^n$ with $n > 1$ has one end.

40:E. Find the number of ends for the universal covering space of the bouquet of two circles.

40:F. Does there exist a 2-manifold with a finite number of ends which cannot be embedded into a compact 2-manifold?

40:G. Prove that for any compact set $K \subset S^2$ with connected complement $S^2 \setminus K$ there is a natural map of the set of ends of $S^2 \setminus K$ to the set of connected components of K.

Let W be an open set of X. The set of ends $U_1 \subset \dots \subset U_n \subset \dots$ of X such that $U_n \subset W$ for sufficiently large n is said to be *open.*

40:H. Prove that this defines a topological structure in the set of ends of X.

The set of ends of X equipped with this topological structure is called the *space of ends* of X. Denote this space by $\mathcal{E}(X)$.

40.1:1. Construct non-compact connected manifolds of dimension two without boundary and with isomorphic infinitely generated fundamental group, but with non-homeomorphic spaces of ends.

40.1:2. Construct non-compact connected manifolds of dimension two without boundary and with isomorphic infinitely generated fundamental group, but with different number of ends.

40.1:3. Construct non-compact connected manifolds of dimension two without boundary with isomorphic infinitely generated fundamental group and the same number of ends, but with different topology in the space of ends.

40.1:4. Let K be a completely disconnected closed set in S^2. Prove that the map $\mathcal{E}(S^2 \setminus K) \to K$ defined in *40:G* is continuous.

40.1:5. Construct a completely disconnected closed set $K \subset S^2$ such that this map is a homeomorphism.

40.B. Prove that there exists an uncountable family of pairwise nonhomeomorphic connected 2-manifolds without boundary.

The examples of non-compact manifolds dimension 2 presented above show that there are too many non-compact connected 2-manifolds. This makes impossible any useful topological classification of non-compact 2-manifolds. Theorems reducing the homeomorphism problem for 2-manifolds of this type to the homeomorphism problem for their spaces of ends do not seem to be really useful: spaces of ends look not much simpler than the surfaces themselves.

However, there is a special class of non-compact 2-manifolds, which admits a simple and useful classification theorem. This is the class of simply connected non-compact 2-manifolds without boundary. We postpone its consideration to the end of this section. Now we turn to the case, which is the simplest and most useful for applications.

Closed Surfaces

40.C. *Any connected closed manifold of dimension two is homeomorphic either to sphere S^2, or sphere with handles, or sphere with crosscaps.*

Recall that according to Theorem *34.M* the basic surfaces represent pairwise distinct topological (and even homotopy) types. Therefore, *34.M* and *40.C* together give topological and homotopy classifications of closed 2-dimensional manifolds.

We do not recommend to prove Theorem *40.C* immediately and, especially, in the formulation given here. All known proofs of *40.C* can be decomposed into two main stages: firstly, a manifold under consideration is equipped with some additional structure (like triangulation or smooth structure); then using this structure a required homeomorphism is constructed. Although the first stage appears in the proof necessarily and is rather difficult, it is not useful outside the proof. Indeed, any closed 2-manifold, which we meet in a concrete mathematical context, is either equipped, or can be easily equipped with the additional structure. The methods of imposing the additional structure are much easier, than a general proof of existence for this structure in arbitrary 2-manifold.

Therefore, we suggest for the first case to restrict ourselves to the second stage of the proof of Theorem *40.C*, prefacing it with general notions related to the most classical additional structure, which can be used for this purpose.

Triangulations of Surfaces

By an *Euclidean triangle* we mean the convex hall of three non-collinear points of Euclidean space. Of course, it is homeomorphic to disk D^2, but not only the topological structure is relevant for us now. The boundary of a triangle contains three distinguished points, its *vertices,* which separates the boundary into three pieces, its *sides.* A *topological triangle* in a topological space X is an embedding of an Euclidean triangle into X. A *vertex* (respectively, *side*) of a topological triangle $T \to X$ is the image of a vertex (respectively, side) of T in X.

A set of topological triangles in a 2-manifold X is a *triangulation* of X provided the images of these triangles comprise a fundamental cover of X and any two of the images either are disjoint or intersect in a common side or in a common vertex.

40.D. Prove that in the case of compact X the former condition (about fundamental cover) means that the number of triangles is finite.

40.E. Prove that the condition about fundamental cover means that the cover is locally finite.

Two Properties of Triangulations of Surfaces

Triangulations of surfaces are not ramified

40.F. Let E be a side of a triangle involved into a triangulation of a 2-manifold X. Prove that there exist exactly two triangles of this triangulation for which E is a side. Cf. *36.G*, *36.H* and *36.P*.

Local strong connectedness

40.G. Let V be a vertex of a triangle involved into a triangulation of a 2-manifold X and T, T' be two triangles of the triangulation adjacent to V. Prove that there exisits a sequence $T = T_1, T_2, \dots, T_n = T'$ of triangles of the triangulation such that V is a vertex of each of them and triangles T_i, T_{i+1} have common side for each $i = 1, \dots, n-1$.

Triangulations present a surface combinatorially.

Scheme of Triangulation

Let X be a 2-manifold and $\mathcal{T}$ a triangulation of X. Denote the set of vertices of $\mathcal{T}$ by V. Denote by Σ_2 the set of triples of vertices, which are vertices of a triangle of $\mathcal{T}$. Denote by Σ_1 the set of pairs of vertices, which are vertices of a side of $\mathcal{T}$. Put $\Sigma_0 = S$. This is the set of vertices of $\mathcal{T}$. Put $\Sigma = \Sigma_2 \cup \Sigma_1 \cup \Sigma_0$. The pair (V, Σ) is called the *(combinatorial) scheme* of $\mathcal{T}$.

40:I. Prove that the combinatorial scheme (V, Σ) of a triangulation of a 2-manifold has the following properties:

(a) Σ is a set consisting of subsets of V,
(b) each element of Σ consists of at most 3 elements of V,
(c) three-element elements of Σ cover V,
(d) any subset of an element of Σ belongs to Σ,
(e) intersection of any collection of elements of Σ belongs to Σ,
(f) for any two-element element of Σ there exist exactly two three-element elements of Σ containing it.

Let V be a set and Σ is a set of finite subsets of V. The pair (V, Σ) is called a *triangulation scheme* if

- any subset of an element of Σ belongs to Σ,
- intersection of any collection of elements of Σ belongs to Σ,
- any one element subset of V belongs to Σ.

There is a natural way to associate a topological space (in fact, a cellular space) to any triangulation scheme. Namely, for a triangulation scheme (V, Σ) consider the set $S(V, \Sigma)$ of all functions $c : V \to I\,(= [0, 1])$ such that

$$\operatorname{Supp}(c) = \{v \in V \,:\, c(v) \neq 0\}$$

belongs to Σ and $\sum_{v \in V} c(v) = 1$. Equip $S(V, \Sigma)$ with the compact open topology.

40:J. Prove that $S(V, \Sigma)$ is a cellular space with cells $\{c \in S \,:\, \operatorname{Supp}(c) = \sigma\}$ with $\sigma \in \Sigma$.

40:K. Prove that if (V, Σ) is the combinatorial scheme of a triangulation of a 2-manifold X then $S(V, \Sigma)$ is homeomorphic to X.

40:L. Let (V, Σ) be a triangulation scheme such that

(a) V is countable,
(b) each element of Σ consists of at most 3 elements of V,
(c) three-element elements of Σ cover V,
(d) for any two-element element of Σ there exist exactly two three-element elements of Σ containing it

Prove that (V, Σ) is a combinatorial scheme of a triangulation of a 2-manifold.

Examples

40.2. Consider the cover of torus obtained in the obvious way from the cover of the square by its halves separated by a diagonal of the square. Is it a triangulation of torus? Why not?

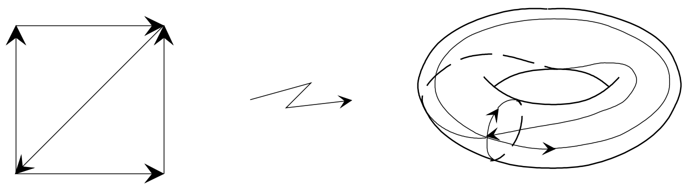


FIGURE 1

40.3. Prove that the simplest triangulation of S^2 consists of 4 triangles.

40.4*. Prove that a triangulation of torus $S^1 \times S^1$ contains at least 14 triangles, and a triangulation of the projective plane contains at least 10 triangles.

A lot! Just say NO to triangulations.

Families of Polygons

The problems considered above show that triangulations provide a combinatorial description of 2-dimensional manifolds, but this description is usually too bulky. Here we will study other, more practical way to present 2-dimensional manifolds combinatorially. The main idea is to use larger building blocks.

Let $\mathcal{F}$ be a collection of convex polygons $P_1, P_2, \dots$. Let the sides of these polygons be oriented and paired off. Then we say that this is a *family of polygons.* There is a natural quotient space of the sum of polygons involved in a family: one identifies each side with its pair-mate by a homeomorphism, which respects the orientations of the sides. This quotient space is called just the *quotient of the family.*

40.H. Prove that the quotient of the family of polygons is a 2-manifold without boundary.

40.I. Prove that the topological type of the quotient of a family does not change when the homeomorphism between the sides of a distinguished pair is replaced by other homeomorphism which respects the orientations.

40.J. Prove that any triangulation of a 2-manifold gives rise to a family of polygon whose quotient is homeomorphic to the 2-manifold.

A family of polygons can be described combinatorially: Assign a letter to each distinguished pair of sides. Go around the polygons writing down the letters assigned to the sides and equipping a letter with exponent -1 if the side is oriented against the direction in which we go around the polygon. At each polygon we write a word. The word depends on the side from which we started and on the direction of going around the polygon. Therefore it is defined up to cyclic permutation and inversion. The collection of words assigned to all the polygons of the family is called a *phrase associated with the family of polygons.* It describes the family to the extend sufficient to recovering the topological type of the quotient.

> ***40.5.*** Prove that the quotient of the family of polygons associated with phrase $aba^{-1}b^{-1}$ is homeomorphic to $S^1 \times S^1$.
>
> ***40.6.*** Identify the topological type of the quotient of the family of polygons associated with phrases
>
> (a) aa^{-1};
> (b) ab, ab;
> (c) aa;
> (d) $abab^{-1}$;
> (e) $abab$;
> (f) $abcabc$;
> (g) $aabb$;
> (h) $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\dots a_gb_ga_g^{-1}b_g^{-1}$;
> (i) $a_1a_1a_2a_2\dots a_ga_g$.

40.K. A collection of words is a phrase associated with a family of polygons, iff each letter appears twice in the words.

A family of polygons is called *irreducible* if the quotient is connected.

40.L. A family of polygons is irreducible, iff a phrase associated with it does not admit a division into two collections of words such that there is no letter involved in both collections.

Operations on Family of Polygons

Although any family of polygons defines a 2-manifold, there are many families defining the same 2-manifold. There are simple operations which change a family, but do not change the topological type of the quotient of the family. Here are the most obvious and elementary of these operations.

(a) Simultaneous reversing orientations of sides belonging to one of the pairs.
(b) Select a pair of sides and subdivide each side in the pair into two sides. The orientations of the original sides define the orderings of the halves. Unite the first halves into one new pair of sides, and the second halves into the other new pair. The orientations of the original sides define in an obvious way orientations of their halves. This operation is called 1-*subdivision.* In the quotient it effects in subdivision of a 1-cell (which is

the image of the selected pair of sides) into two 1-cells. This 1-cells is replaced by two 1-cells and one 0-cell.

(c) The inverse operation to 1-subdivision. It is called 1-*consolidation.*

(d) Cut one of the polygons along its diagonal into two polygons. The sides of the cut comprise a new pair. They are equipped with an orientation such that gluing the polygons by a homeomorphism respecting these orientations recovers the original polygon. This operation is called 2-*subdivision.* In the quotient it effects in subdivision of a 2-cell into two new 2-cells along an arc whose end-points are 0-cells (may be coinciding). The original 2-cell is replaced by two 2-cells and one 1-cell.

(e) The inverse operation to 2-subdivision. It is called 2-*consolidation.*

Topological and Homotopy Classification of Closed Surfaces

40.M Reduction Theorem. *Any finite irreducible family of polygons can be reduced by the five elementary operations to one of the following standard families:*

(a) aa^{-1}

(b) $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\dots a_gb_ga_g^{-1}b_g^{-1}$

(c) $a_1a_1a_2a_2\dots a_ga_g$ *for some natural* g.

40.N Corollary. *Any triangulated closed connected* 2*-manifold is homeomorphic to either sphere, or sphere with handles, or sphere with crosscaps.*

Theorems *40.N* and *34.M* provide classifications of triangulated closed connected 2-manifolds up to homeomorphisms and homotopy equivalence.

40.M.1 ***Reduction to Single Polygon.*** Any finite irreducible family of polygons can be reduced by elementary operations to a family consisting of a single polygon.

40.M.2 ***Cancellation.*** A family of polygons corresponding to a phrase containing a fragment aa^{-1} or $a^{-1}a$, where a is any letter, can be transformed by elementary operations to a family corresponding to the phrase obtained from the original one by erasing this fragment, unless the latter is the whole original phrase.

40.M.3 ***Reduction to Single Vertex.*** An irreducible family of polygons can be turned by elementary transformations to a polygon such that all its vertices are projected to a single point of the quotient.

40.M.4 ***Separation of Crosscap.*** A family corresponding to a phrase consisting of a word $XaYa$, where X and Y are words and a is a letter, can be transformed to the family corresponding to the phrase $bbY^{-1}X$.

40.M.5. If a family, whose quotient has a single vertex in the natural cell decomposition, corresponds to a phrase consisting of a word $XaYa^{-1}$, where X and Y are nonempty words and a is a letter, then $X = UbU'$ and $Y = Vb^{-1}V'$.

40.M.6 ***Separation of Handle.*** A family corresponding to a phrase consisting of a word $UbU'aVb^{-1}V'a^{-1}$, where U, U', V, and V' are words and a, b are letters, can be transformed to the family presented by phrase $dcd^{-1}c^{-1}UV'VU'$.

40.M.7 ***Handle plus Crosscap Equals 3 Crosscaps.*** A family corresponding to phrase $aba^{-1}b^{-1}ccX$ can be transformed by elementary transformations to the family corresponding to phrase $abdbadX$.

Recognizing Closed Surfaces

40.O. What is the topological type of the 2-manifold, which can be obtained as follows: Take two disjoint copies of disk. Attach three parallel strips connecting the disks and twisted by π. The resulting surface S has a connected boundary. Attach a copy of disk along its boundary by a homeomorphism onto the boundary of the S. This is the space to recognize.

40.P. *Euler characteristic of the cellular space obtained as quotient of a family of surfaces is invariant under homotopy equivalences.*

> ***40.7.*** How can *40.P* help to solve *40.O*?
>
> ***40.8.*** Let X be a closed connected surface. What values of $\chi(X)$ allow to recover the topological type of X? What ambiguity is left for other values of $\chi(X)$?

Orientations

By an *orientation of a segment* one means an ordering of its end points (which one of them is initial and which one is final). By an *orientation of a polygon* one means orientation of all its sides such that each vertex is the final end point for one of the adjacent sides and initial for the other one. Thus an orientation of a polygon includes orientation of all its sides. Each segment can be oriented in two ways, and each polygon can be oriented in two ways.

An orientation of a family of polygons is a collection of orientations of all the polygons comprising the family such that for each pair of sides one of the pair-mates has the orientation inherited from the orientation of the polygon containing it while the other pair-mate has the orientation opposite to the inherited orientation. A family of polygons is said to be *orientable* if it admits an orientation.

> ***40.9.*** Which of the families of polygons from Problem *40.6* are orientable?
>
> ***40.10.*** Prove that a family of polygons associated with a word is orientable iff each letter appear in the word once with exponent -1 and once with exponent 1.

40.Q. *Orientability of a family of polygons is preserved by the elementary operations.*

A surface is said to be *orientable* if it can be presented as the quotient of an orientable family of polygons.

40.R. A surface S is orientable, iff any family of polygons whose quotient is homeomorphic to S is orientable.

40.S. Spheres with handles are orientable. Spheres with crosscaps are not.

More About Recognizing Closed Surfaces

40.11. How can the notion of orientability and *40.Q* help to solve *40.O*?

40.T. *Two closed connected manifolds of dimension two are homeomorphic iff they have the same Euler characteristic and either are both orientable or both nonorientable.*

Compact Surfaces with Boundary

As in the case of one-dimensional manifolds, classification of compact two-dimensional manifolds with boundary can be easily reduced to the classification of closed manifolds. In the case of one-dimensional manifolds it was very useful to double a manifold. In two-dimensional case there is a construction providing a closed manifold related to a compact manifold with boundary even closer than the double.

40.U. *Contracting to a point each connected component of the boundary of a two-dimensional compact manifold with boundary gives rise to a closed two-dimensional manifold.*

40.12. A space homeomorphic to the quotient space of *40.U* can be constructed by attaching copies of D^2 one to each connected component of the boundary.

40.V. *Any connected compact manifold of dimension two with nonempty boundary is homeomorphic either to sphere with holes, or sphere with handles and holes, or sphere with crosscaps and holes.*

40.W. Riddle. Generalize orientabilty to the case of nonclosed manifolds of dimension two. (Give as many generalization as you can and prove that they are equivalent. The main criterium of success is that the generalized orientability should help to recognize the topological type.)

40.X. *Two compact connected manifolds of dimension two are homeomorphic iff they have the same Euler characteristic, are both orientable or both nonorientable and their boundaries have the same number of connected components.*

Simply Connected Surfaces

40:M Theorem*. *Any simply connected non-compact manifold of dimension two without boundary is homeomorphic to* $\mathbb{R}^2$.

41. One-Dimensional $\mathbb{Z}_2$-Homology of Surfaces

Polygonal Paths on Surface

Let F be a triagulated surface. A path $s : I \to F$ is said to be *polygonal* if $s(I)$ is contained in the one-dimensional skeleton of the triangulation of F, the preimage of any vertex of the triangulation is finite, and the restriction of s to a segment between any two consequitive points which are mapped to vertices is an affine homeomorphism onto an edge of the triangulation. In terms of kinematics, a polygonal path represents a moving point, which goes only along edges, does not stay anywhere, and, whenever it appears on an edge, it goes along the edge with a constant speed to the opposite end-point. A circular loop $l : S^1 \to F$ is said to be *polygonal* if the corresponding path $I \xrightarrow{t \mapsto \exp(2\pi i t)} S^1 \xrightarrow{l} F$ is polygonal.

41:A. Let F be a triagulated surface. Any path $s : I \to F$ connecting vertices of the triangulation is homotopic to a polygonal path. Any circular loop $l : S^1 \to F$ is freely homotopic to a polygonal one.

A polygonal path is a combinatorial object:

41:B. To describe a polygonal path up to homotopy, it is enough to specify the order in which it passes through vertices.

On the other hand, pushing a path to the one-dimensional skeleton can create new double points. Some edges may appear several time in the same edge.

> ***41:1.*** Let F be a triangulated surface and α be an element of $\pi_1(F)$ different from 1. Prove that there exists a natural N such that for any $n \geq N$ each polygonal loop representing α^n passes through some edge of the triangulation more than once.

Subdivisions of Triangulation

To avoid a congestion of paths on edges, one can add new edges, i.e., subdivide the triangulation. Although an elementary operation on families of polygons applied to a triangulation, gives rise to a family, which is not a triangulation, making several elementary operations, one can get a new triangulation with more edges.

One triangulation of a surface is called a *refinement* of another one if each triangle of the former is contained in a triagle of the latter. There are several standard ways to construct a refinement of a triangulation.

For example, add a new vertex, which is located inside of a triangle τ of a given triangulation, connect it with the vertices of this triangle with segments, which are three new edges. The triangle is subdivided into three new triangles. The other triangles of the original triangulations are kept intact. This is called the *star subdivision centered at* τ. See Figure 2.

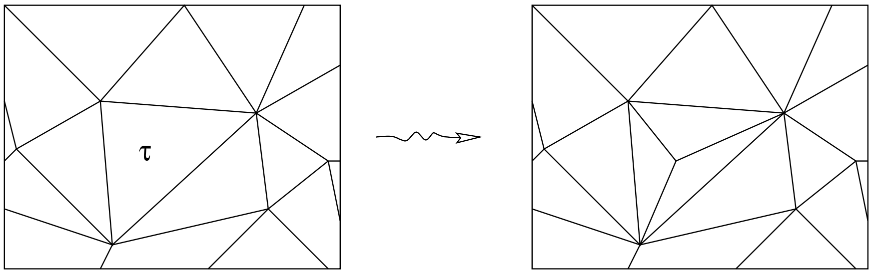


FIGURE 2. Star subdivision centered at triangle τ

Another kind of local subdivision: add a new vertex located on an edge ε of a given triangulation, connect by new edges this vertex to the vertices opposite to ε of the triangles adjacent to ε. Each of the adjacent triangles is subdived into two new triangles. Leave the other triangles intact. This is a *star subdivision centered at* ε. See Figure 3.

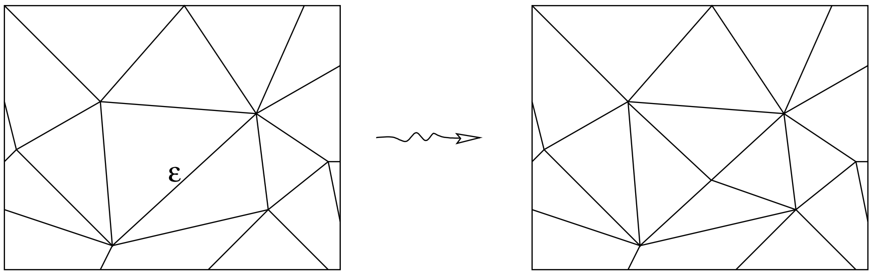


FIGURE 3. Star subdivision centered at edge ε

41:2. Construct a triangulation and its subdivision which cannot be obtained as a composition of star subdivisions centered at edges and triangles.

41:3. Prove that a subdivision of a triangulation of a compact surface can be presented as a result of a finite sequence of star sudivisions centered at triangles and edges and operations inverse to operations of these types.

Bringing Loops to General Position

41:C. Let F be a triangulated and u, v polygonal circular loops on F. Then there exist a subdivision of the triangulation of F and polygonal loops u', v' homotopic to u and v, respectively, such that $u'(I) \cap v'(I)$ is finite.

41:D. Let F be a triangulated and u a polygonal circular loop on F. Then there exist a subdivision of the triangulation of F and a polygonal loop v homotopic to u such that v maps the preimage $v^{-1}(\varepsilon)$ of any edge $\varepsilon \subset v(I)$ homeomorphically onto ε. (In other words, v passes along each edge at most once).

Let u, v be polygonal circular loops on a triangulated surface F and a be an isolated point of $u(I) \cap v(I)$. Suppose $u^{-1}(a)$ and $v^{-1}(a)$ are one point sets. One says that u intersects v *translversally* at a if there exist a neighborhood U of a in F and a homeomorphism $U \to \mathbb{R}^2$ which maps $u(I) \cap U$ onto the x-axes and $v(I) \cap U$ to y-axes.

Polygonal circular loops u, v on a triangulated surface are said to be in *general position* to with respect each other, if $u(I) \cap v(i)$ is finite, for each point $a \in u(i) \cap v(I)$ each of the sets $u^{-1}(a)$ and $v^{-1}(a)$ contains a single point and u, v are transversal at a.

41:E. Any two circular loops on a triangulated surface are homotopic to circular loops, which are polygonal with respect to some subdivision of the triangulation and in general position with respect to each other.

For a map $f : X \to Y$ denote by $S_k(f)$ the set

$$\{a \in X \mid f^{-1}f(a) \text{ consists of } k \text{ elements}\}$$

and put

$$S(f) = \{a \in X \mid f^{-1}f(a) \text{ consists of more than 1 element}\}.$$

A polygonal circular loop l on a triangulated surface F is said to be *generic* if

(a) $S(l)$ is finite,
(b) $S(l) = S_2(l)$,
(c) at each $a \in l(S_2(l))$ the two branches of $s(I)$ intersecting at a are transversal, that is a has a neighborhood U in F such that there exists a homeomorphism $U \to \mathbb{R}^2$ mapping the images under s of the connected components of $s^{-1}(U)$ to the coordinate axis.

41:F. Any circular loop on a triangulated surface is homotopic to a circular loop, which is polygonal with respect to some subdivision of the triangulation and generic.

Generic circular loops are especially suitable for graphic representation, because the image of a circular loop defines it to a great extend:

41:G. Let l be a generic polygonal loop on a triangulated surface. Then any generic polygonal loop k with $k(S^1) = l(S^1)$ is homotopic in $l(S^1)$ to either l or l^{-1}.

Thus, to describe a generic circular loop up to a reparametrization homotopic to identity, it is sufficient to draw the image of the loop on the surface and specify the direction in which the loop runs along the image.

The image of a generic polygonal loop is called a *generic (polygonal) closed connected curve.* A union of a finite collection of generic closed connected polygonal curves is called a *generic (polygonal) closed curve.* A generic closed connected curve without double points (i.e., an embedded oriented circle contained in the one-dimensional skeleton of a triangulated surface) is called a *simple polygonal closed curve.*

The adjective *closed* in the definitions above appears because there is a version of the definitions with (non-closed) paths instead of loops.

41:H. Riddle. What modifications in Problems *41:C* – *41:G* and corresponding definitions should be done to replace loops by paths everywhere?

By a *generic polygonal curve* we will mean a union of a finite collection of pairwise disjoint images of generic polygonal loops and paths.

Cutting Surface Along Curve

41:I Cutting Surface Along Curve. Let F be a triangulated two-dimensional manifold and $C \subset F$ a one-dimensional manifold contained in the 1-skeleton of the triangulation of F. Assume that $\partial C = \partial F \cap C$. Prove that there exists a two-dimensional manifold T and a surjective continuous map $p : T \to F$ such that:

(a) $p| : T \smallsetminus p^{-1}(C) \to F \smallsetminus C$ is a homeomorphism,
(b) $p| : p^{-1}(C) \to C$ is a two-fold covering.

Such T and p are unique up to a homeomorphism: if $\tilde{T}$ and $\tilde{p}$ are other manifold and mapping satisfying the same conditions then there exists a homeomorphism $h : \tilde{T} \to T$ such that $p \circ h = \tilde{p}$.

The surface T described in *41:I* is called the *result of cutting F along C*. It is denoted by F ✂ C. This is not the complement $F \smallsetminus C$, though a copy of $F \smallsetminus C$ is contained in F ✂ C as a dense subset, which is homotopy equivalent to the whole F ✂ C.

41:J Triangulation of F ✂ C. There exists a unique triangulation of F ✂ C such that the natural map F ✂ $C \to F$ maps edges onto edges and triangles onto triangles homeomorphically.

41:4. Describe the topological type of F ✂ C for the following F and C:

(a) F is Möbius band, C its core circle (deformation retract);
(b) $F = S^1 \times S^1$, $C = S^1 \times 1$;
(c) F is $S^1 \times S^1$ standardly embedded into $\mathbb{R}^3$, C the trefoil knot on F, that is $\{(z, w) \in S^1 \times S^1 \mid z^2 = w^3\}$;
(d) F is Möbius band, C is a segment: show that there are two possible placements of C in F and describe F ✂ C for both of them;
(e) $F = \mathbb{R}P^2$, $C = \mathbb{R}P^1$.
(f) $F = \mathbb{R}P^2$, C is homeomorphic to circle: show that there are two possible placements of C in F and describe F ✂ C for both of them.

41:5 Euler Characteristic and Cutting. Find the Euler characteristic of F ✂ C when $\partial C = \varnothing$. What if $\partial C \neq \varnothing$?

Curves on Surfaces and Two-Fold Coverings

Let F be a two-dimensional triangulated surface and $C \subset F$ a manifold of dimension one contained in the 1-skeleton of the triangulation of F. Let $\partial C = \partial F \cap C$. Since the preimage $\tilde{C}$ of C under the natural projection $F \text{✂} C \to F$ is a two-fold covering space of C, there is an involution $\tau : \tilde{C} \to \tilde{C}$ which is the only nontrivial automorphism of this covering. Take two copies of $F \text{✂} C$ and identify each $x \in \tilde{C}$ in one of them with $\tau(x)$ in the other copy. The resulting space is denoted by $F^{\approx C}$.

41:K. The natural projection $F \text{✂} C \to F$ defines a continuous map $F^{\approx C} \to F$. This is a two-fold covering. Its restriction over $F \setminus C$ is trivial.

One-Dimensional $\mathbb{Z}_2$-Cohomology of Surface

By *35:G*, a two-fold covering of F can be thought of as an element of $H^1(F; \mathbb{Z}_2)$. Thus any one-dimensional manifold C contained in the 1-skeleton of F and such that $\partial C = \partial F \cap C$ defines a cohomology class of F with coefficients in $\mathbb{Z}_2$. This class is said to be *realized* by C.

41:L. The cohomology class with coefficients in $\mathbb{Z}_2$ realized by C in a compact surface F is zero, iff C divides F, that is, $F = G \cup H$, where G and H are compact two-dimensional manifolds with $G \cap H = C$.

Recall that the cohomology group of a path-connected space X with coefficients in $\mathbb{Z}_2$ is defined above in Section 35 as $\operatorname{Hom}(\pi_1(X), \mathbb{Z}_2)$.

41:M. Let F be a triangulated connected surface, let $C \subset F$ be a manifold of dimension one with $\partial C = \partial F \cap C$ contained in the 1-skeleton of F. Let l be a polygonal loop on F which is in general position with respect to C. Then the value which the cohomology class with coefficients in $\mathbb{Z}_2$ defined by C takes on the element of $\pi_1(F)$ realized by l equals the number of points of $l \cap C$ reduced modulo 2.

One-Dimensional $\mathbb{Z}_2$-Homology of Surface

41:N $\mathbb{Z}_2$-Classes via Simple Closed Curves. Let F be a triangulated connected two-dimensional manifold. Every homology class $\xi \in H_1(F; \mathbb{Z}_2)$ can be represented by a polygonal simple closed curve.

41:O. A $\mathbb{Z}_2$-homology class of a triangulated two-dimensional manifold F represented by a polygonal simple closed curve $A \subset F$ is zero, iff there exists a compact two-dimensional manifold $G \subset F$ such that $A = \partial G$.

Of course, the "if" part of *41:O* follows straightforwardly from *35:L*. The "only if" part requires trickier arguments.

41:O:1. If A is a polygonal simple closed curve on F, which does not bound in F a compact 2-manifold, then there exists a connected compact 1-manifold $C \subset F$ with $\partial C = \partial F \cap C$, which intersects A in a single point transversally.

41:O:2. Let F be a two-dimensional triangulated surface and $C \subset F$ a manifold of dimension one contained in the 1-skeleton of the triangulation of F. Let $\partial C = \partial F \cap C$. Any polygonal loop $f : S^1 \to F$, which intersects C in an odd number of points and transversally at each of them, is covered in $F^{\approx C}$ by a path with distinct end-points.

41:O:3. See *35:6*.

Poincaré Duality

To be written!

One-Sided and Two-Sided Simple Closed Curves on Surfaces

To be written!

Orientation Covering and First Stiefel-Whitney Class

To be written!

Relative Homology

To be written!

42. Surfaces Beyond Classification

To be written!

Genus of Surface

To be written!

Systems of disjoint curves on a surface

To be written!

Polygonal Jordan and Schönflies Theorems

To be written!

Polygonal Annulus Theorem

To be written!

Dehn Twists

To be written!

Coverings of Surfaces

To be written!

Branched Coverings

To be written!

Mapping Class Group of Torus

To be written! Lifting homeomorphisms to the universal covering space. Nielsen and Baer Theorems for torus. $GL(2,\mathbb{Z})$. Dehn twists along meridian and longitude and relation between them. Center of the mapping class group.

Braid Groups

To be written!

43. Three-Dimensional Manifolds

To be written!

Poincaré Conjecture

Lens Spaces

Seifert Manifolds

Fibrations over Circle

Heegaard Splitting and Diagrams

CHAPTER 7

Smooth Manifolds

Although manifolds provide a scene for almost all geometric branches of Mathematics, the topological structure of a manifold does not decorate this scene enough. It is not sufficient to discuss most of phenomena of Analysis and Geometry.

Usually in applications, manifolds arise equipped with various additional structures. One of them, *smooth* or *differential* structure, appears more often than others. The goal of this Chapter is to introduce the smooth structure and develope the basic theory.

While topological structures provide a basis for discussing phenomena related to continuity, smooth structures provide a basis for discussing phenomena related to differentiability.

The traditional definition of smooth structures is quite long and different from definitions of similar, and, in fact, closely related structures which are studied in algebraic geometry and topology. Furthermore, smooth structures are traditionally defined only on manifolds. This deprives us of flexibility that we enjoy in general topology, where any set-theoretic construction has a topological counter-part: a subset turns into a subspace, a quotient set turns into a quotient space, etc. The image of a differential manifold under a differentiable map may happen to be not a manifold, and hence not eligible to bear any trace of a differential structure.

Therefore we dare to change the very basic definitions of the differential topology. The notion of differential manifold becomes a special case of more general notions of differential space and differential variety. Of course, specialists are aware about the possibility of these generalizations. However as far as we know, nobody did a serious attempt to rely on the generalizations in a textbook written for beginners. We try to overcome the phobia about singularities, which was a characteristic property of texts on differential topology. We believe, this makes the subject simpler, although introduces possibilty to speak about pathological objects.

As we claimed above, we think on teaching the elementary topology as about teaching a language. This is a great language, one of the main parts of the language of Mathematics. It is not our goal to teach only "politically correct" words: we do not want to exclude a single word just because it can be used in a description of "bad", "pathological" objects.

Of course, the standard approach to smooth manifolds is also presented, right after the new one. But first, we must refresh the background from Multivariable Calculus.

44. Analytic Digression: Differentiable Functions in Euclidean Space

Differentiability and Differentials

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called *differentiable* at $a \in \mathbb{R}$ if there exists a number $f'(a)$ such that

$$\lim_{x\to 0} \frac{f(a+x) - f(a)}{x} = f'(a). \tag{21}$$

This definition does not admit immediate generalization to the case of a map $\mathbb{R}^n \to \mathbb{R}^k$, but can be reformulated in a way that does. Namely, denote by L the linear map $\mathbb{R} \to \mathbb{R} : x \mapsto f'(a)x$. Then (21) is equivalent to

$$\lim_{x\to 0} \frac{f(a+x) - f(a) - L(x)}{x} = 0.$$

Let f be a map defined in a neighborhood of a point $a \in \mathbb{R}^n$ and taking values in $\mathbb{R}^k$. One says that f is *differentiable at a* if there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^k$ such that

$$\lim_{x\to 0} \frac{|f(a+x) - f(a) - L(x)|}{|x|} = 0.$$

In this case L is called the *differential* of f at a.

44:A. If f is differentiable at a, then its differential at a is unique.

The differential of f at a is denoted by d_af.

> ***44:1.*** Prove that for any linear map $L : \mathbb{R}^n \to \mathbb{R}^k$ different from d_af there exists a neighborhood U of a such that
>
> $$|f(x) - f(a) - d_af(x-a)| < |f(x) - f(a) - L(x-a)|$$
>
> for $x \in U \smallsetminus a$.
>
> Theorem *44:1* means that the affine map $x \mapsto f(a) + d_af(x-a)$ approximates f in a neighborhood of a better than any other affine map.
>
> ***44:2.*** Prove that if the dimensions of both source and target are equal to 1 then d_af is multiplication by $\dfrac{df}{dx}(a)$.

Derivative Along Vector

The image of a vector v under d_af, i.e., $d_af(v)$ is denoted also by $D_vf(a)$ and called the *derivative of f at a in direction v.*

> ***44:3.*** Prove that $D_vf(a) = \lim_{t\to 0} \dfrac{f(a+tv) - f(a)}{t}$.
>
> Thus, $D_vf(a)$ is the velocity of changing of f when a moves with velocity v.

44:4. Prove that if v is the i-th standard base vector (i.e., all the components of v, but i-th, equal 0, and the i-th component is 1), then $D_v f(a)$ is equal to $(\frac{\partial f_1}{\partial x_i}(a), \frac{\partial f_2}{\partial x_i}(a), \dots \frac{\partial f_k}{\partial x_i}(a))$, where f_j is the j-th component of f.

44:5. The differential $d_a f$ of a map $f : \mathbb{R}^n \to \mathbb{R}^k$ has matrix $\left(\frac{\partial f_j}{\partial x_i}\right)$ (the Jacobian matrix of f at a).

Main Properties of Differential

44:B. Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^k$, $W \subset \mathbb{R}^m$. If a map $f : U \to V$ is differentiable at $a \in U$ and $g : V \to W$ is differentiable at $f(a)$ then $g \circ f : U \to W$ is differentiable at a and $d_a(g \circ f) = d_{f(a)} g \circ d_a f$. In other words, the differential of composition is the composition of differentials.

44:6. Recognize Theorem *44:B* as a reformulation of the Chain Rule.

44:C. The differential of the identity map is the identity map.

44:7 ***Generalization of 44:C.*** Differential of a linear map L at each point coinsides with L.

Higher Order Derivatives

If $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^k$ and a map $f : U \to V$ is differentiable at each point of U, the differentials $d_a f$ give rise to map $U \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^k) : a \mapsto d_a f$. This generalizes the notion of derivative function.

There is a map closely related to this one, and more convinient for generalizations. It is defined as follows: $U \times \mathbb{R}^n \to \mathbb{R}^k : (a, v) \mapsto D_v f(a)$. The relation is provided by the definition $D_v f(a) = d_a f(v)$.

44:D. Prove that $U \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^k) : a \mapsto d_a f$ is contnuous iff $U \times \mathbb{R}^n \to \mathbb{R}^k : (a, v) \mapsto D_v f(a)$ is continuous.

44:E. Prove that $U \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^k) : a \mapsto d_a f$ is differentiable at a iff $U \times \mathbb{R}^n \to \mathbb{R}^k : (a, v) \mapsto D_v f(a)$ is differentiable at (a, v) for each $v \in \mathbb{R}^n$.

44:8. ***Riddle.*** How does this look like in the case of $n = 1$ or even $n = k = 1$?

Is it possible to reduce in *44:E* the set of v, for which $U \times \mathbb{R}^n \to \mathbb{R}^k : (a, v) \mapsto D_v f(a)$ is differentiable at (a, v)?

The map $U \times \mathbb{R}^n \to \mathbb{R}^k : (a, v) \mapsto D_v f(a)$ is called the *derivative* map for f and denoted by Df. Since it is also a map of the same kind, one can iterate the construction and define the second derivative $D^2 f : U \times (\mathbb{R}^n)^2 \to \mathbb{R}^k$, third derivative $D^3 f : U \times (\mathbb{R}^n)^3 \to \mathbb{R}^k$ and r-th derivative $D^r f : U \times (\mathbb{R}^n)^r \to \mathbb{R}^k$.

44:9. Prove that $D^r f(a, v_1, \dots v_r)$ does not change when one interchange $v_1, \dots\ v_r$.

44:10. Express $D^r f(a, v_1, \dots v_r)$ in "classical" terms, i.e., write down an expression for $D^r f(a, v_1, \dots v_r)$ in terms of partial derivatives of components of f and coordinates of $v_1, \dots\ v_r$.

44:11. Let $e_1, \dots\ e_n$ be the standard basis of $\mathbb{R}^n$. Prove that for $x = \sum_{i=1}^n x^i e_i$

$$\lim_{x \to 0} \frac{|f(a+x) - f(a) - \sum_{r=1}^s \frac{1}{r!} \sum_{i_1,\dots,i_r=1}^n D^r f(a, e_{i_1}, \dots, e_{i_r}) x^{i_1} \dots x^{i_r}|}{|x|^s} = 0$$

C^r-Maps

Let U be an open subset of $\mathbb{R}^n$ and r a non-negative integer or ∞. A map $f : U \to \mathbb{R}^k$ is said to be of *class* C^r or a C^r*-map* if at each point of U it has all the derivatives of order $\le r$ and all of them are continuous. A map is of class C^∞ if it is of class C^r for all finite r.

44:F. A map $f : U \to \mathbb{R}^k$ is of class C^r, iff its components $f_1, \dots\ f_k$ have all the partial derivatives of order $\le r$ and these partial derivatives are continuous.

44:12. Construct a map which has all the partial derivatives of order $\le r$ at each point, but is not of class C^r.

Let U be an open subset of R^n. A map $f : U \to \mathbb{R}^k$ is said to be *real analytic* or *of class* C^a at $x_0 \in U$ if there exists a neighborhood V of x_0 in U such that the Taylor series $\sum_{r=0}^\infty \frac{1}{r!} \sum_{i_1,\dots,i_r=1}^n D^r f(x_0, e_{i_1}, \dots, e_{i_r}) x^{i_1} \dots x^{i_r}$ converges to $f(x_0 + x)$ for $x_0 + x \in V$.

44:G. A real analytic map is of class C^∞.

A map of class C^0 is just a continuous map. It is convinient to assume $a > \infty$ and speak about classes C^r with $0 \le r \le a$.

Diffeomorphisms

Let U, V be open subsets of $\mathbb{R}^n$ and r be a natural number, or ∞ or a. A map $f : U \to V$ is called a *diffeomorphism of class* C^r, or C^r*-diffeomorphism* or just *diffeomorphism*(of U to V) if f is of class C^r at each point of U, invertible, and f^{-1} is of class C^r at each point of V.

44:H. The differential of a diffeomorphism at any point is an isomorphism.

44:I. Composition of C^r-diffeomorphisms is a C^r-diffeomorphism. The map inverse to a C^r-diffeomorphism is a C^r-diffeomorphism.

44:13. Which of the following maps are diffeomorphisms and what are the classes of the diffeomorphisms:

(a) $\mathbb{R} \to \mathbb{R} : x \mapsto x^2$,
(b) $\mathbb{R} \to \mathbb{R} : x \mapsto x^3$,
(c) $(0,1) \to (0,1) : x \mapsto x^3$,
(d) $(0,1) \to (0,1) : x \mapsto x^2$,

(e) $\mathbb{C} \to \mathbb{C} : x \mapsto x^3$,
(f) $\mathbb{R} \to \mathbb{R} : x \mapsto x + x^3$,
(g) $\mathbb{R} \to \mathbb{R} : x \mapsto x + x^2$,
(h) $\mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} x + x^2, \text{ if } x \geq 0 \\ x - x^2, \text{ if } x < 0 \end{cases}$,
(i) $\mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} x + x^5, \text{ if } x \geq 0 \\ x - x^{44}, \text{ if } x < 0 \end{cases}$,
(j) $\mathbb{R} \to \mathbb{R} : x \mapsto x^{5/3}$,
(k) $\mathbb{R} \to \mathbb{R} : x \mapsto x + x^{101/3}$?

Inverse Function Theorem

The following important and famous theorem is a sort of inverse to *44:H*.

44:J Inverse Function Theorem. *If $f : U \to \mathbb{R}^n$ is a C^r-map with $r \geq 1$ defined in a neighborhood U of $a \in \mathbb{R}^n$ and $d_a f : \mathbb{R}^n \to \mathbb{R}^n$ is invertible then there exist neighborhoods $V \subset U$ of a and $W \subset \mathbb{R}^n$ of $f(a)$ such that $f| : V \to W$ is a C^r-diffeomorphism.*

44:K Corollary. Let U and V be open sets in $\mathbb{R}^n$. A map $f : U \to V$ is a C^r-diffeomorphism iff it is a bijective C^r-map and $d_a f$ is an isomorphism for any $a \in U$.

Implicit Function Theorem

44:L Implicit Function Theorem. *Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^k$ a C^r-differentiable map. If $d_a f$ is surjective then the point a has a neighborhood $V \subset U$ such that V can be presented (by a renumeration of coordinates) as $A \times B$ with $A \subset \mathbb{R}^{n-k}$, $B \subset \mathbb{R}^k$ and $f^{-1}f(a) \cap V$ is the graph of some C^r-map $\varphi : A \to B$, i.e., $f^{-1}f(a) \cap V = \{(x, y) \in A \times B \mid y = \varphi(x)\}$.*

C^r-Functions

The set of all the C^r-functions $U \to \mathbb{R}$ is denoted by $\mathcal{C}^r(U)$.

44:M. If $f \in \mathcal{C}^r(U)$ then $f|_V \in \mathcal{C}^r(V)$ for any open $V \subset U$. In other words, for open sets $V \subset U \subset \mathbb{R}^n$ formula $f \mapsto f|V$ defines a map $\mathcal{C}^r(U) \to \mathcal{C}^r(V)$.

Useful C^∞-Function

44:N Bell-Shape Function. There exists a C^∞-function $\beta : \mathbb{R}^n \to \mathbb{R}$ which takes value 1 on the unit ball $D^n \subset \mathbb{R}^n$, takes value 0 on the complement of the ball B of radius 2 centered at 0 takes values in $(0, 1)$ on $\operatorname{Int} B \smallsetminus D^n$. A C^a-function with these properties does not exists.

44:N:1. The function

$$\alpha_1 : x \mapsto \begin{cases} \exp(\frac{1}{(x-1)(x-2)}), & \text{if } x \in (1, 2), \\ 0, & \text{if } x \neq\in (1, 2) \end{cases}$$

is a C^∞-function on $\mathbb{R}$.

44:N:2 ***Lemma on Smooth Step Function.*** The function

$$\alpha_2(x) = \frac{\int_0^x \alpha_1(t)\,dt}{\int_1^2 \alpha_1(t)\,dt}$$

is a C^∞-function on $\mathbb{R}$. It takes value 0 on $[0,1]$ and 1 on $[2,\infty)$. A C^a-function $f:\mathbb{R}\to\mathbb{R}$ with $f[0,1]=0$ and $f[2,\infty)=1$ does not exists.

Applications of Bell-Shape Function

44:O Retreat Ensures Expansion. Let $U\subset\mathbb{R}^n$ be an open set and $f:U\to\mathbb{R}$ a C^r-function with $r\le\infty$. Prove that any point $a\in U$ has a neighborhood $V\subset U$ such that $f|V$ is a restiction of a C^r-function $g:\mathbb{R}^n\to\mathbb{R}$.

44:P $\mathcal{C}^r(\mathbb{R}^n)$ ***Knows All the*** $\mathcal{C}^r(U)$***.*** A function $f:U\to\mathbb{R}$ is of class C^r iff any point $a\in U$ has a neighborhood $V\subset U$ such that $f|_V$ is a restiction of a C^r-function $\mathbb{R}^n\to\mathbb{R}$.

C^r-Maps

Consider open sets $U,V\subset\mathbb{R}^n$. Any map $f:U\to V$ defines a map of the set of all the functions $V\to\mathbb{R}$ to the set of functions $U\to\mathbb{R}$:

$$(\varphi:V\to\mathbb{R})\mapsto(f\circ\varphi:U\to\mathbb{R}).$$

If f is a C^r-map then this maps C^r-functions to C^r-functions.

44:Q. A map $f:U\to V$ is a C^r-diffeomorphism, iff it defines a bijection $\mathcal{C}^r(V)\to\mathcal{C}^r(U)$.

45. Differential Spaces

Motivation: Topological Structure via Continuous Functions

Let X be a topological space. Consider the set of all continuous functions $X \to \mathbb{R}$. It is denoted by $\mathcal{C}(X)$.

45:A. $\mathcal{C}(X)$ is an algebra over $\mathbb{R}$ with respect to the pointwise addition of functions, multiplication of function by numbers and multiplication of functions. In other words, if $f, g \in C(X)$, then $(x \mapsto \alpha f(x) + \beta g(x)) \in C(X)$ and $(x \mapsto f(x)g(x)) \in C(X)$ and these operations satisfy the axioms of algebra over $\mathbb{R}$.

Besides these linear operations, there are other operations, with respect to which $\mathcal{C}(X)$ is closed.

45:B. Let $f_1, \dots, f_n \in \mathcal{C}(X)$ and $f : X \to \mathbb{R}^n$ be a map defined by f_1, $\dots$,f_n. Let $f(X) \subset A$ and let $g : A \to \mathbb{R}$ be a continuous function. Then $g \circ f| \in \mathcal{C}(X)$.

45:C. For any topological space X there is a minimal topological structure φ on X such that $\mathcal{C}(X) = \mathcal{C}(X, \varphi)$. Prove that if X is a metrizable space then φ coincides with the original topology of X. Find a topological space such that these topological structures are different.

Metrizable topological spaces comprise a large and important class of topological spaces. The class of topological spaces, which are recoverable from algebras of continuous functions on them, is even larger. For spaces of this class the whole theory could be rebuilt on the basis of $\mathcal{C}(X)$, which would be a replacement for the topological structure (i.e., the set of open sets) of X.

We describe this opportunity because of its similarity with our approach to differential structures. Exactly as the notion of topological structure extends the notion of continuous function to a more general situation, the notion of differential structure is to extend the notion of differentiable function.

However, differential structures were never defined in a generality comparable to the generality of topological spaces. Maybe this is why the approach via distinguishing an algebra of "good" real valued functions, which in the case of continuity looks more restrictive than the standard approach, fits so well in the case of differentiability: it is applied to the situations, in which general topology could be perfectly based on algebras of continuous functions.

45:D. Prove that a topological space X can be embedded to $\mathbb{R}^n$, iff:

(a) the topological structure of X is defined by $\mathcal{C}(X)$,
(b) the algebra $\mathcal{C}(X)$ contains n functions f_1, $\dots$, f_n such that any $f \in \mathcal{C}(X)$ can be obtained from $f_1, \dots\ f_n$ by an operation described in *45:B* and

(c) for any different $a, b \in X$ there exists f_i with $f_i(a) \neq f_i(b)$.

Differential Spaces

Let X be a set and r be a natural number or infinity. A *differential structure* of class C^r on X is an algebra $\mathcal{C}^r(X)$ of functions $X \to \mathbb{R}$ satisfying the following two conditions:

(a) For any $f_1, \dots, f_n \in \mathcal{C}^r(X)$ such that the image of the map $f : X \to \mathbb{R}^n$ defined by $f_1, \dots, f_n$ is contained in an open set $A \subset \mathbb{R}^n$ and any C^r-map $g : A \to \mathbb{R}$ the composition $g \circ (f|) : X \to \mathbb{R}$ belongs to $\mathcal{C}^r(X)$. (Cf. *45:B* above.)

(b) A function $f : X \to \mathbb{R}$ belongs to $\mathcal{C}^r(X)$ if for each $a \in X$ there exist $g, h \in \mathcal{C}^r(X)$ such that $h(a) > 0$ and $f(x) = g(x)$ for each x with $h(x) > 0$.

A set equipped with a *differential structure* of class C^r is called a *differential space* of class C^r or *C^r-space.* Elements of $\mathcal{C}^r(X)$ are called *C^r-functions* on X.

Any differential space has a natural topological structure: the smallest one with respect to which all the functions belonging to $\mathcal{C}^r(X)$ are continuous. It is called the *underlying topological structure* and X equipped with this structure is called the *underlying topological space* of the C^r-space. The terms from general topology applied to a C^r-space are understood as being applied to the underlying topological space. For example, "Hausdorff C^r-space" means "C^r-space whose underlying topological space is Hausdorff".

45:E. The underlying topological structure has the basis consisting of the sets which are defined by finite systems of inequalities $f(x) > 0$ with $f \in \mathcal{C}^r(X)$.

> ***45:1.*** Let X be a C^r-space with $r \leq \infty$, let $f_1, \dots, f_r \in \mathcal{C}^r(X)$, and $U_i = f_i^{-1}(0, +\infty)$ for $i = 1, \dots, r$. Construct $f, g \in \mathcal{C}^r(X)$ such that $\cap_{i=1}^r U_i = f^{-1}(0, +\infty)$ and $\cup_{i=1}^r U_i = g^{-1}(0, +\infty)$.
>
> ***45:2.*** The underlying topological structure of a C^r-space with $r \leq \infty$ has the basis consisting of the sets each of which is defined by an inequality $f(x) > 0$ with $f \in \mathcal{C}^r(X)$.

45:F. In terms of the underlying topology, the second condition in the definition of differential structure is formulated as follows: the property of belonging to $\mathcal{C}^r(X)$ is local, i.e., a function $f : X \to \mathbb{R}$ belongs to $\mathcal{C}^r(X)$, provided in a neighborhood of each point of X it coincides with some $g \in \mathcal{C}^r(X)$.

> ***45:3.*** For a given set X, what is a differential C^r-structure on X with the indiscrete (underlying) topology? Does it exist? Is it unique?
>
> ***45:4.*** For a given set X, what is a differential C^r-structure on X with the discrete topology? Does it exist? For which X is it unique?
>
> ***45:5.*** Prove that any C^r-space satisfying the first separation axiom is Hausdorff.

45:6. Prove that any C^r-space satisfying the first separation axiom is regular.

45:7. Let X be a C^r-space with $r \le \infty$. Let $f, g \in \mathcal{C}^r(X)$ and $A = f^{-1}[0, +\infty)$, $B = g^{-1}[0, +\infty)$. Prove that if $A \cap B = \varnothing$ then there exists a function $h \in \mathcal{C}^r(X)$ such that $h(X) \subset [0, 1]$, $h^{-1}(0) = A$ and $h^{-1}(1) = B$.

45:G. The set of all the functions of class C^r on an open subset $U \subset \mathbb{R}^n$ is a differential structure of class C^r on U. This C^r-structure is the minimal one which contains all the n coordinate projections $U \to \mathbb{R}$.

Differential Structure of a Metric Space

Let X be a metric space with metric $\rho : X \times X \to \mathbb{R}_+$. A function $f : X \to \mathbb{R}$ is said to be *differentiable at* $a \in X$ if for any neighborhood U of a one can find points $b_1, \dots, b_k \in U \setminus a$ and numbers $\beta_1, \dots, \beta_k \in \mathbb{R}$ such that

$$\lim_{x \to a} \frac{|f(x) - f(a) - \sum_{i=1}^k \beta_i(\rho(b_i, x) - \rho(b_i, a))|}{\rho(x, a)} = 0.$$

45:1. Prove that the function $X \to \mathbb{R} : x \mapsto \rho(a, x)$ may be non-differentiable at some $x \neq a$. Prove that this can be the case for $X = S^1$ with some metric. On the other hand, if X is a subspace of $\mathbb{R}^n$ equipped with the metric which is the restriction of the standard metric of $\mathbb{R}^n$ then $X \to \mathbb{R} : x \mapsto \rho(a, x)$ is differentiable at each $x \neq a$.

45:2. Prove that for any metric space X function $X \to \mathbb{R} : x \mapsto \rho(a, x)^r$ with integer $r > 1$ is differentiable at $a \in X$.

45:A. Prove that if X is an open subspace of $\mathbb{R}^n$ then the notion of differentiability introduced above coincides with the classical differentiability discussed in Section 44.

Let X be a metric space. A function $f : X \to \mathbb{R}$ is said to be *continuously differentiable at* $a \in X$ if for any neighborhood U of a there exists a neighborhood V of a and continuous functions $b_1, \dots, b_k : V \to U$, $\beta_1, \dots, \beta_k : V \to \mathbb{R}$ (for some k) such that for any point $c \in V$

$$\lim_{x \to c} \frac{|f(x) - f(c) - \sum_{i=1}^k \beta_i(\rho(b_i(c), x) - \rho(b_i(c), c))|}{\rho(x, c)} = 0.$$

Denote by $\mathcal{C}^1(X)$ the set of functions $X \to \mathbb{R}$ continuously differentiable at each point of X.

45:B. Prove that $\mathcal{C}^1(X)$ is a differential structure of class C^1 for any metric space X.

45:3. What is $\mathcal{C}^1(X)$ if

(a) $X = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$ with metric induced from $\mathbb{R}^2$,
(b) $X = \{(x, y, z) \in \mathbb{R}^3 \mid xy = 0\}$ with metric induced from $\mathbb{R}^3$,

(c) $X = S^2$ with metric induced from $\mathbb{R}^3$
(d) X is the Cantor set with metric induced from $\mathbb{R}$?

Let X be a metric space and r a natural number. A function $f : X \to \mathbb{R}$ is called a *function of class C^r at $a \in X$* if for any neighborhood U of a one can find a neighborhood V of a and continuous functions $b_1, \dots, b_k : V \to U$, $p : V \to \mathbb{R}[x_1, \dots, x_k]$, where p takes values in polynomials of degree $\le r$ such that for any point $c \in V$

$$\lim_{x \to c} \frac{|f(x) - f(c) - p(\rho(b_1(c), x), \dots, \rho(b_k(c), a))|}{\rho(x, c)^r} = 0.$$

Denote by $\mathcal{C}^r(X)$ the set of functions $X \to \mathbb{R}$ of class C^r at each point of X.

45:C. Prove that $\mathcal{C}^r(X)$ is a differential structure of class C^r for any metric space X.

Differential Subspaces

45:H. Let X be a C^r-space and A its subset. Consider the set of functions $f : A \to \mathbb{R}$ such that for each $b \in A$ there exist $g, h \in \mathcal{C}^r(X)$ with $h(b) > 0$ and $f(x) = g(x)$ for each $x \in A$ with $h(x) > 0$. Prove that this is a differential structure of class C^r on A.

This set of function is denoted by $\mathcal{C}^r(A)$ and called the *C^r-structure induced by $\mathcal{C}^r(X)$*. The set A equipped with $\mathcal{C}^r(A)$ is called a *C^r-subspace* of X.

45:I. (Cf. *44:O*) Let $U \subset \mathbb{R}^n$. Prove that the set of functions belonging to the C^r-structure on U induced by the standard C^r-structure of R^n coincides with the set of C^r-functions $U \to \mathbb{R}$.

Below all the subsets of $\mathbb{R}^n$ are considered as C^r-spaces with the structure induced, as on subspaces, by the standard C^r-structure of $\mathbb{R}^n$, unless the opposite is stated explicitely.

45:J. Prove that if A is a subset of a C^r-space X and $f \in \mathcal{C}^r(A)$ then $f|_B : B \to \mathbb{R}$ belongs to $\mathcal{C}^r(B)$ for any $B \subset A$.

45:8. Prove that the C^r-structure induced on $\mathbb{R}$ from the standard C^r-structure of $\mathbb{R}^2$ coincides with the standard C^r-structure of $\mathbb{R}$.

45:9. Show that the map $\mathcal{C}^r(X) \to \mathcal{C}^r(A) : f \mapsto f|A$ may be nonsurjective. Under what conditions it is surjective?

C^r-Structures on Subspace of Metric Space

45:D. Prove that for an open subset A of a metric space X the C^r-structure induced from the metric C^r-structure of X coincides with the C^r-structure induced by the restriction to A of the metric of X.

45:E. For any subset A of a metric space X the C^r-structure induced by the restriction to A of the metric of X is contained in the C^r-structure induced on A from the metric C^r-structure of X.

45:F. Prove that for $A \subset \mathbb{R}^n$ the C^r-structure induced from the standard C^r-structure of $\mathbb{R}^n$ coincides with the C^r-structure induced by the restriction to A of the metric of $\mathbb{R}^n$.

45:G. Construct a subset A of a metric space X such that the C^r-structure induced on A from the metric C^r-structure of X does not coincide with the C^r-structure induced by the restriction to A of the metric of X.

> *45:G:1.* Embed isometrically $\mathbb{R}^1$ with the standard metric to a metric space X such that function $\mathbb{R} \to \mathbb{R} : x \mapsto |x|$ is diffrentiable with respect to the C^1-structure induced from the metric C^1-structure on X.

Differentiable Maps

Let X, Y be C^r-spaces. A map $f : X \to Y$ is called a *differentiable map of class C^r* or a *map of class C^r* or or just a *C^r-map* if $\varphi \circ f \in \mathcal{C}^r(X)$ for each $\varphi \in \mathcal{C}^r(Y)$. A C^r-map $f : X \to Y$ defines a homomorphism $\mathcal{C}^r(Y) \to \mathcal{C}^r(X)$.

45:K General Properties of C^r-Maps. Prove that:

(a) The composition of C^r-maps is a C^r-map.
(b) The identity map of a C^r-space is a C^r-map.
(c) The inclusion of a C^r-subspace into the C^r-space is a C^r-map.
(d) A submap of a C^r-map is a C^r-map.

> ***45:10.*** Let X be a C^r-space. Prove that $f \in \mathcal{C}^r(X)$, iff $f : X \to \mathbb{R}$ is a C^r-map (with respect to the standard C^r-structure of $\mathbb{R}$).
>
> ***45:11.*** Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^k$ be open sets. Prove that $f : U \to V$ is a C^r-map with respect to the C^r-structures induced from the standard structures of the ambient spaces $\mathbb{R}^n$ and $\mathbb{R}^k$ iff it is a C^r-map in the sense defined in Section 44 (that is the compostions of f with all the coordinate projections $U \to \mathbb{R}$ are r times continuously differentiable).

Diffeomorphisms

Let X, Y be C^r-spaces. A map $X \to Y$ is called a *diffeomorphism of class C^r* or *C^r-differomorphism* if it is an invertible C^r-map, and the inverse map is also of class C^r. C^r-spaces are said to be (C^r-)*diffeomorphic* if there exists a C^r-diffeomorphism $X \to Y$.

45:L General Properties of Diffeomorphisms. Prove that:

(a) The composition of C^r-diffeomorphisms is a C^r-diffeomorphism.
(b) The identity map of a C^r-space is a C^r-diffeomorphism.

(c) The inverse map to a C^r-diffeomorphism is a C^r-diffeomorphism.

45:M. The diffeomorphism relation of C^r-spaces is an equivalence relation.

45:N. Prove that C^r-diffeomorphisms of open subsets of $\mathbb{R}^n$ defined in Section 44 are C^r-diffeomorphisms in the sense discussed here.

45:12. Prove that any diffeomorphism of a semi-cubic parabola

$$C = \{(x,y) \in \mathbb{R}^2 \mid x^3 = y^2\}$$

onto itself preserves $(0,0) \in C$).

45:13. Consider the angle $A = \{(x,y) \in \mathbb{R}^2 \mid x = 0, y \geq 0\} \cup \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y = 0\}$, semi-cubic parabola $C = \{(x,y) \in \mathbb{R}^2 \mid x^3 = y^2\}$ and line $\mathbb{R}$. Prove that there exist C^r-bijections $A \to \mathbb{R}$, $\mathbb{R} \to C$ and $C \to \mathbb{R}$, but these C^r-spaces are pairwise nondiffeomorphic.

Differentiale Embeddings

Recall that a *topological embedding* is a map $f : X \to Y$ of a topological space X to a topological space Y such that its submap $f : X \to f(X)$ is a homeomorphism. In the setup of differential spaces this definition has an obvious counter-part: a map $f : X \to Y$ of a C^r-space X to a C^r-space Y is called a *C^r-embedding* if its submap $f : X \to f(X)$ is a C^r-diffeomorphism.

45:O. The inclusion of a smooth submanifold to the smooth manifold is a differentiable embedding.

45:P. (Cf. *45:D.*) Prove that a C^r-space X can be embedded to $\mathbb{R}^n$, iff the algebra $\mathcal{C}^r(X)$ contains n functions $f_1, \dots, f_n$ such that

(a) $\mathcal{C}^r(X)$ is the minimal C^r-structure containing $f_1, \dots f_n$,
(b) for any different $a, b \in X$ there exists f_i with $f_i(a) \neq f_i(b)$.

45:14. Which of the following maps are differentiable embeddings:

(a) $\mathrm{id} : \mathbb{R} \to \mathbb{R}$,
(b) $S^1 \to \mathbb{R}^2 : (\cos 2\pi t, \sin 2\pi t) \mapsto (\sin 2\pi t, \sin 4\pi t)$,
(c) $S^1 \to S^1 : z \mapsto z^2$,
(d) $\mathbb{R}^1 \to \mathbb{R}^2 : t \mapsto (t^2, t^3)$,
(e) $\mathbb{R}^1 \to \mathbb{R}^3 : t \mapsto (t^2, t^3, t)$,
(f) $\mathbb{R} \to \mathbb{R}^2 : t \mapsto (t^2, t^4)$,
(g) $I \to \mathbb{R}^2 : t \mapsto (\sin \pi t, \sin 2\pi t)$,
(h) $[0,1) \to \mathbb{R}^2 : t \mapsto (\sin \pi t, \sin 2\pi t)$,
(i) $(0,1) \to \mathbb{R}^2 : t \mapsto (\sin \pi t, \sin 2\pi t)$,
(j) $\mathbb{R} \to S^1 \times S^1 : t \mapsto (e^{it}, e^{\pi i t})$,
(k) $S^1 \to S^1 \times S^1 : z \mapsto (z^3, z^2)$,
(l) $S^1 \to S^1 \times S^1 : z \mapsto (z^4, z^2)$,
(m) $\mathbb{R} \to \mathbb{R}^2 : x \mapsto \begin{cases} (x, x + x^5), & \text{if } x \geq 0 \\ (x, x - x^{44}), & \text{if } x < 0 \end{cases}$,
(n) $\{(x,y) \in \mathbb{R}^2 \mid xy = 0\} \to \mathbb{R}^2 : (x,y) \mapsto \begin{cases} (x, 2x), & \text{if } y = 0, \\ (2y, y), & \text{if } x = 0 \end{cases}$,

(o) $\{(x,y) \in \mathbb{R}^2 \mid y(y-x^2)=0\} \to \mathbb{R}^2 : (x,y) \mapsto \begin{cases} (x,0), & \text{if } y=0, \\ (0,x), & \text{if } y \neq 0 \end{cases}$,

(p) $\{(x,y) \in \mathbb{R}^2 \mid y^2 = x^3\} \to \mathbb{R}^3 : (x,y) \mapsto (x,y,y^{1/3})$?

Semicubic Parabola

45:Q. The set of all C^∞-functions on $\mathbb{R}$ with the first derivative vanishing at 0 is a C^∞-structure on $\mathbb{R}$.

45:R. Prove that for any C^∞-function $f : \mathbb{R} \to \mathbb{R}$ with $\frac{df}{dx}(0) = 0$ there exist C^∞-functions $\phi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \phi(x^2) + \psi(x^3)$.

45:S. The C^∞-space of Problem *45:Q* is C^∞-diffeomorphic to the subspace of $\mathbb{R}^2$ defined by equation $x^3 = y^2$. The map $x \mapsto (x^2, x^3)$ is the differential embedding.

46. Constructing Differential Spaces

Multiplication of Differentiable Spaces

Let X and Y be C^r-spaces. Denote by $\mathcal{C}^r(X \times Y)$ the minimal C^r-structure on $X \times Y$ which contains the compositions of the natural projections $X \times Y \to X$ and $X \times Y \to Y$ with C^r-functions on X and Y, respectively. The set $X \times Y$ equipped with the C^r-structure $\mathcal{C}^r(X \times Y)$ is called a *product of C^r-spaces* X and Y.

46:1. Prove that, from the point of view of C^r-spaces, $\mathbb{R}^p \times \mathbb{R}^q = \mathbb{R}^{p+q}$.

46:2. Let X, Y, A and B be C^r-spaces and $f : X \to Y$, $g : A \to B$ be C^r-maps. Prove that

(a) the Cartesian product $f \times g : X \times A \to Y \times B$ is a C^r-map,
(b) if f, g are C^r-diffeomorphisms then $f \times g$ is a C^r-diffeomorphisms,
(c) if f, g are C^r-embeddings then $f \times g$ is a C^r-embedding.

46:3. Let A, X and Y be C^r-spaces and $f : A \to X$, $g : A \to Y$ be C^r-maps. Prove that

(a) the map $h : A \to X \times Y : a \mapsto (f(a), g(a))$ is a C^r-map,
(b) if f is a C^r-embeddings then h is a C^r-embedding, too.

Quotient Differential Spaces

46:A. Let X be a C^r-space and S a partition of X. Prove that the set of functions $f : X/_{S} \to \mathbb{R}$ such that $f \circ \mathrm{pr} \in \mathcal{C}^r(X)$, where pr is the canonical projection $X \to X/_{S}$, is a differential structure of class C^r on the quotient set $X/_{S}$.

This set of function is denoted by $\mathcal{C}^r(X/_{S})$ and called the *quotient* of $\mathcal{C}^r(X)$. The quotient set $X/_{S}$ equipped with $\mathcal{C}^r(X/_{S})$ is called a *quotient C^r-space* of X.

46:4. Let S be the partition of $\mathbb{R}^2$ into vertical lines (i.e., sets of the form $a \times \mathbb{R}$). Prove that $\mathbb{R}^2/_{S}$ is diffeomorphic to $\mathbb{R}$.

46:5. Prove that the quotient space of the segment $[-1, 1]$ obtained by identifying the end points -1 and 1 is not diffeomorphic to the circle S^1 (with the C^r-structure induced from the standard C^r-structure of the ambient plane $\mathbb{R}^2$). Find a C^r-subspace of $\mathbb{R}^2$ diffeomorphic to this quotient C^r-space.

46:6. Prove that the quotient space of the segment $[-1, 1]$ obtained by identifying x with $x + 3/2$ for $x \le -1/2$ is diffeomorphic to S^1.

46:7. Prove that the orbit space of involution $\mathbb{R}^2 \to \mathbb{R}^2 \mid (x, y) \mapsto (-x, -y)$ is diffeomorphic to the cone $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2, z \ge 0\}$.

46:8. Prove that the orbit space of involution $\mathbb{R}^2 \to \mathbb{R}^2 \mid (x, y) \mapsto (x, -y)$ is diffeomorphic to the half-plane $\mathbb{R}^2_+$.

46:9. Prove that the quotient space D^2/S^1 (the boundary circle S^1 of the disk D^2 is contracted to a single point) is not diffeomorphic to a subset of Euclidean space of any dimension.

There is a natural way to introduce a C^r-structure into a disjoint sum of C^r-spaces.

46:B. Describe explicitely the natural construction of disjoint sum of differential spaces.

As in the case of topological spaces, by gluing C^r-spaces one means composition of two constructions: disjoint summation followed by factorization.

46:10. Prove that the result of gluing of two copies of the half-space $\mathbb{R}^n_+$ by the identity map of the boundary hyperplane is diffeomorphic to $\{(x, y, z) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} \mid y \geq 0 \text{ and } z = 0\} \cup \{(x, y, z) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} \mid z \geq 0 \text{ and } y = 0\}$

46:11. Prove that the result of gluing of two copies of the half-space $\mathbb{R}^n_+$ by the map of the subset $\{(x_1, \dots, x_n) \mid 0 \geq x_1 \geq 1\}$ onto itself defined by $(x_1, x_2, \dots, x_n) \mapsto (1 - x_1, x_2, \dots, x_n)$ is diffeomorphic to $\mathbb{R}^n$.

Classical Lie Groups and Homogeneous Spaces

To be written

Space of n-Point Subsets of Surface

To be written

Toric Varieties

To be written

47. Smooth Manifolds

C^r-Manifolds

A C^r-space X is said to be *locally modelled* on a C^r-space Y if any point of X has a neighborhood diffeomorphic to an open C^r-subspace of Y.

A C^r-space is called a *smooth,* or *differential,* or *differentiable*[1] *manifold of class* C^r or just C^r*-manifold* of dimension n if it is modelled on a half-space $\mathbb{R}^n$ and the underlying topological space is Hausdorff and second countable. As it follows immediately from the definition, the underlying topological space of a C^r-manifold of dimension n is a (topological) manifold of dimension n.

47:1. Consider the following subsets of the plane $\mathbb{R}^2$.

(a) $\{(x,y)\in\mathbb{R}^2 \mid x\ge 0, y=0\}\cup\{(x,y)\in\mathbb{R}^2 \mid x=0, y\ge 0\}$,
(b) $\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\}$,
(c) $\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1, x\ge 0, y\ge 0\}\cup$
$\{(x,y)\in\mathbb{R}^2 \mid x=1, y\le 0\}\cup\{(x,y)\in\mathbb{R}^2 \mid x\le 0, y=1\}$,
(d) $\{(x,y\in\mathbb{R}^2 \mid y=x^6, x\ge 0)\}\cup\{(x,y)\in\mathbb{R}^2 \mid y=0, x\le 0\}$,
(e) $\{(x,y)\in\mathbb{R}^2 \mid y^3=x^5\}$,
(f) $\{(x,y)\in\mathbb{R}^2 \mid y^{103}=x^3\}$,
(g) $\{(x,y)\in\mathbb{R}^2 \mid y^{103}\ge x^3\}$.

Equip them with the C^r-structure induced from $\mathbb{R}^2$. For which r is each of them a C^r-manifold?

47:2. Which of the following C^r-subspaces of an Euclidean space are C^r-manifolds?

(a) S^n,
(b) D^n,
(c) I^n,
(d) $\mathbb{R}^n_+$,
(e) $\{(x,y,z)\in\mathbb{R}^3 \mid xyz\ge 0\}$,
(f) $\{(x,y,z)\in\mathbb{R}^3 \mid xyz> 0\}$.

A diffeomorphism $\xi : U\to G$ of an open set U of a C^r-manifold X onto an open set of $\mathbb{R}^n_+$ or $\mathbb{R}^n$ is called a *(local) coordinate system* or a *chart* in X. The compositions of ξ and the coordinate projections $G\to\mathbb{R}$ are denoted by ξ^1, ξ^2, … ξ^n and called *coordinates* in the coordinate system ξ. The value of ξ^i at $a\in U$ is called the i*-th coordinate* of a in ξ.

Let $\xi : U\to G$ and $\eta : V\to H$ be charts in a C^r-manifold X. Then there appear charts $\xi| : U\cap V\to\xi(U\cap V)$, $\eta| : U\cap V\to\eta(U\cap V)$ and a map $(\xi|)\circ(\eta|)^{-1} : \eta(U\cap V)\to\xi(U\cap V)$. The latter is called the *transition map* from η to ξ and denoted by t^{ξ}_{η}. This map calculates the coordinates of a point in ξ given the coordinates of this point in η.

47:A. Prove that the transition map between any two charts of a C^r-manifold is a C^r-diffeomorphism.

[1] A funny term: nobody is able to differentiate this differenti*able* manifold!

47:B. Prove that the C^r-structure of a C^r-manifold can be recovered from a collection of its local coordinate systems if the supports of these local coordinate systems cover the whole manifold.

47:C. Prove that the boundary of the underlying manifold of a C^r-manifold equipped with the induced C^r-structure is a C^r-manifold.

47:D. Under what conditions the product of two C^r-manifolds (considered as a product in the category of C^r-spaces) is a C^r-manifold?

Manifolds with Corners

A C^r-space is called an *n-dimensional smooth manifold of class C^r with corners* if it is modelled on $(\mathbb{R}_+)^n$ and the underlying topological space is Hausdorff and second countable. As it follows immediately from the definition, its underlying topological space is a (topological) manifold of dimension n. Of course, any smooth manifold of class C^r is a smooth manifold of class C^r with corners.

47:A. Prove that there exists a smooth manifold of class C^r with corners which is not a smooth manifold of class C^r.

47:B. Prove that the product of any two C^r-manifolds with corners is a C^r-manifold with corners.

In particular, the product of any two C^r-manifolds (without corners) is a C^r-manifold with corners.

Traditional Approach to Smooth Manifolds

The theory presented in the previous section is a natural generalization of the traditional theory which treats only smooth manifolds. The traditional theory was first developped in full generality by H. Whitney (Differentiable manifolds, Annals of Mathematics, 37 (1936) 645–680), athough one should mention also a book by Hermann Weyl (Die Idee der Riemannschen Flächen, Teubner, Leipzig, Berlin, 1923) where the same scheme was applied in the case of one-dimensional complex manifolds. Now it is commonly accepted.

Here we sketch it. Doing this, we redefine several notions introduced above. Eventually, the newly introduced notions will be identified with their previously introduced versions, but for a while, to avoid confusion, we will refer to the notions definined above adding the words *in the sense of differential spaces.*

Let X be a manifold of dimension n, let $U \subset X$ an open set, $\xi : U \to G$ a homeomorphism onto an open set of either $\mathbb{R}^n$ or $\mathbb{R}^n_+$. Then ξ is called a *chart* or *(local) coordinate system* in X, the set U is called the *support* of ξ. The compositions of ξ and the coordinate projections $G \to \mathbb{R}$ are denoted by $\xi^1, \xi^2, \dots \xi^n$ and called *coordinates* in the coordinate system ξ. The value of ξ^i at $a \in U$ is called the *i-th coordinate* of a in ξ.

Let $\xi : U \to G$ and $\eta : V \to H$ be charts in X. Then there appear charts $\xi| : U \cap V \to \xi(U \cap V)$, $\eta| : U \cap V \to \eta(U \cap V)$ and homeomorphism $(\xi|) \circ (\eta|)^{-1} : \eta(U \cap V) \to \xi(U \cap V)$. The latter is called the *transition map* from η to ξ and denoted by t^{ξ}_{η}. This map calculates the coordinates of a point in ξ given the coordinates of this point in η. Since $\eta(U \cap V)$ and $\xi(U \cap V)$ are open subsets of $\mathbb{R}^n$ or $\mathbb{R}^n_+$, all the notions developed in Calculus can be applied to t^{ξ}_{η}. In particular, t^{ξ}_{η} may be of class C^r. If t^{ξ}_{η} is a C^r-diffeomorphism then ξ and η are said to be *C^r-related.* If $U \cap V = \varnothing$, the charts also are C^r-related.

47:E. C^r-related charts are C^s-related for any $s < r$.

A collection of mutually C^r-related charts whose supports cover X is called a *C^r-atlas* of X.

Two C^r-atlases are said to be *C^r-equivalent* if their union is a C^r-atlas.

47:F. Reformulate the definition of C^r-equivalence of atlases in terms of transition maps.

47:G. Prove that C^r-equivalence of C^r-atlases is an equivalence relation.

Only transitivity in *47:G* is not absolutely obvious, is it?

A class of C^r-equivalent atlases of a manifold X is called a *differential structure of class C^r* on X, or *differentiable*[2] *structure of class C^r* on X, or *smooth structure of class C^r* on X, or just *C^r-structure.* A pair consisting of a manifold X and a C^r-structure on X is called a smooth (or differential, or differentiable) manifold of class C^r or a *C^r-manifold.*

47:H. (Cf. *47:B.*) A differential structure is determined by any atlas belonging to it.

47:I. A differential structure contains a maximal atlas. This is the unon of all the atlases of this structure.

Sometimes the maximal C^r-atlas is called C^r-structure. Although we do not identify them, we say that a chart belongs to a C^r-structure and is a coordinate system of the corresponding C^r-manifold if it belongs to the maximal C^r-atlas.

47:J. Let X be a C^r-manifold, $a \in X$, and $f : X \to \mathbb{R}$ a function. Let $\xi : U \to G$ and $\eta : V \to H$ be charts with supports containing a. Then for any $s \leq r$ if the composition $G @ > \xi^{-1} >> U @ > f|_U >> \mathbb{R}$ is a C^s-function at $\xi(a)$ then $H @ > \eta^{-1} >> V @ > f|_V >> \mathbb{R}$ is a C^s-function at $\eta(a)$.

[2]Of course, nobody differentiates this *differentiable* structure!

A function $f : X \to \mathbb{R}$ defined on a C^r-manifold X is said to be of class C^s (with $s \le r$) at $a \in X$ if for some (and hence, by *47:J*, for any) chart $\xi : U \to G$ with $U \ni a$ the composition $G@>\xi^{-1}>>U@>f|_U>>\mathbb{R}$ is a function of class C^s at $\xi(a)$. A function is said to be a *C^s-function* if it is of class C^s at each $a \in X$.

Equivalence of the Two Approaches

47:K. All the C^r-functions on a C^r-manifold X comprise a C^r-structure on X in the sense of differential spaces. With respect to this C^r-structure, all the charts of X are charts in the sense of differential spaces. In particular, as a differential space, X is a C^r-manifold in the sense of differential spaces.

47:L. Let X be a C^r-manifold in the sense of differential spaces. Then its charts in the sense of differential spaces comprise a C^r-atlas turning X into a C^r-manifold in the traditional sense. Cf. *47:A* and *47:B*.

Thus we have two conversions: any C^r-manifold can be converted as indicated in *47:K* to a C^r-space which is a C^r-manifold in the sense of differential spaces, and any C^r-manifold in the sense of differential spaces can be converted as indicated in *47:L* to a C^r-manifold in the traditional sense.

47:M. These two conversions are inverse two each other: both of their compositions are identity.

Revision of Boundary

Let X be a smooth manifold of class C^r and dimension n, and let $U@>\xi>>G \subset \mathbb{R}^n_+$ be a chart belonging to its C^r-structure. Then $\xi(U \cap \partial X) = G \cap \mathbb{R}^{n-1}$ is an open set of the boundary hyperplane $\mathbb{R}^{n-1}$ of $\mathbb{R}^n$ and $\xi| : U \cap \partial X \to G \cap \mathbb{R}^{n-1}$ is a local coordinate system in ∂X.

47:N. Local coordinate systems in ∂X obtained in this way from local coordinate systems belonging to the C^r-structure of X define a C^r-structure on ∂X.

47:O. The C^r-structure on ∂X defined in *47:N* coincides with the one induced on ∂X as on differential subspace of X, cf. *47:C*.

Revision of Multiplication

Let X and Y be smooth manifolds of class C^r and dimensions p and q, respectively. Let $\partial Y = \varnothing$. For charts $\xi : U \to G$ and $\eta : V \to H$ belonging to the C^r-structures of X and Y, respectively, define map

$$\xi \times \eta : U \times V \to G \times H : (a, b) \mapsto (\xi(a), \eta(b)).$$

This is a chart in $X \times Y$.

47:P. All the charts of this sort are C^r-related to each other.

The C^r-structure defined by an atlas, which consists of charts of this type, is meant when one says on $X \times Y$ as on manifold of class C^r.

47:Q. The C^r-structure on $X \times Y$ defined by an atlas, which consists of charts of the type described above, coincides with the C^r-structure defined as on a product of differential spaces. Cf. *46:1* and *47:D*.

Revision of Differentiable Maps

Let X, Y be smooth manifolds of class C^r and $f : X \to Y$ a map. Suppose f is continuous at $a \in X$.

47:R. Let $\eta : V \to H$ be a chart of Y with $f(a) \in V$. Prove that there exists a chart $\xi : U \to G$ of X with $a \in U$ and $f(U) \subset V$.

The map $\eta \circ (f|_{U,V}) \circ \xi^{-1}$ is called a *representative of f in local coordinate systems ξ and η*. We denote it by f_ξ^η. The map f is said to be *of class* C^s (with $s \le r$) at a if there a representative f_ξ^η of $f : X \to Y$ is of class C^s with $s \le r$ at $\xi(a)$.

47:S. Prove that this does not depend on the choice of coordinate systems: if there is a representative f_ξ^η of $f : X \to Y$ which is of class C^s at $\xi(a)$ then any other representative $f_{\tilde\xi}^{\tilde\eta}$ of f is of class C^s at $\tilde\xi(a)$.

Hence, the class of a map at a point is well-defined. A map $f : X \to Y$ is said to be of class C^s if at each $a \in X$ it is of class C^s.

47:T. Let X, Y be C^r-manifolds. A map $f : X \to Y$ is of class C^r in the sense defined above, iff it is a C^r-map in the sense of differential spaces.

Rank of Mapping

47:U. Prove that the rank of the Jacobian matrix (the matrix of the first order partial derivatives) at $\xi(a)$ of a representative f_ξ^η of f does not depend on the choice of ξ and η.

This rank is denoted by $\operatorname{rk}_a f$ and called the *rank of f at a*. Let $f : X \to Y$ be a C^r-map. A point $b \in Y$ is called a *regular value* of f if $\operatorname{rk}_a f = \dim Y$ at each point $a \in f^{-1}(b)$.

Differential Topology

Let X and Y be smooth manifolds of class C^r. Recall that a map $f : X \to Y$ is a *diffeomorphism of class* C^r if it is of class C^r, invertible and the inverse is also of class C^r. (Cf. above.)

Information: If X, Y are C^r-manifolds and there exists a diffeomorphism $X \to Y$ of class C^1 then there exists a C^r-diffeomorphism $X \to Y$.

Smooth manifolds XS, Y are said to be *diffeomorphic* if there exists a diffeomorphism $X \to Y$.

Information: There exists homeomorphic, but not diffeomorphic smooth manifolds. The lowest dimension of such manifolds is four.

> ***47:3.*** Prove that if two one-dimensional smooth manifolds are homeomorphic, they are also diffeomorphic.

Differential topology is a branch of mathematics which studies properties of smooth manifolds preserved by diffeomorphisms.

Submanifolds

In Section 45 any subset of a C^r-space was equipped with the induced C^r-structure. If we consider only smooth manifolds then a subset, which can receive a structure, must satisfy strong restrictions. It must be a manifold and positioned in such a nice way that the structure of C^r-space induced as it was described in Section 45 would turn it to a smooth manifold. Moreover, for some reasons usually one imposes extra conditions on the position in the ambient manifold.

Let X be a smooth manifold of class C^r and dimension n, and A be a subset of X. One says that A is a smooth k-dimensional subset of X if at each $b \in A$ there exists a chart $U@ > \xi >> G$ of X such that the pair $(\xi(U), \xi(U \cap A))$ coincides with one of the following pairs: $(\mathbb{R}^n, \mathbb{R}^k)$, $(\mathbb{R}^n, \mathbb{R}^k_+)$, $(\mathbb{R}^n_+, \mathbb{R}^k_+)$. The submap $\xi| : U \cap A \to \xi(U \cap A)$ is a chart of A.

47:V. Prove that all the charts obtained in this way from charts belonging to the same C^r-structure of X are C^r-related.

47:W. The C^r-structure on a smooth subset described above coincides with the smooth structure induced on the subset from the ambient C^r-manifold as it was defined in Section 45.

The smooth subset A equipped with the C^r-structure which is defined by the charts of this sort is called a *(smooth C^r-) submanifold* of X.

The definition of smooth subset gives a clear idea of what a smooth subset is. It says that in a neighborhood of each of its points a smooth subset looks like and placed in the ambient manifold either as $\mathbb{R}^k$ in $\mathbb{R}^n$, or $\mathbb{R}^k_+$ in $\mathbb{R}^n$, or $\mathbb{R}^k_+$ in $\mathbb{R}^n_+$. However, this definition is not convinient when one wants to check if some special set is smooth. Now we consider its reformulations more adapted for this kind of problems. For the sake of simplicity we restrict ourselves to the case of *proper* smooth subsets, i.e., smooth subsets with $\partial A = \partial X \cap A$. In the definition of proper smooth sets one can skip the pair $(\mathbb{R}^n, \mathbb{R}^k_+)$.

47:X. Prove that A is a proper smooth k-dimensional subset of a smooth manifold X, iff for each $b \in A$ there exists a local coordinate system ξ :

$U \to \mathbb{R}^n_{(+)}$ of X (where $\mathbb{R}^n_{(+)}$ denotes either $\mathbb{R}^n$ or $\mathbb{R}^n_+$) and a differentiable map $f : \mathbb{R}^k_{(+)} \to \mathbb{R}^{n-k}$ such that $\xi(A \cap U)$ is the graph of f.

47:Y. Prove that A is a proper smooth k-dimensional subset of a smooth manifold X, iff for each $b \in A$ there exists a local coordinate system $\xi : U \to \mathbb{R}^n_{(+)}$ of X and a differentiable map $f : \mathbb{R}^n_{(+)} \to \mathbb{R}^{n-k}$ such that $0 \in \mathbb{R}^{n-k}$ is a regular value of f and, if $\mathbb{R}^n_{(+)} = \mathbb{R}^n_+$, a regular value of $f|_{\partial \mathbb{R}^n_+}$ and $\xi(A \cap U) = f^{-1}(0)$

Cf. Implicit Function Theorem *44:L*.

48. Immersions and Embeddings

Immersions

Let X and Y be C^r-manifolds. *immersion* if its rank at each point of X is equal to the dimension of X.

48:1. Which of the following differentiable maps are immersions:

(a) $\mathrm{id} : \mathbb{R} \to \mathbb{R}$,
(b) a constant map $\mathbb{R} \to \mathbb{R}$,
(c) the projection $\mathbb{R}^2 \to \mathbb{R}$,
(d) $S^1 \to \mathbb{R}^2 : (\cos 2\pi t, \sin 2\pi t) \mapsto (\sin 2\pi t, \sin 4\pi t)$,
(e) $S^1 \to S^1 : z \mapsto z^2$,
(f) $\mathbb{R}^1 \to \mathbb{R}^2 : t \mapsto (t^2, t^3)$,
(g) $\mathbb{R} \to \mathbb{R}^2 : t \mapsto (t^2, t^4)$,
(h) $\mathbb{R} \to S^1 \times S^1 : t \mapsto (e^{it}, e^{\pi i t})$,
(i) $I \to \mathbb{R}^2 : t \mapsto (\sin \pi t, \sin 2\pi t)$,
(j) $\mathbb{R}^2 \to \mathbb{R}^3 : (x, y) \mapsto (\cos x, \sin x \cos y, \sin x \sin y)$,
(k) $\mathbb{R}^2 \to \mathbb{R}^3 : (x, y) \mapsto (\cos x(2 + \cos y), \sin x(2 + \cos y), \sin y)$?

48:2. Prove that an immersion of a closed smooth manifold to a closed connected smooth manifold of the same dimension is a covering with a finite number of sheets.

48:3. Is the same true for compact manifolds with boundary?

48:4. How to generalize *48:2* to the case of compact manifolds with boundary, anyway?

48:5. Does there exist an immersion $S^2 \to \mathbb{R}^2$? What about immersion $S^1 \times S^1 \to \mathbb{R}^2$? Find a generalization for the answers to these questions.

48:6. Does there exist an immersion of a handle (i.e., torus with a hole) to the plane?

Differentiable Embeddings

Recall that a map $f : X \to Y$ of a C^r-space X to a C^r-space Y is called a C^r*-embedding* if its submap $f : X \to \varphi(X)$ is a C^r-diffeomorphism.

In the traditional approach to smooth manifolds, one should add to this an additional condition, because the image $f(X)$ is not a smooth manifold automatically. Thus the definition looks as follows: A C^r-map $f : X \to Y$ is called a *differentiable embedding* if $f(X)$ is a smooth subsmanifold of Y and $f| : X \to f(X)$ is a diffeomorphism.

48:A. The inclusion of a smooth submanifold to the smooth manifold is a differentiable embedding.

48:7. Which of the following differentiable maps are differentiable embeddings:

(a) $\mathrm{id} : \mathbb{R} \to \mathbb{R}$,
(b) $S^1 \to \mathbb{R}^2 : (\cos 2\pi t, \sin 2\pi t) \mapsto (\sin 2\pi t, \sin 4\pi t)$,
(c) $S^1 \to S^1 : z \mapsto z^2$,

(d) $\mathbb{R}^1 \to \mathbb{R}^2 : t \mapsto (t^2, t^3)$,
(e) $\mathbb{R} \to \mathbb{R}^2 : t \mapsto (t^2, t^4)$,
(f) $I \to \mathbb{R}^2 : t \mapsto (\sin \pi t, \sin 2\pi t)$,
(g) $[0, 1) \to \mathbb{R}^2 : t \mapsto (\sin \pi t, \sin 2\pi t)$,
(h) $(0, 1) \to \mathbb{R}^2 : t \mapsto (\sin \pi t, \sin 2\pi t)$,
(i) $\mathbb{R} \to S^1 \times S^1 : t \mapsto (e^{it}, e^{\pi it})$,
(j) $S^1 \to S^1 \times S^1 : z \mapsto (z^3, z^2)$,
(k) $S^1 \to S^1 \times S^1 : z \mapsto (z^4, z^2)$,
(l) $\mathbb{R} \to \mathbb{R}^2 : x \mapsto \begin{cases} (x, x + x^5), & \text{if } x \geq 0 \\ (x, x - x^{44}), & \text{if } x < 0 \end{cases}$?

Immersions Versus Embeddings

48:B Embedding Is Immersion. *A differentiable embedding of a smooth manifold to a smooth manifold is an immersion.*

48:C Immersion Is Embedding Locally. *Let $f : X \to Y$ be an immersion. Prove that each point $a \in X$ has a neighborhood U such that $f|_U : U \to Y$ is a differentiable embedding, unless $f(a) \in \partial Y$.*

48:8. ***Riddle.*** What if, under the conditions of *48:C*, $f(a) \in \partial Y$?

48:D Diff. Embedding = Top. Embedding + Immersion. *Let X and Y be C^r-manifolds. A map $f : X \to Y$ is a C^r-differentiable embedding, iff f is a topological embedding and C^r-immersion.*

48:D:1. Let X and Y be C^r-manifolds, $a \in X$ and $f : X \to Y$ be a topological embedding. Then for any neighborhoods U and V of a and $f(a)$, respectively, there exist open subsets $U_0 \subset U$ and $V_0 \subset V$ such that $f(U_0) = V_0 \cap f(X)$.

48:D:2 ***Straightening Immersion.*** Let U be an open set of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^k$ be a C^r-map such that $\operatorname{rk}_a f = n$ for some $a \in U$. Then there exist a neighborhood $V \subset U$ of a and a neighborhood W of $f(a)$ and C^r-diffeomorphisms $g : V \to \tilde{V} \subset \mathbb{R}^n$ and $h : W \to \tilde{W} \subset \mathbb{R}^k$ such that $W \cap f(U) = f(V)$ and $h \circ (f|) \circ g^{-1}$ is the linear embedding $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0, \dots, 0)$.

Embeddability to Euclidean Spaces

In early years of topology (say, in papers by Henry Poincaré) by a smooth manifold one meant what is called smooth submanifolds of an Euclidean space. It was not convinient, because the embedding usually is quite irrelevant, and sometimes is not easy to find. For example, $\mathbb{R}P^n$ with $n = 2^k$ can be embedded into $\mathbb{R}^2 n$, but does not admit an embedding into $\mathbb{R}^{2n-1}$. The standard smooth structure is easier to get from the well-known natural two-fold covering space, which is S^n, than to describe a smooth embedding into an Euclidean space. So, the transition to Whitney's abstract definition of smooth structures was well-motivated. However the transition poses the question if the set of objects really changes. The answer is negative: any smooth manifold can be smoothly embedded into an Euclidean space. This was proved for C^r-manifolds with

$r \leq \infty$ by Whitney in the very same paper (Differentiable manifolds, Annals of Mathematics, 37 (1936) 645–680), where he introduced differential structures. The real analytic case was done about twenty years later by Grauert and Remmert.

Here we consider the simplest case of the embedding theorems.

48:E. *Any compact C^r-manifold can be C^r-differentiably embedded into Euclidean space of sufficiently high dimension.*

> *48:E:1.* To embed a compact C^r-manifold X to $\mathbb{R}^N$ it is necessary and sufficient to construct N real valued functions $f_1, \dots, f_N$ of class C^r on X such that for any $a \in X$ there exist $i_1 < i_2 < \dots < i_n$ with $\operatorname{rk}\left(\frac{\partial f_{i_r}}{\partial x_s}(a)\right) = n$ and there exist i, j with $f_i(a) \neq f_j(a)$.
>
> *48:E:2.* Each point of a C^r-manifold of dimension n has a neighborhood which admits a C^r-embedding to $\mathbb{R}^n$.
>
> *48:E:3.* (Cf. *44:N*) Let X be an n-dimensional C^r-manifold, $\xi : U \to G$ a local coordinate system such that G contains a ball of radius 2 centered at the origin of $\mathbb{R}^n$. Let g be a C^r-function defined on U. Then there exists a C^r-function $h : X \to \mathbb{R}$ such that $h|_{\xi^{-1}(D^n)} = g|_{\xi^{-1}(D^n)}$ and $h(x) = 0$ for $x \neq\in U$.

The proof of *48:E* sketched above does not work for $r = a$, i.e., for real analytic functions.

> ***48:9.*** Where does it not work for real analytic embedding?

Theorem *48:E* is correct for real analytic case too, but requires arguments of absolutely different nature. As it was mentioned above, these arguments were found by Grauert and Remmert in the fifties.

Information: Any n-dimensional C^r-manifold can be C^r-embedded into $\mathbb{R}^{2n}$. For $r \leq \infty$ existense of a C^r-embedding to $\mathbb{R}^{2n+1}$ was proved by H. Whitney in Differentiable manifolds, Annals of Mathematics, 37 (1936) 645–680. Eight years later he managed to decrease the dimension of the Euclidean space by one. The same Whitney's results combined with the Grauert-Remmert technique give embedding to $\mathbb{R}^{2n}$ for a real analytic manifold of dimension n.

49. Tangent Vectors

As smooth manifolds generalize smooth surfaces lying in Euclidean space, tangent vectors to a smooth manifold generalize vectors in Euclidean space applied to a point of a surface and tangent to it.

In literature there are at least three completely different ways of defining vectors tangent to a smooth manifold. Of course, the results are equivalent, but this appears as a surprise. The variety of definitions can be partially explained by advantages of different definitions in different situations, but the main reason is a difference in pedagogical principles and experience.

Different people think about vectors in different ways. A school math teacher thinks that it is a directed segment, or a class of parallel equally directed segments of the same length.

A physisist would laugh at this: he knows for sure that the electric field strength is a vector, but has next to nothing to do with a directed segment. A usual definition of a vector for physisists is that this is a quantity which is characterized by a direction in the space and magnitude (the latter is a number depending on the choice of unit of measurement).

For a mathematician, vector is just an element of a linear space. At first glance, this is the most general point of view. Both vectors of school teacher and physisist are vectors for mathematician, because one can sum them and multiply by a number and these operations are subject of the same axioms. But when one needs to extend a definition of vector to a new situation, the axiomatic point of view is not creative. It is good mostly for throwghing away wrong candidates.

Coordinate Definition

A physisist would probably agree with the following definition of a vector: a *vector* is a quantity which can be characterised by n real numbers (its *coordinates*) if it is taken in n-dimensional space and a coordinate system is fixed. When the coordinate system changes, the coordinates of a vector change accordingly. The first definition of tangent vector that we consider fits to this scheme.

Let X be a smooth manifold of class C^r and dimension n and a be a point of X. Denote by $\mathcal{C}_a$ the set of local coordinate systems of X with supports containing a. A *tangent vector* of X at a is a map $v : \mathcal{C}_a \to \mathbb{R}^n$ such that for any $\xi, \eta \in \mathcal{C}_a$

(22)
$$v(\xi) = \left(\frac{\partial \xi}{\partial \eta}\right)_a v(\eta),$$

where $\left(\frac{\partial \xi}{\partial \eta}\right)_a$ is the Jacobian matrix of the transition function from η to ξ at $\eta(a)$.

If $v(\xi) = (x^1, \dots, x^n)$ and $v(\eta) = (y^1, \dots, y^n)$ then formula (22) can be rewritten as follows:

$$x^i = \sum_{j=1}^{n} \frac{\partial \xi^i}{\partial \eta^j}(a) y^j, \tag{23}$$

Here the upper indices are just indices, not exponents. The unusual position is determined by the Einstein notations for multilinear algebra which are explained below. The main goals of these notations is to exclude numerous summation signs and encode the difference between elements of a vector space and the conjugate space. In the Einstein notations formula (23) looks as follows:

$$x^i = \frac{\partial \xi^i}{\partial \eta^j}(a) y^j,$$

see Digression on Einstein Notations below.

The set of all the tangent vectors of X at a is denoted by T_aX and called the *tangent space of X at a.*

49:A. T_aX is a vector space with respect to coordinatewise operations $(v + w)(\xi) = v(\xi) + w(\xi)$ and $(av)(\xi) = a(v(\xi))$.

49:B. Any coordinate system $\xi \in \mathcal{C}_a$ defines a map

$$T_aX \to \mathbb{R}^n : v \mapsto v(\xi).$$

This map is a linear isomorphism.

In particular, a vector $v \in T_aX$ is determined by $v(\xi)$, and $v(\xi)$ can be any element of $\mathbb{R}^n$. The coordinates of $v(\xi)$ (with respect to the canonical coordinate system in $\mathbb{R}^n$) are called the *coordinates of v in (or with respect to) the local coordinate system ξ.*

Digression on Einstein Notations

To be written

Differentiation of Functions

To be written

Differential of Map

To be written

Tangent Bundle

Consider the set of all the tangent vectors of a manifold X, i.e., $\cup_{a\in X} T_a X$. It is denoted by TX and called the *tangent bundle* of X.

For any local coordinate system $X \supset U@>\xi>>G\subset R^n$ put $TU = \cup_{a\in U} T_a X$ and define a bijection $TU \to G\times\mathbb{R}^n$ by formula $T_x X \ni v \mapsto (\xi(x), v(\xi))$. By this bijection one introduces to TU topology and smooth structure from $G\times\mathbb{R}^n$

To be finished

Tangent Vectors in Euclidean Space

To be written

Vectors as Velocities

To be written

50. Vector Bundles

To be written

General Terminology of Fibrations

Trivial and Locally Trivial

Induced Fibrations

Vector Bundles

Constructions with Vector Bundles

Tautological Bundles

Homotopy Classification of Vector Bundles

Low-Dimensional

51. Orientation

To be written

Linear Algebra Digression: Orientations of Vector Space

Related Orientations

Orientation of Vector Bundle

Orientation and Orientability of Smooth Manifold

Orientation of Boundary

Orientation Covering

Projective Spaces

52. Transversality and Cobordisms

To be written

Sard Theorem

Transversality

Embedding to $\mathbb{R}^{2n+1}$

Normal Bundle and Tubular Neighborhood

Pontryagin Construction

Degree of Map

Linking Numbers

Hopf Invariant

Thom Construction

Cobordisms