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Introduction to Supersymmetric Gauge Theories

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Abstract

In these lectures I present a basic introduction to supersymmetry, especially to $N = 1$ supersymmetric gauge theories and their renormalization, in the Wess-Zumino gauge. I also discuss the various ways supersymmetry may be broken in order to account for the lack of exact supersymmetry in the actual world of elementary particles.

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Introduction

This course intends to be an introduction to supersymmetry, and in particular to $N = 1$ supersymmetric gauge theories. I shall present here the formalism in components, in the so-called Wess-Zumino gauge (WZ-gauge). Whereas many books [1, 2, 3, 4, 5, 6] and reviews [7, 8, 9, 10] on the superspace formalism are available, few are especially devoted to the formalism in the WZ-gauge. There are however two good reasons for studying the latter, despite of the simplicity and of the beauty of the superspace formalism. The first one is that in the physically interesting case where supersymmetry is broken, e.g. by the presence of nonsupersymmetric masses, a good part of the advantages of superspace is lost. The second motivation is that there is no simple superspace formalism for extended ($N > 1$) supersymmetry[3]: these lectures may thus provide also a basis for a further study including these extended theories.

The plan of the lectures is as follows. I shall begin, in Section 1, with the description of a simple nonrelativistic supersymmetric quantum model, in order to familiarize ourselves with some of the features pecular to supersymmetry. Then, after some generalities about N-extended supersymmetry and its representations in the spaces of one-particle states, in Section 2, I shall introduce the $N = 1$ chiral supermultiplet, in the component formalism, in the fist part of Section 3, and apply it to a simple model. In the second part of Section 3, I shall introduce the gauge supermultiplet in the Wess-Zumino gauge and describe the general supersymmetric gauge invariant action. Gauge fixing, BRS invariance and renormalization of supersymmetric gauge theories in the Wess-Zumino gauge will be discusssed in Section 4. In Section 5, I shall finally give some comments on the different ways one may break supersymmetry (explicitly or spontaneously) in order to account for the absence of exact superymmetry in nature, and present a case of spontaneous breakdown of supersymmetry accompagnied by the phenomenon of radiative mass generation.

1 Nonrelativistic Supersymmetry

In order to get familiarized with the concept of supersymmetry, let us introduce it first in the framework of nonrelativistic quantum mechanics. Although various applications may be found in the literature [12], I shall restrict myself to the simple model of the *supersymmetric harmonic oscillator*. It consists of a bosonic and of a fermionic part.

1.1 The Bosonic Oscillator

The bosonic part is the usual one-dimensional harmonic oscillator, defined by the Hamiltonian

$$H_B = \frac{1}{2}(p^2 + \omega_B^2) \ . \tag{1.1}$$

[3]The superspace of extended supersymmetry is the "harmonic superspace" [11]. It is not as simple to handle with as the $N = 1$ superspace.

In terms of the annihilation and creation operators

$$a = \frac{1}{\sqrt{2\omega_B}}(p - i\omega_B x) \ , \quad a^+ = \frac{1}{\sqrt{2\omega_B}}(p + i\omega_B x)$$

the Heisenberg commutation relation $[p, x] = -i$ writes[4]

$$[a, a^+] = 1 \ , \tag{1.2}$$

and the Hamiltonian takes the form

$$H_B = \frac{\omega_B}{2}(a^+ a + a a^+) = \omega_B \left(N_B + \frac{1}{2} \right) \ , \tag{1.3}$$

where $N_B = a^+ a$ is a "counting operator", with eigenvalues $0, 1, 2, \cdots$. The energy spectrum is thus given by

$$E_B = \omega_B \left(n_B + \frac{1}{2} \right) \ , \quad n_B = 0, 1, 2, \cdots \ . \tag{1.4}$$

1.2 The Fermionic Oscillator

One can define in an analogous way a fermionic harmonic oscillator, by introducing fermionic annihilation and creation operators b, b^+ obeying the anticommutation rules

$$\{b, b^+\} = 1 \ , \quad \{b, b\} = \{b^+, b^+\} = 0 \ , \tag{1.5}$$

and a Hamiltonian

$$H_F = \frac{\omega_F}{2}(b^+ b - b b^+) = \omega_F \left(N_F - \frac{1}{2} \right) \ , \tag{1.6}$$

where $N_F = b^+ b$ is the fermionic counting operator, with eigenvalues $0, 1$. The fermionic energy spectrum thus is

$$E_F = \omega_F \left(n_F - \frac{1}{2} \right) \ , \quad n_F = 0, 1, 2, \cdots \ . \tag{1.7}$$

An explicit representation of the fermionic operators, preserving the anticommutation rules (1.5), is given in terms of Pauli matrices by[5]

$$b = \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \quad b^+ = \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ .$$

The fermionic Hamiltonian then takes the form of the Hamiltonian for a system consisting of one spin 1/2:

$$H_F = \frac{\omega_F}{2}\sigma_3 \ . \tag{1.8}$$

[4]We take $\hbar = 1$. The superscript $^+$ means Hermitean conjugation.

[5]$\sigma_\pm = \sigma_1 \pm i\sigma_2$, $[\sigma_+, \sigma_-] = \sigma_3$.

1.3 The Supersymmetric Oscillator

The supersymmetric oscillator is obtained by combining one bosonic and one fermionic oscillators (1.3), (1.6) with the same frequency $\omega = \omega_B = \omega_F$. Its Hamiltonian is given by

$$\begin{aligned} H = H_B + H_F &= \frac{\omega}{2}(a^+a + aa^+) + \frac{\omega}{2}(b^+b - bb^+) \\ &= \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2 + \frac{1}{2}\omega\sigma_3 \ . \end{aligned} \tag{1.9}$$

The spectrum is obtained as the sum of (1.4) and (1.7):

$$E_{n_B,n_F} = \omega\,(n_B + n_F) \ , \quad n_B = 0,1,2,\cdots \ , \ n_F = 0,1 \ . \tag{1.10}$$

It corresponds to a degenerate groundstate

$$\psi_{0,0} \ , \quad \text{of energy } E_{0,0} = 0 \ ,$$

and of a sequence of doubly degenerated of states

$$\psi_{n,0} \text{ and } \psi_{n-1,1} \ , \quad \text{of energy } E_{n,0} = E_{n-1,1} = n\omega \ , \quad n = 1,2,\cdots \ .$$

As one may expect, the degeneracy of the excited states is due to a symmetry. Let us indeed introduce two Hermitean *supercharges*

$$Q_1 = \sqrt{\omega}\left(a^+b + ab^+\right) \ , \quad Q_2 = \sqrt{\omega}i\left(a^+b - ab^+\right) \ . \tag{1.11}$$

One easily sees that, together with the Hamiltonian (1.9), they obey the *superalgebra*[6]

$$\{Q_i, Q_j\} = 2\delta_{ij}H \ , \quad [Q_i, H] = 0 \ . \tag{1.12}$$

In particular, we have

$$H = (Q_1)^2 = (Q_2)^2 \ . \tag{1.13}$$

The second of eqs. (1.12), namely the commutativity of the supercharges with the Hamiltonian, follows from the special case (1.13) of the first one. It characterizes them as the generators of a symmetry, called *supersymmetry*. The infinitesimal supersymmetry transformations of the fundamental variables x, p, σ_1 and σ_2 are easily calculated. They read:

$$\begin{aligned} &\delta_k x = i[Q_k, x] = \frac{1}{\sqrt{2}}\sigma_k \ , \quad \delta_k p = i[Q_k, p] = \frac{i\omega}{\sqrt{2}}\varepsilon_{kl}\sigma_l \ , \\ &\delta_k \sigma_l = i\{Q_k, \sigma_l\} = \delta_{kl}\frac{1}{\sqrt{2}}p + \varepsilon_{kl}\sqrt{2}\omega x \ , \end{aligned} \tag{1.14}$$

with $\varepsilon_{kl} = -\varepsilon_{lk}$, $\varepsilon_{12} = 1$.

It is worth emphasizing three important properties of this system: the nonnegativity of the energy, the double degeneracy of the energy levels, and the vanishing ground state energy. These properties are generic properties of supersymmetry. As the following proposition shows, they follow merely from the superalgebra (1.12):

Proposition.

[6]With the notation $\{A, B\} = AB + BA$, $\quad [A, B] = AB - BA$.

(i) The energy eigenvalues are nonnegative.

(ii) The ground state energy is vanishing if and only if the ground state is invariant under supersymmetry transformations. It is then a supersymmetry singlet.

(iii) If the ground state energy is nonzero, then the ground state is not invariant under supersymmetry. In other words, supersymmetry is then spontaneously broken.

Proof: (i) Energy positivity follows from the equalities, valid for any state vector $|\psi\rangle$:

$$\langle\psi| H |\psi\rangle = \langle\psi| Q^1 Q^1 |\psi\rangle = \|Q^1 |\psi\rangle\|^2 \geq 0 \ . \tag{1.15}$$

which are a consequence of (1.13). (ii) Let us suppose that the ground state $|0\rangle$ is supersymmetric: $Q^i |0\rangle = 0$ for $i = 1, 2$. $H |0\rangle = 0$ then follows from (1.13). Conversely, $\langle 0| H |0\rangle = 0$ implies $Q^i |0\rangle = 0$ due to (1.15) and the positive definiteness of the norm.

(iii) is a corollary of (i) and (ii). □

Remark. In our exemple of the supersymmetric oscillator, we saw explicitly in (1.10) how the bosonic and fermionic contributions to the ground state energy, $\omega/2$ and $-\omega/2$, respectively, exactly cancel. As point (ii) of the proposition above shows, this boson-fermion cancellation in the ground state energy, following from the superalgebra alone, is quite general.

2 Relativistic Supersymmetry. Generalities

In 1967, Coleman and Mandula [13] proved that there is no nontrivial way to unify an internal symmetry with the relativistic space-time symmetry. In more precise words, any symmetry group $\mathcal{G}$ containing the Poincaré group and an arbitrary internal symmetry group has the form of a direct product

$$\mathcal{G} = \text{ Poincaré } \times \text{ Internal Symmetry } .$$

This theorem concerns the symmetries of the S-matrix. It is based on quite general hypothese such as locality and unitarity, and on the assumption that the symmetries are described by Lie groups. Coleman-Mandula's theorem has put an end to numerous tentatives of unification of internal and space-time symmetries. However, in 1975, Haag, Łopuszański and Sohnius [14] found a way out of this "no-go" theorem, through the relaxation of the assumption that the group of symmetries is a Lie group, whose generators obey a Lie algebra in the form of *commutation* rules. Allowing indeed for the presence of *anticommutaton* rules as well, they showed that the so-called Poincaré supersymmetry allows for a nontrivial unification. Their result is nevertheless very restrictive, in the sense that beyond the usual bosonic scalar generators – Lie group generators obeying commutation rules – the only other possible ones are fermionic generators of spin 1/2 – obeying anticommutation rules.

2.1 Poincaré Supersymmetry Algebras

According to the result of Haag, Łopuszański and Sohnius [14], the basis of the super-Poincaré algebra with N supersymmetry generators ("N-supersymmetry") consists of:

- Bosonic (*even*) Hermitean scalar generators T_a, $a = 1 \cdots \dim(G)$, of some internal symmetry Lie group G,
- The (*even*) generators P_μ and $M_{[\mu\nu]}$ of the 4-dimensional Poincaré group.
- Fermionic (*odd*) Weyl spinor generators Q^i_α, $\alpha = 1, 2$; $i = 1, \cdots, N$, belonging to a dimension N representation of G, and their conjugates $\bar{Q}^{\dot\alpha}_i$.
- Central charges $Z^{[ij]}$, i.e. bosonic operators commuting with all the T_a's and all the Q_α's, as well as with the Poincaré generators.

The T_a's and $Z^{[ij]}$'s are scalars, whereas the Q^i_α's belong to the representation $(1/2, 0)$ of the Lorentz group and the $\bar{Q}^{\dot\alpha}_i$'s to the conjugate representation $(0, 1/2)$. The Q_α's are Weyl spinors, with two complex components[7].

Moreover, the general superalgebra of N-extended supersymmetry, also called the N-super-Poincaré algebra, reads (we write only the nonvanishing (anti)commutators):

$$\begin{aligned}
&[M_{\mu\nu}, M_{\rho\sigma}] = -i(g_{\mu\rho}M_{\nu\sigma} - g_{\mu\sigma}M_{\nu\rho} + g_{\nu\sigma}M_{\mu\rho} - g_{\nu\rho}M_{\mu\sigma}) \ , \\
&[M_{\mu\nu}, P_\lambda] = i(P_\mu g_{\nu\lambda} - P_\nu g_{\mu\lambda}) \ ,
\end{aligned} \tag{2.1}$$

$$[T_a, T_b] = i f_{ab}{}^c T_c \ , \tag{2.2}$$

$$\begin{aligned}
&\left\{Q^i_\alpha, Q^j_\beta\right\} = \varepsilon_{\alpha\beta} Z^{[ij]} \ , \\
&\left\{Q^i_\alpha, \bar{Q}^j_{\dot\alpha}\right\} = 2\delta^{ij}\sigma^\mu_{\alpha\dot\alpha} P_\mu \ ,
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
&[Q^i_\alpha, M_{\mu\nu}] = \frac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q^i_\beta \ , \\
&[Q^i_\alpha, T_a] = (R_a)^i{}_j Q^j_\alpha \ .
\end{aligned} \tag{2.4}$$

This result is the most general one for a massive theory. In a massless theory, another set of fermionic charges, S^i_α (and their conjugates), may be present. Then, in this case, the Lie group G is restricted to be U(N), for $N \neq 4$, and either U(4) or SU(4), for N=4. The superalgebra moreover contains all the generators of the conformal group – which contains the Poincaré group as a subgroup: one calls it the N-superconformal algebra.

In these lectures we shall restrict ourselves to the case N=1, where the part of the superalgebra (2.1)-(2.4) which involves the spinor charges reduces to the original Wess-Zumino algebra

$$\left\{Q_\alpha, \bar{Q}_{\dot\alpha}\right\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \ , \quad \{Q_\alpha, Q_\beta\} = 0 \ , \quad \left\{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\right\} = 0 \ , \tag{2.5}$$

[7]The notations and conventions are given in the Appendix.

and

$$[Q_\alpha, M_{\mu\nu}] = \frac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta \ , \quad [\bar{Q}^{\dot\alpha}, M_{\mu\nu}] = -\frac{1}{2}(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\bar{Q}^{\dot\beta} \ ,$$
$$[Q_\alpha, P_\mu] = 0 \ , \quad [\bar{Q}_{\dot\alpha}, P_\mu] = 0 \ , \tag{2.6}$$

$$[Q_\alpha, R] = -Q_\alpha \ , \quad [\bar{Q}_{\dot\alpha}, R] = \bar{Q}_{\dot\alpha} \ . \tag{2.7}$$

Here, R is the infinitesimal generator of an Abelian group into which the internal symmetry group G has shrunk.

Remark. The elements of the Pauli matrices σ^μ, $\sigma_{\mu\nu}$ and $\bar\sigma_{\mu\nu}$ appear as the structure constants of this superalgebra.

2.2 Representations in the Space of Single Particle States

Bibliography: [7, 8]

Massive particles:

Massive particles are defined as irreducible representations (irrep.) of the Poincaré group, labeled, in the massive case, by their mass m and their spin j. One-particle states can be taken as eigenstates of the four momentum $p = (p_\mu,\, \mu = 0, \cdots, 3)$, with $p^2 = m^2$, and of the spin component j_3, with $j_3 = -j, -j+1, \cdots, j$. It is convenient to work in the rest frame $p = (m, \mathbf{0})$, where the Wess-Zumino algebra (2.5) takes the simple form

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2m\delta_{\alpha\dot\alpha} \ , \quad \{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0 \ .$$

We see that the operators

$$a_\alpha = \frac{1}{\sqrt{2m}}Q_\alpha \ , \quad a^+_\alpha = \frac{1}{\sqrt{2m}}\bar{Q}_{\dot\alpha}$$

obey the algebra of two pairs of fermionic annihilation and creation operators:

$$\{a_\alpha, a^+_\beta\} = \delta_{\alpha\beta} \ , \quad \{a_\alpha, a_\beta\} = \{a^+_\alpha, a^+_\beta\} = 0 \ , \quad \alpha, \beta = 1, 2 \ .$$

From the form of this algebra it follows that one can construct all its irreducible representations by acting with the creation operators a^+_α on a state $|\Omega\rangle$ – analogous to the vacuum state in usual Fock space – characterized by

$$a_\alpha\,|\Omega\rangle = 0 \ , \quad \alpha = 1, 2 \ .$$

The states of the irrep. defined from $|\Omega\rangle$ are thus:

$$|\Omega\rangle \ , \qquad a^+_\alpha\,|\Omega\rangle \ , \ \alpha = 1, 2 \ , \qquad a^+_1 a^+_2\,|\Omega\rangle = -a^+_1 a^+_2\,|\Omega\rangle \ . \tag{2.8}$$

Let us assume that $|\Omega\rangle$ is a one-particle state of spin j, with $j_3 = \mu$. Then, from the commutation relation

$$[J^i, a^+_\alpha] = \frac{1}{2}\sigma^{i}_{\alpha}{}^{\beta} a^+_\beta \ ,$$

which is a special case of (2.6) for the rotation group generators $J^i = \frac{1}{2}\varepsilon^{ijk}M_{jk}$, it follows that the states in (2.8) have the spin components μ, $\mu \pm 1/2$ and μ, respectively. If one takes the $2j+1$ possible states Ω with $-j \leq \mu \leq j$, the multiplicities of the three "levels" in (2.8) are $2j+1$, $2(2j+1)$ and $2j+1$, respectively. One observes that the numbers of bosonic and fermionic degrees of freedom are equal, each, to $2(2j+1)$. In the special case $j = 0$, we have two spin 0 particles and one spin 1/2 particle (with $\mu = \pm 1/2$).

Remark. These representations, of dimension $4(2j+1)$, are irreducible as representations of the algebra generated by Q_α, $\bar{Q}_{\dot{\alpha}}$, P_μ and $M_{\mu\nu}$. As representations of the Wess-Zumino algebra (2.5), generated by Q_α, $\bar{Q}_{\dot{\alpha}}$ and P_μ only, they are irreducible only for $j = 0$.

Massless particles:

Massless one-particle states are characterized by their 3-momentum $\mathbf{p}$ and their helicity $\Lambda = \mathbf{J} \cdot \mathbf{p}/|\mathbf{p}|$.

Let us consider states $|E, \lambda\rangle$ of 4-momentum $p = (E, 0, 0, E)$ and of helicity eigenvalue λ, with $\Lambda\,|E, \lambda\rangle = \lambda\,|E, \lambda\rangle$, choosing a frame where the 3-momentum is parallel to the z-axis. On such states, the Q, $\bar{Q}$ anticommutation rule (2.5) reduces to

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\left(\sigma^0 + \sigma^3\right)_{\alpha\dot{\alpha}} E \ ,$$

i.e.:

$$\{Q_1, \bar{Q}_1\} = 4E \ , \quad \{Q_1, \bar{Q}_2\} = \{Q_2, \bar{Q}_1\} = \{Q_2, \bar{Q}_2\} = 0 \ .$$

It follows in particular, from the last anticommutation relation and from the positivity of the operator $Q_2\bar{Q}_2$, that $Q_2 = 0$ on these states. The nontrivial part of the algebra has thus the form

$$\{a, a^+\} = 1 \ , \quad \{a, a\} = \{a^+, a^+\} = 0 \ , \quad \text{with} \quad a = \frac{1}{2\sqrt{E}}Q_1 \ , \quad a^+ = \frac{1}{2\sqrt{E}}\bar{Q}_1 \ .$$

We have now only one pair of fermionic creation and annihilation operators: the irrep. are thus two-dimensional. Starting with a state $|\Omega\rangle$ annihilated by a: $a\,|\Omega\rangle = 0$, we get the doublet

$$|\Omega\rangle \ , \quad a^+\,|\Omega\rangle \ . \tag{2.9}$$

If $|\Omega\rangle$ is a state of helicity λ, then, from the commutation relation

$$[J^3, a^+] = \frac{1}{2}a^+ \ ,$$

follows that the helicity of its partner $a^+ |\Omega\rangle$ is $\lambda + 1/2$.

Remark. Considering only the proper and orthochronous component of the Lorentz group, which leaves helicity invariant, we see that the doublets are irreducible representations of the supergroup generated by the whole algebra including the Lorentz generators. Of course, if parity is included, then an irrep. is 4-dimensional, consisting of the states of helicity λ, $\lambda+1/2$, $-\lambda$ and $-\lambda - 1/2$.

Generalization for $N > 1$ (Massless particles):

The generalization to extended supersymmetry, $N > 1$, is straightforward if we assume the absence of central charges. We shall moreover restrict ourselves to the case of massless particles. With the same kinematics assumptions as in the case $N = 1$ above, the extended superalgebra reads

$$\{Q_1^i, \bar{Q}_1^j\} = 4\delta_{ij}E \ , \quad \{Q_1^i, \bar{Q}_2^j\} = \{Q_2^i, \bar{Q}_1^j\} = \{Q_2^i, \bar{Q}_2^j\} = 0 \ , \ i, j = 1, \cdots, N \ .$$

Again, Q_2^i vanishes on the helicity states $|E, \lambda\rangle$, and the nontrivial part of the algebra takes the form

$$\{a_i, a_j^+\} = \delta_{ij}, \quad \{a_i, a_j\} = \{a_i^+, a_j^+\} = 0 \ , \text{with} \quad a_i = \frac{1}{2\sqrt{E}} Q_1^i \ , \quad a_i^+ = \frac{1}{2\sqrt{E}} \bar{Q}_1^i \ .$$

Starting from the state $|\Omega\rangle = |E, \lambda\rangle$ obeying the conditions $a_i |\Omega\rangle = 0$, we construct the supermultiplet

$$\begin{aligned}
&|\Omega\rangle = |E, \lambda\rangle \\
&a_i^+ |\Omega\rangle = \left| E, \lambda + \frac{1}{2}; i \right\rangle \\
&a_i^+ a_j^+ |\Omega\rangle = |E, \lambda + 1; [ij]\rangle \\
&\cdots \\
&a_{i_1}^+ a_{i_2}^+ \cdots a_{i_N}^+ |\Omega\rangle = \left| E, \lambda + \frac{N}{2}; [i_1 \cdots i_N] \right\rangle
\end{aligned} \tag{2.10}$$

If one wants to specialize to renormalizable theories, on has to restrict the helicity values to $|\lambda| \leq 1$ (supersymmetric gauge theories). One sees, by putting $\lambda = -1$ in the equations above, that the number of supercharges is restricted to $N \leq 4$. If one allows for helicities up to 2 (supergravity), one gets the restriction $N \leq 8$.

3 Field Supermultiplets. Models

Bibliography: [1, 2, 3, 4, 6, 10]

The main object of this Section, after an introduction to the chiral supermultiplet and of the model of Wess-Zumino exhibiting the self-interaction of such a supermultiplet, will be the study of supersymmetric gauge theories, in the Wess-Zumino gauge, and of their renormalizability.

Let us begin with a brief historical review. Since their discovery [15], most of the efforts towards the renormalization of the super-Yang-Mills (SYM) theories, for many years, have been using the formalism of superspace and superfields, where the supersymmetry is realized linearly. Despite of earlier attempts [16], it is only recently [17, 18, 19, 20, 21] that their renormalization in the Wess-Zumino gauge has been successfully achieved[8]. The latter formalism implies a supersymmetry which is realized nonlinearly and whose commutation relations with the gauge transformations are nontrivial. But it has the advantage of involving a minimum number of unphysical gauge of freedom, on the one hand, and of being more suitable for applications to theories with extended ($N > 1$) supersymmetry] [17, 19], on the other hand.

3.1 The Chiral Supermultiplet

As it is known since a long time, in field theory the number of independent fields generally exceeds the number of physical degrees of freedom. (This is particularly striking in gauge theories, where all sorts of ghost fields appear.) We shall encounter this already in the simplest supersymmetric model in 4-dimensional space-time, namely the Wess-Zumino model with a "chiral supermultiplet".

The chiral supermultiplet consists of one complex scalar field $A(x)$, one Weyl spinor field $\psi_\alpha(x)$, $\alpha = 1,2$ and a second complex scalar field $F(x)$. The infinitesimal supersymmetry transformations of these fields read (with infinitesimal anticommuting parameters ε_α, $\bar\varepsilon_{\dot\alpha}$)

$$\delta_{\rm SUSY}\varphi = i\left[\varepsilon^\alpha Q_\alpha + \bar\varepsilon_{\dot\alpha}\bar Q^{\dot\alpha},\, \varphi\right] = \left(\varepsilon^\alpha\delta_\alpha + \bar\varepsilon_{\dot\alpha}\bar\delta^{\dot\alpha}\right)\varphi\ , \quad \varphi = A, \psi, F\ ,$$

with

$$\begin{aligned} &\delta_\alpha A = \psi_\alpha\ , && \bar\delta_{\dot\alpha} A = 0\ , \\ &\delta_\alpha \psi^\beta = 2\delta_\alpha^\beta F\ , && \bar\delta_{\dot\alpha}\psi_\alpha = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu A\ , \\ &\delta_\alpha F = 0\ , && \bar\delta_{\dot\alpha} F = i\partial_\mu\psi^\alpha\sigma^\mu_{\alpha\dot\alpha}\ . \end{aligned} \tag{3.1}$$

and similarly for the complex conjugate components $\bar A$, $\bar\psi$ and $\bar F$. One easily checks that the Wess-Zumino algebra (2.5) holds:

$$\{\delta_\alpha, \bar\delta_{\dot\alpha}\} = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\ , \quad \{\delta_\alpha, \delta_\beta\} = \{\bar\delta_{\dot\alpha}, \bar\delta_{\dot\beta}\} = 0\ , \tag{3.2}$$

[8] Let us mention however the pioneering works of Ref. [22] and of [23], where strong arguments showing the ultraviolet finiteness of the $N = 4$ SYM theory have been presented for the first time.

the infinitesimal translations $x^\mu \to x^\mu + \xi^\mu$ being represented by

$$\delta_{\mathrm{transl}}\varphi = i\left[\xi^\mu P_\mu, \varphi\right] \ , \quad \text{with } i\left[P_\mu, \varphi\right] = \partial_\mu\varphi \ .$$

Remarks.

1. The space-time integral of the last component of a chiral supermultiplet, i.e. of its F-component, is invariant under supersymmetry:

$$\delta_{\mathrm{SUSY}}\int d^4x \ F(x) = 0 \ . \tag{3.3}$$

2. The condition

$$\bar\delta_{\dot\alpha}A = 0 \tag{3.4}$$

defines the *chiral* supermultiplet. The remaining transformation rules follow uniquely from the algebra. Indeed, let us first define the fields ψ_α and $F_{\alpha\beta}$ by

$$\delta_\alpha A = \psi_\alpha \ , \quad \delta_\alpha\psi_\beta = F_{\alpha\beta} = -2\varepsilon_{\alpha\beta}F + G_{(\alpha\beta)} \ ,$$

where we have separated the bi-spinor $F_{\alpha\beta}$ into its antisymmetric and symmetric parts. It follows immediately from the anticommutativity of the operators δ_a, that any product of three or more of them is identically vanishing. Hence $\delta_\alpha F_{\alpha\beta} = 0$, or:

$$\delta_a F = 0 \ , \quad \delta_a G_{(\alpha\beta)} = 0 \ .$$

On the other hand, the anticommutation rule of $\bar\delta_{\dot\alpha}$ with δ_α implies

$$\begin{aligned} \bar\delta_{\dot\alpha}\psi_\alpha &= \bar\delta_{\dot\alpha}\delta_\alpha A = \left\{\bar\delta_{\dot\alpha}, \delta_\alpha\right\} A = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu A \ , \\ \bar\delta_{\dot\alpha}F_{\alpha\beta} &= \bar\delta_{\dot\alpha}\delta_\alpha\psi_\beta = \left\{\bar\delta_{\dot\alpha}, \delta_\alpha\right\}\psi_\beta - \delta_a\bar\delta_{\dot\alpha}\psi_\beta \\ &= 2i\left(\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\psi_\beta - \sigma^\mu_{\beta\dot\alpha}\partial_\mu\psi_\alpha\right) \ , \end{aligned}$$

The right-hand side of the second equation being antisymmetric in the indices α and β, we can write

$$\bar\delta_{\dot\alpha}G_{(\alpha\beta)} = 0 \quad \text{and} \quad \bar\delta_{\dot\alpha}F = i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\psi^\alpha \ ,$$

where we have used that $F = \frac{1}{4}\varepsilon^{\alpha\beta}F_{\alpha\beta}$. We have thus obtained the transformation rules (3.1) for the fields A, ψ and F. Moreover, the field $G_{(\alpha\beta)}$ has turned out to be invariant under both δ_α and $\bar\delta_{\dot\alpha}$, which implies, through the algebra, that it is a constant: being uninterested in constant fields, we may as well set it to zero.

The structure of the chiral supermultiplet and of its complex conjugate, the antichiral supermultiplet $\{\bar A, \bar\psi_{\dot\alpha}, \bar F\}$ defined by the condition conjugate to (3.4): $\delta_\alpha\bar A = 0$, are illustrated in Tables 1 and 2.

$$
\begin{array}{ccccc}
 & & Q & & Q & \\
 & A & \longrightarrow & \psi_\alpha & \longrightarrow & F \\
\bar{Q} & \downarrow & & & & \\
 & 0 & & & &
\end{array}
$$

Table 1: Chiral supermultiplet. The horizontal and vertical arrows indicate the action of the supercharges Q_α and $\bar{Q}_{\dot\alpha}$, respectively (see (3.1)). Only the transformations leading to new fields, and not to derivatives of them, are represented.

$$
\begin{array}{cccc}
 & & Q & \\
 & \bar{A} & \longrightarrow & 0 \\
\bar{Q} & \downarrow & & \\
 & \bar{\psi}_{\dot\alpha} & & \\
\bar{Q} & \downarrow & & \\
 & \bar{F} & &
\end{array}
$$

Table 2: Antichiral supermultiplet. The horizontal and vertical arrows indicate the action of the supercharges Q_α and $\bar{Q}_{\dot\alpha}$, respectively (see (3.1)). Only the transformations leading to new fields, and not to derivatives of them, are represented.

The R-transformation belonging to the Wess-Zumino superalgebra (2.5)–(2.7) acts on the chiral supermultiplet and on its conjugate as

$$
i\,[\eta R, \varphi] = \delta_R \varphi \ , \quad \varphi = A, \psi, F \text{ and conjugates} \ ,
$$

where η is an infinitesimal parameter, and

$$
\begin{aligned}
&\delta_R A = inA \ , && \delta_R \bar{A} = -in\bar{A} \ , \\
&\delta_R \psi_\alpha = i(n+1)\psi_\alpha \ , && \delta_R \bar{\psi}_{\dot\alpha} = -i(n+1)\bar{\psi}_{\dot\alpha} \ , \\
&\delta_R F = i(n+2)F \ , && \delta_R \bar{F} = -i(n+2)\bar{F} \ .
\end{aligned}
\tag{3.5}
$$

The real number n appearing in the R-transformations is called the *R-weight* of this chiral supermultiplet.

Construction of invariant actions:

The construction of supersymmetry invariants proceeds from the observation that the last component of a supermultiplet always transforms under supersymmetry (3.1) as a total derivative, which implies that its space-time integral is invariant. We have seen this explicitely for the chiral multiplet (see Equ. (3.3)). This remains true for the last component – the "D-component" – of the most general supermultiplet, called the *vector supermultiplet*,

$$
\phi = \left\{ C,\, \chi_\alpha,\, \bar{\chi}_{\dot\alpha},\, M,\, \bar{M},\, v_\mu,\, \lambda_\alpha,\, \bar{\lambda}_{\dot\alpha},\, D \right\} \ , \tag{3.6}
$$

with the transformation rules

$$
\begin{aligned}
&\delta_\alpha C = \chi \ , \quad \delta_\alpha \chi^\beta = \delta_\alpha^\beta M \ , \quad \delta_\alpha \bar\chi_{\dot\alpha} = \sigma^\mu_{\alpha\dot\alpha}(v_\mu + i\partial_\mu C) \ , \\
&\delta_\alpha M = 0 \ , \quad \delta_\alpha \bar M = \lambda_\alpha - i(\sigma^\mu \partial_\mu \bar\chi)_\alpha \ , \quad \delta_\alpha v_\mu = \frac{1}{2}(\sigma_\mu \bar\lambda)_\alpha - \frac{i}{2}(\sigma^\nu \bar\sigma_\mu \partial_\nu \chi)_\alpha \ , \\
&\delta_\alpha \lambda^\beta = \delta_\alpha^\beta D + i(\sigma^\nu \bar\sigma^\mu)_\alpha{}^\beta \partial_\nu v_\mu \ , \quad \delta_\alpha \bar\lambda_{\dot\alpha} = i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu M \ , \quad \delta_\alpha D = -i(\sigma^\mu \partial_\mu \bar\lambda)_\alpha \ .
\end{aligned}
\tag{3.7}
$$

(We consider here the *real* vector supermultiplet). It is not difficult to check that these transformation rules are the most general ones compatible with the Wess-Zumino algebra, the reality condition being taken into account. The latter condition allows to compute the transformations under $\bar Q_{\dot\alpha}$ as the complex conjugate of the transformations above. The result is illustrated in Table 3.

$$
\begin{array}{ccccccc}
 & & Q & & Q & \\
 & C & \longrightarrow & \chi_\alpha & \longrightarrow & M \\
\bar Q & \downarrow & & \downarrow & & \downarrow \\
 & \bar\chi_{\dot\alpha} & \longrightarrow & v_\mu & \longrightarrow & \bar\lambda_{\dot\alpha} \\
\bar Q & \downarrow & & \downarrow & & \downarrow \\
 & \bar M & \longrightarrow & \lambda_\alpha & \longrightarrow & D
\end{array}
$$

Table 3: Real vector supermultiplet. The horizontal and vertical arrows indicate the action of the supercharges Q_α and $\bar Q_{\dot\alpha}$, respectively (see (3.7)). Only the transformations leading to new fields, and not to derivatives of them, are represented.

Remark. One easily checks that any supermultiplet whose first component A, $\bar A$ or C is zero, is identically vanishing. This implies that any supermultiplet is uniquely defined by its first conponent.

In order to write invariant actions one has to consider supermultiplets which are themselves products of elementary supermultiplets, then take their last component F, $\bar F$ or D, and integrate. The supermultiplet composition rules for chiral supermultiplets are listed here:

i) The product A_1A_2 of two chiral supermultiplets A_1 and A_2 is the – unique – chiral supermultiplet[9]

$$
\begin{aligned}
&\{A_1A_2,\ A_1A_2|_\psi,\ A_1A_2|_F\} \\
&\quad = \left\{ A_1A_2,\ A_1\psi_{2\alpha} + \psi_{1\alpha}A_2,\ A_1F_2 + F_1A_2 - \frac{1}{2}\psi_1^\alpha\psi_{2\alpha} \right\} \ .
\end{aligned}
\tag{3.8}
$$

[9]We often denote a supermultiplet with the name of its first component.

ii) The product $A\bar{A}$ of a chiral supermultiplet A by its conjugate $\bar{A}$ is the – unique – real vector supermultiplet (we only write out explicitely its first and last component)

$$
\begin{aligned}
&\left\{A\bar{A},\ \cdots,\ A\bar{A}|_D\right\} = \\
&\quad\left\{A\bar{A},\ \cdots,\right. \\
&\qquad\left. 2\partial^\mu A\partial_\mu\bar{A} - A\partial^2\bar{A} - \partial^2 A\bar{A} + i\psi^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\bar\psi^{\dot\alpha} - i\partial_\mu\psi^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\psi^{\dot\alpha} + 4F\bar{F}\right\}\ .
\end{aligned}
\tag{3.9}
$$

The Wess-Zumino model:

Let us consider now, as an illustration, the theory of one chiral supermultiplet A. Assigning canonical dimensions 1 and 3/2 to the fields A and ψ, respectively – then F has dimension 2 – the most general supersymmetric action which is power-counting renormalizable, i.e. of dimension bounded by four[10] reads

$$
\begin{aligned}
\Sigma(A,\bar{A}) &= \int d^4x\ \left(\frac{1}{4}\ A\bar{A}\Big|_D - \frac{m}{2}\left(A^2\Big|_F + \bar{A}^2\Big|_{\bar F}\right) - \frac{\lambda}{12}\left(A^3\Big|_F + \bar{A}^3\Big|_{\bar F}\right)\right) \\
&= \int d^4x\ \Big(\partial^\mu A\partial_\mu\bar{A} + \frac{i}{2}\psi\sigma^\mu\partial_\mu\bar\psi + F\bar{F} \\
&\qquad -\frac{m}{4}(4AF - \psi^2 + \text{ conj.}) - \frac{\lambda}{8}(2A^2F - A\psi^2 + \text{ conj.})\Big)
\end{aligned}
\tag{3.10}
$$

This action is moreover invariant under the R-transformations (3.5), up to the mass terms, if we assign the R-weight $n = -2/3$ to the chiral multiplet A.

The field equations are[11]

$$
\begin{aligned}
\frac{\delta\Sigma}{\delta A} &= -\partial^2\bar{A} - mF - \frac{\lambda}{2}AF = 0\ , \\
\frac{\delta\Sigma}{\delta\psi} &= \frac{i}{2}\partial\!\!\!/\bar\psi + \frac{m}{2}\psi + \frac{\lambda}{4}A\psi = 0\ , \\
\frac{\delta\Sigma}{\delta F} &= \bar{F} - mA - \frac{\lambda}{4}A^2 = 0\ .
\end{aligned}
\tag{3.11}
$$

It is worth noticing that F and $\bar{F}$ are auxiliary fields, i.e. that their equations of motion are purely algebraic. It follows that they can be eliminated and thus effectively do not represent independent degrees of freedom. Substituting the expression given by (3.11) for F and its conjugate in the action (3.10), we get the equivalent action

$$
\begin{aligned}
&\Sigma(A,\bar{A}) = \\
&\int d^4x\ \left(\frac{i}{2}\psi\sigma^\mu\partial_\mu\bar\psi + \partial^\mu A\partial_\mu\bar{A} + \frac{m}{4}\psi^2 + \frac{m}{4}\bar\psi^2 + \frac{\lambda}{8}A\psi^2 + \frac{\lambda}{8}\bar{A}\bar\psi^2 - V(A,\bar{A})\right)\ ,
\end{aligned}
\tag{3.12}
$$

[10] Our units are chosen such that $c = \hbar = 1$. Dimensions are counted in mass units. The "dimension" of a space-time integral such as the action is by definition the dimension of its integrand – here: the Lagrangian density.

[11] **Notation:** $(\partial\!\!\!/\bar\psi)_\alpha = \sigma^\mu_{\alpha\dot\alpha}\partial_\mu\bar\psi^{\dot\alpha}$.

with a potential given by

$$V(A,\bar{A}) = F(\bar{A})\bar{F}(A) = \left| mA + \frac{\lambda}{4}A^2 \right|^2 \ , \tag{3.13}$$

which turns out to be positive. This positivity is a pecularity of supersymmetric theories.

Remarks.

1. The new action (3.12) is invariant under the transformation laws (3.1) where F has been eliminated as well. The algebra (3.2) is however modified: the anticommutator of two supersymmetries is no more a simple translation, but there is an additional term involving a field equation. This happens for the anticommutator as applied to the spinor component only:
$$\left\{\delta_\alpha, \bar{\delta}_{\dot{\alpha}}\right\} \psi^\beta = 2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\psi^\beta + 4\delta^\beta_\alpha \frac{\delta\Sigma}{\delta\bar{\psi}^{\dot{\alpha}}} \ . \tag{3.14}$$

2. This nonclosure, and the fact that the transformations are now nonlinear, could complicate the renormalization procedure: the fields transform now into composite fields, which may be the source of new ultraviolet divergences. It is thus useful, in certain cases, to keep oneself from eliminating the auxiliary field F.

3.2 The Vector Supermultiplet. Gauge Invariance

The transformation rules of the "real vector supermultiplet" have been given by (3.7) and illustrated in Table 3. This multiplet containing a vector field, v_μ, is a good candidate for a supersymmetric generalization of the concept of gauge vector field. There is however another possibility of implementing supersymmetry in gauge theories, which involves a shorter supermultipet, the Wess-Zumino gauge supermultiplet, in short: the *WZ-multiplet*. It consists of a vector field A_μ, of a Weyl spinor field λ_α and of an auxiliary pseudoscalar field D. Such a supermultiplet only makes sense, as we shall see, in the context of a gauge theory.

Let us write the supersymmetry transformation laws in the case we are dealing with a general nonabelian gauge theory. The generators X_a of the gauge group G obey the Lie algebra

$$[X_a, X_b] = i f_{ab}{}^c X_c \ , \tag{3.15}$$

and the fields of the WZ-multiplet take an index a, the infinitesimal gauge transformations being

$$\delta_{\rm gauge} A^a_\mu = \nabla_\mu\omega^a = \partial_\mu\omega^a + f_{bc}{}^a A^b_\mu\omega^c \ , \delta_{\rm gauge}\varphi^a = f_{bc}{}^a\varphi^b\omega^c \ , \quad \varphi = \lambda, \bar{\lambda}, D \ , \tag{3.16}$$

where $\omega^a(x)$ is the infinitesimal local parameter. Then the supersymmetry transformations read

$$
\begin{aligned}
\delta_\alpha A^a_\mu &= \frac{1}{4}\sigma_{\mu\alpha\dot\alpha}\bar\lambda^{a\dot\alpha} \ , \\
\delta_\alpha \lambda^{a\beta} &= \delta^\beta_\alpha D^a + 2\sigma^{\mu\nu}{}_\alpha{}^\beta F^a_{\mu\nu} \ , \\
\delta_\alpha \bar\lambda^a_{\dot\alpha} &= 0 \ , \\
\delta_\alpha D^a &= -i\sigma^\mu_{\alpha\dot\alpha}\nabla_\mu\bar\lambda^{a\dot\alpha} \ ,
\end{aligned}
\tag{3.17}
$$

with the Yang-Mills field strength and the covariant derivative defined by

$$
\begin{aligned}
F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f_{bc}{}^a A^b_\mu A^c_\nu \ , \\
\nabla_\mu \varphi^a &= \partial_\mu\varphi + f_{bc}{}^a A^b_\mu \varphi^c \ , \quad \varphi = \lambda, \bar\lambda, D \ .
\end{aligned}
\tag{3.18}
$$

These transformation rules are illustrated in Table 4. The two main differences between the

$$
\begin{array}{cccc}
 & & Q & \\
 & A_\mu & \longrightarrow & \bar\lambda_{\dot\alpha} \\
\bar Q & \downarrow & & \downarrow \\
 & \lambda_\alpha & \longrightarrow & D
\end{array}
$$

Table 4: Wess-Zumino vector supermultiplet. The horizontal and vertical arrows indicate the action of the supercharges Q_α and $\bar Q_{\dot\alpha}$, respectively (see (3.17)). Only the transformations leading to new fields – and not to derivatives of them – are represented.

latter transformation rules and the rules (3.7) are that they are nonlinear in the fields, on the one hand, and that their algebra is not exactly of the Wess-Zumino type, as in (3.2), on the other hand. Concerning the second point, one checks indeed that these transformations obey the following superalgebra, which we write down here for every component of the WZ-multiplet:

$$
\begin{aligned}
\left\{\delta_\alpha, \bar\delta_{\dot\alpha}\right\} A^a_\nu &= 2i\sigma^\mu_{\alpha\dot\alpha}F^a_{\mu\nu} = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu A^a_\nu - i\nabla_\nu\omega^a_{\alpha\dot\alpha} \ , \\
\left\{\delta_\alpha, \bar\delta_{\dot\alpha}\right\} \varphi^a &= 2i\sigma^\mu_{\alpha\dot\alpha}\nabla_\mu\varphi^a = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\varphi^a + f_{bc}{}^a\omega^b_{\alpha\dot\alpha}\varphi^c \ , \quad \varphi = \lambda, \bar\lambda, D \ ,
\end{aligned}
\tag{3.19}
$$

with

$$
\omega^a_{\alpha\dot\alpha} = 2\sigma^\mu_{\alpha\dot\alpha}A^a_\mu \ . \tag{3.20}
$$

One recognizes in the right-hand sides the superposition of a translation and of a gauge transformation (3.16), but with a *field dependent parameter* $\omega^a_{\alpha\dot\alpha}$ as given by (3.20).

Remarks.

1. The algebra generated by these supersymmetry transformations is not closed. Indeed, if we compute the commutator of the right-hand sides of (3.19) with supersymmetry transformations, we obtain gauge transformations whose parameter is the supersymmetry transform of (3.20), and so on, repeating the operation. This finally leads to an

infinite chain of new gauge transformations, which sets an apparently insolvable problem when coming to the quantum corrections, each new gauge transformation involving a new composite field which ought to be renormalized [16]. It is however clear that, at the classical level, when applied to gauge invariant quantities, this algebra reduces to the usual Wess-Zumino (3.2) one. A solution of this puzzle, applicable to the quantum theory, will be presented in Subsection 4.1.

2. It is possible to work with the full vector supermultiplet (3.6), with the advantage that its supersymmetric transformations (3.7) are linear, which allows the use of the formalism of superfields and superspace [1, 6, 10]. However, this multiplet contains supplementary fields C, χ_α and M, which represent unphysical gauge degrees of freedom associated to a"supergauge invariance". Two options are then possible:

   a) Either work with all these variables, taking profit of the power of superspace formalism. But this gain is less obvious if one wants to treat theories with supersymmetry breaking.

   b) Or eliminate the spurious variables by choosing a special supergauge called the *Wess-Zumino gauge*. The resulting version of the theory is precisely the one we are considering in these lectures.

Matter fields:

In the class of theories we are going to consider, matter is described by chiral supermultiplets $\{A^i, \psi^i, F^i\}$, with gauge transformations

$$\delta_{\rm gauge}\varphi^i(x) = -iT^i_{aj}\omega^a(x)\varphi^j(x)\ , \quad \varphi^i = A^i, \psi^i, F^i\ , \tag{3.21}$$

corresponding to a certain unitary representation of the gauge group, the matrices T_a obeying the commutation rules (3.15). The supersymmetry transformations are the ones we have given previously, (3.1), but with derivatives replaced by gauge covariant derivatives

$$\nabla_\mu\varphi^i = \partial_\mu\varphi^i + iT^i_{aj}A^a_\mu\varphi^j\ . \tag{3.22}$$

They read:

$$\begin{array}{ll} \delta_\alpha A^i = \psi^i_\alpha\ , & \bar\delta_{\dot\alpha} A^i = 0\ , \\ \delta_\alpha\psi^{i\beta} = 2\delta^\beta_\alpha F\ , & \bar\delta_{\dot\alpha}\psi^i_\alpha = 2i\sigma^\mu_{\alpha\dot\alpha}\nabla_\mu A^i\ , \\ \delta_\alpha F^i = 0\ , & i\bar\delta_{\dot\alpha} F^i = i\nabla_\mu\psi^{i\alpha}\sigma^\mu_{\alpha\dot\alpha}\ . \end{array} \tag{3.23}$$

As for the WZ-multiplet, the algebra closes on the translations, accompanied by field dependent gauge transformations:

$$\left\{\delta_\alpha, \bar\delta_{\dot\alpha}\right\}\varphi^i = 2i\sigma^\mu_{\alpha\dot\alpha}\nabla_\mu\varphi^i = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\varphi^i + iT^i_{aj}\omega^a_{\alpha\dot\alpha}\varphi^j\ , \quad \varphi^i = A^i, \psi^i, F^i\ , \tag{3.24}$$

with $\omega^a_{\alpha\dot\alpha}$ given by (3.20).

Super-Yang-Mills action:

The most general supersymmetric gauge invariant action of dimension bounded by 4, involving the WZ-multiplet and the matter chiral supermultiplets, with supersymmetry transformations as given by (3.17) and (3.23), reads

$$
\begin{aligned}
S_{\mathrm{SYM}} = \int d^4x \, \Big(& \frac{1}{g^2}\Big(-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + i\lambda^{a\alpha}\sigma^{\ \mu}_{\alpha\dot\beta}\nabla_\mu\bar\lambda^{a\dot\beta} + \frac{1}{2}D^a D_a\Big) \\
& + \nabla_\mu A^i \nabla_\mu \bar A_i + \frac{i}{2}\psi^{i\alpha}\sigma^{\ \mu}_{\alpha\dot\beta}(\nabla_\mu\bar\psi^{\dot\beta})_i + F^i\bar F_i \\
& + \Big(-\frac{1}{2}m_{(ij)}(2A^iF^j - \frac{1}{2}\psi^i\psi^j) - \frac{1}{4}(\lambda_{(ijk)}(A^iA^jF^k - \frac{1}{2}\psi^{i\alpha}\psi^j_\alpha A^k) + \text{ conj.}\Big) \\
& + \Big(-i\bar\psi_{i\dot\beta}T^i_{aj}A^j\bar\lambda^{a\dot\beta} + \text{ conj.}\Big) \\
& + \frac{1}{4}\bar A_i T^i_{aj}A^j D^a\Big) \ ,
\end{aligned}
\tag{3.25}
$$

The mass matrix $m_{(ij)}$ and the Yukawa coupling constants $\lambda_{(ijk)}$ are invariant symmetric tensors of the gauge group.

This action is also invariant, up to the mass terms, under the R-transformations, given by (3.5) with the weight $n = -2/3$ for the chiral multiplets, and by

$$
\delta_R A_\mu = 0 \ , \quad \delta_R\lambda_\alpha = -\lambda_\alpha \ , \quad \delta_R\bar\lambda_{\dot\alpha} = \bar\lambda_{\dot\alpha} \ , \tag{3.26}
$$

for the WZ-multiplet. The generator R still obeys the commutation rules (2.7) with the supersymmetry generators, and commutes with the translation generator.

Remark. The supersymmetric extensions of the Standard Model of elementary interactions are given by actions of the type (3.25), with specific gauge group and representations. This is in particular the case of the "Minimal Supersymmetric Standard Model" and of the Supersymmetric Grand Unified Theories [24].

As for the model of subsection 3.1, we can eliminate the auxiliary fields F, $\bar F$ and D by substituting their equations of motion:

$$
\begin{aligned}
\bar F^i(A) &= m_{ij}A^j + \frac{1}{4}\lambda_{ijk}A^jA^k \ , \\
D_a(A,\bar A) &= -\frac{1}{4}\bar A_i T^i_{aj}A^j \ ,
\end{aligned}
\tag{3.27}
$$

in the action (3.25) and in the supersymmetry transformation laws (3.7) and (3.23). Again, the action contains a potential term for the scalar fields which is positive:

$$
V(A,\bar A) = \frac{1}{2}D^a(A,\bar A)D_a(A,\bar A) + F^i(\bar A)\bar F_i(A) \ . \tag{3.28}
$$

Also, the superalgebra does close only modulo equations of motions. Explicitly, the anticommutator $\{\delta_\alpha, \bar\delta_{\dot\alpha}\}$ applied on the spinor fields yields (c.f. (3.14))

$$
\begin{aligned}
\left\{\delta_\alpha, \bar\delta_{\dot\alpha}\right\} \psi^\beta &= 2i\sigma^\mu_{\alpha\dot\alpha}\nabla_\mu\psi^\beta + 4\delta^\beta_\alpha \frac{\delta\Sigma_{\rm SYM}}{\delta\bar\psi^{\dot\alpha}} \ , \\
\left\{\delta_\alpha, \bar\delta_{\dot\alpha}\right\} \lambda^\beta &= 2i\sigma^\mu_{\alpha\dot\alpha}\nabla_\mu\lambda^\beta - 2\delta^\beta_\alpha \frac{\delta\Sigma_{\rm SYM}}{\delta\bar\lambda^{\dot\alpha}} \ .
\end{aligned}
\tag{3.29}
$$

4 Renormalization of SYM Theories in the Wess-Zumino Gauge

The problem to be solved is to show the renormalizability of the super-Yang-Mills (SYM) theory, i.e. to show that it is possible to construct a perturbative quantum version of it, preserving both gauge invariance and supersymmetry. Since we don't dispose of any regularization procedure preserving both invariances together, it is convenient to use a general, regularization independent renormalization scheme. Such a scheme, called the "algebraic renormalization" [25], has been exposed by Silvio Sorella in his lectures at the present School [26]. I shall only summarize the procedure as I want to apply it here, only emphasizing the points which are specific to the supersymmetric gauge models.

In order to simplify at most the exposition, I shall consider the pure SYM case, without matter fields, the auxiliary field D_a eliminated – its equation of motion reducing now to the trivial equation $D_a = 0$.

4.1 BRS Algebra. Gauge Fixing

The starting point is the gauge and supersymmetry invariant action (3.25), without the matter chiral supermultiplets A^i, and with the auxiliary fields D_a eliminated:

$$
S_{\rm SYM} = \int d^4x \, \frac{1}{g^2}\left(-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + i\lambda^{a\alpha}\sigma^{\ \mu}_{\alpha\dot\beta}\nabla_\mu\bar\lambda^{a\dot\beta}\right)
\tag{4.1}
$$

Being masssless, this action is exactly invariant under the R-transformations (3.26).

Remark. The action (4.1) looks like that of an ordinary gauge theory, the "gaugino" field λ playing the role of a matter field minimally coupled to the gauge field. The only specificity, which makes the theory supersymmetric, is that λ belongs to the adjoint representation of the gauge group.

We know that the superalgebra of the symmetries of the theory is not closed (c.f. the first of the remarks following Equ.(3.16)). In particular, we have seen that the anticommutator of two supersymmetry transformations is a translation accompanied by a field dependent gauge transformation (and a by a field equation since we have eliminated the auxiliary fields). And we have observed that this leads – apparently – to an infinite algebra involving an infinity of new generators.

The way to solve this puzzle has been found by White [18, 17] (See also [19, 20, 21]). It consists in putting together all the generators of the algebra into one single BRST operator. Let us recall that the usual BRS symmetry of gauge theories is obtained by replacing in the gauge transformations the local infinitesimal parameters $\omega^a(x)$ by *anticommuting* fields $c^a(x)$, the Faddeev-Popov ghost fields, and by defining the transformation laws of these ghost fields in such a way that the transformation be nilpotent. Calling s the *BRS operator* obtained in this way, we can write the result as

$$\begin{aligned} sA^a_\mu &= \nabla_\mu c^a \ , \quad s\lambda^a = f_{bc}{}^a\lambda^b c^c \ , \\ sc^a &= -\frac{1}{2} f_{bc}{}^a c^b c^c \ , \qquad s^2 = 0 \ , \end{aligned} \tag{4.2}$$

with the covariant derivative defined as in (3.18).

Since the superalgebra we have in the present theory mixes gauge with supersymmetry transformations, it is naturel[12] to generalize the BRS operator above, including into it the infinitesimal supersymmetry transformations (3.17), but with the infinitesimal parameters ε^a becoming *commuting* numbers instead of anticommuting ones. These "global ghosts" are constant, of course, supersymmetry not being a local invariance. The translations and the R-transformations (3.26) belonging to the algebra, we also promote their infinitesimal parameters ξ^μ to the rank of global ghosts: they will now be anticommuting numbers. The general rule is to assign to the (local or global) ghosts the statistics opposite to the one of the usual infinitesimal parameters – as shown in Table 5 for the present theory.

Symmetry	**Generators**	**Infinitesimal parameters**
Supersymmetry	Q_α, $\bar{Q}_{\dot\alpha}$	ε^α, $\bar\varepsilon^{\dot\alpha}$ (odd)
Translations	P_μ	ξ^μ (even)
R	R	η (even)
Gauge	X_a	$\omega^a(x)$ (even)

Table 5: The symmetries of the super-Yang-Mills theory, the names of their generators and infinitesimal parameters, and the grading, i.e. the statistics, of the latters.

As in the case of the pure gauge BRS invariance (4.2), we ought to define the transformation rules for the global as well as for for the local ghosts in order to achieve the nilpotency of the generalized BRS operator. It turns out however that, due to the superalgebra (2.5) having field equations in its right-hand sides because of the elimination of the auxiliary fields (see (3.29)), the generalized BRS operator will only be nilpotent up to field equations. Denoting

[12] At my knowledge, [27] is the first paper proposing the use of the BRS techniques for dealing with global symmetries. Dixon [28] has applied this idea to supersymmetry.

the generalized BRS operator with the same symbol s as above, we have

$$
\begin{aligned}
sA^a_\mu &= \nabla_\mu c^a + \varepsilon^\alpha \sigma_{\mu\alpha\dot\beta} \bar\lambda^{a\dot\beta} + \lambda^{a\alpha}\sigma_{\mu\alpha\dot\beta}\bar\varepsilon^{\dot\beta} - i\xi^\nu \partial_\nu A^a_\mu \\
s\lambda^{a\alpha} &= -f^{abc}c^b\lambda^{c\alpha} - \frac{1}{2}\varepsilon^\gamma \sigma^{\mu\nu}{}_\gamma{}^\alpha F^a_{\mu\nu} - \frac{i}{2}g^2\varepsilon^\alpha(\bar A_i T^i_{aj} A_b) - i\xi^\mu\partial_\mu\lambda^{a\alpha} - \eta\lambda^{a\alpha} \\
sc^a &= -\frac{1}{2}f^{abc}c^b c^c - 2i\varepsilon^\alpha \sigma^{\ \mu}_{\alpha\dot\beta}\bar\varepsilon^{\dot\beta} A^a_\mu - i\xi^\mu\partial_\mu c^a \\
s\xi^\mu &= -2\varepsilon^\alpha\sigma^{\ \mu}_{\alpha\dot\beta}\bar\varepsilon^{\dot\beta} \\
s\varepsilon^\alpha &= -\eta\varepsilon^\alpha \\
s\eta &= 0 \ ,
\end{aligned}
\tag{4.3}
$$

with

$$s^2 = \text{ field equations} \ . \tag{4.4}$$

The right-hand side of (4.4) is effectively nonzero for s^2 applied to the spinor fields, as a consequence of (3.29):

$$s^2\lambda^b = 2\varepsilon^\beta\bar\varepsilon^{\dot\alpha}\frac{\delta\Sigma_{\mathrm{SYM}}}{\delta\bar\lambda_{\dot\alpha}} \ .$$

Gauge fixing:

A prerequisite to the construction of a quantum extension of the classical theory defined by the gauge invariant supersymmetric action (4.1) is the fixation of the gauge. In the BRS framework, this is most conveniently done by introducing a *Lagrange multiplier* field $B_a(x)$ and a *Faddeev-Popov antighost* field $\bar c_a(x)$, with BRS transformations

$$s\bar c^a = B^a - i\xi^\mu\partial_\mu\bar c^a \ , \quad sB^a = -2i\varepsilon^\alpha\sigma^{\ \mu}_{\alpha\dot\beta}\bar\varepsilon^{\dot\beta}\partial_\mu\bar c^a - i\xi^\mu\partial_\mu B^a \ . \tag{4.5}$$

The terms in ξ and ε assure the compatibility with the BRS transformations previously defined in (4.3), together with the preservation of the nilpotency.

The gauge is then fixed by adding to the action a suitable BRS invariant piece Σ_{gf} containing the fields B and $\bar c$. A particular choice is the *Landau gauge*, defined by

$$\Sigma_{\mathrm{gf}} = s\int d^4x\ \left(\bar c_a\partial^\mu A^a_\mu\right) = \int d^4x\ \left(B_a\partial^\mu A^a_\mu + \bar c_a\partial^\mu\nabla_\mu c^a\right) + O(\epsilon,\bar\epsilon) \ . \tag{4.6}$$

The BRS invariance of (4.6) follows from the nilpotency of s, which here is exact since no spinor field is involved. The Lagrange multiplier term induces the gauge fixing constraint $\partial^\mu A^a_\mu = 0$ on the gauge field.

The total action

$$\Sigma = \Sigma_{\mathrm{SYM}} + \Sigma_{\mathrm{gf}} \tag{4.7}$$

will be the starting point of the perturbative quantization.

4.2 Renormalization. Anomaly

Bibliography: [18, 20, 21]

The renormalization of a given classical theory – in our case, the theory defined in the preceding Subsection – consists in the perturbative construction of a quantum theory which coincides with it in the limit of vanishing Planck constant [25, 26]. "Perturbative" means here an expansion[13] in the powers of $\hbar$, this power corresponding to the number of loops in the Feynman graphs.

The construction of the quantum theory should preserve the BRS invariance of the classical theory. If the task turns out to be impossible, one speaks of an anomaly.

The quantum BRS invariance is expressed by a *Slavnov-Taylor identity* which, in the classical limit, corresponds to the BRS invariance of the action (4.7). At the quantum level, the relevant object is the Green functional, i.e. the generating functional of the Green functions[14] $\langle 0|\, T\,\varphi_{A_1}\cdots\varphi_{A_N}\,|0\rangle$. An equivalent object – more convenient for the purpose of renormalization – is the *vertex functional* $\Gamma(\varphi)$, which generates the vertex functions $\langle 0|\, T\,\varphi_{A_1}\cdots\varphi_{A_N}\,|0\rangle_{\rm 1PI}$, i.e. the Green functions amputated of their external lines and to which only the one-particle irreducible (1PI) Feynamn graphs[15] are contributing:

$$\left.\frac{\delta\Gamma}{\delta\varphi_{A_1}\cdots\delta\varphi_{A_N}}\right|_{\varphi=0} = \langle 0|\, T\,\varphi_{A_1}\cdots\varphi_{A_N}\,|0\rangle_{\rm 1PI}\ .$$

The perturbative expansion of the vertex functional:

$$\Gamma(\varphi) = \sum_{n\geq 0}\hbar^n\Gamma_n(\varphi)\ , \tag{4.8}$$

is such that it coincides with the classical action in the classical limit[16]:

$$\Gamma_0(\varphi) = \Sigma(\varphi)\ ,\quad \text{or:}\quad \Gamma(\varphi) = \Sigma(\varphi) + O(\hbar)\ . \tag{4.9}$$

Let us write, symbolically[17], the Slavnov-Taylor identity to be proved as

$$s\Gamma = 0\ . \tag{4.10}$$

[13]The expansion in $\hbar$ is equivalent here to the expansion in the powers of the gauge coupling constant g.

[14]The symbol T stands for "time-ordered product", the indices A_k stand for the field type and for the values of the internal and Lorentz indices and of the space-time cooordinates alltogether. Note that we are using the same symbol φ_{A_k} for denoting an argument of the vertex functional – a smooth function – and the corresponding quantum field operator.

[15]A 1PI graph is a Feynman graph with all its external lines suppressed, and which moreover is such that it cannot be decomposed in two disjoint pieces by cutting one single line. Any general Green function has the structure of a tree, the lines representing the full propagators (the 2-point Green functions), and the vertices corresponding to the vertex functions. The potential ultraviolet singularities caused by divergent integrals on the loop momenta are contained in the 1PI graphs contributing to the vertex functions.

[16]Which corresponds to the approximation of the tree Feynman graphs, i.e. of the graphs without loop

[17]A precise formulation [25, 26] would need the introduction of external fields – the Batalin-Vilkovisky "antifields" – coupled to the BRS variations of the various quantum fields. It is in this formalism that the definition of the BRS operator can be extended in such a way that it becomes exactly nilpotent: $s^2 = 0$.

In the classical limit this reduces to the mere BRS invariance of the action:

$$s\Sigma = 0 \ . \tag{4.11}$$

Let us sketch the inductive proof of (4.10). We assume it to hold up to and including the order $n-1$, for $n \geq 1$:

$$s\Gamma = O(\hbar^n) \ . \tag{4.12}$$

In order to proceed, let us use the "quantum action principle" (see e.g. [25]). This theorem states that a variation of the vertex functional – such as the one given here by the BRS operator – yields a local insertion, i.e.:

$$s\Gamma = \Delta \cdot \Gamma = \Delta + \text{loops} \ ,$$

where Δ is the space-time integral of a local field polynomial. In the second equality we have separated out the contributions of the trivial tree graphs consisting only of the single vertices corresponding to the field polynomial Δ, from the loop graph contributions, which are of higher order in $\hbar$. Making explicit the factor in $\hbar$ following from the assumption (4.12), we can thus write

$$s\Gamma = \hbar^n \Delta + O(\hbar^{n+1}) \ .$$

The nilpotency of s implies for Δ the *consistency condition*

$$s\Delta = 0 \ . \tag{4.13}$$

For the present theory, the most general local field polynomial satisfying this condition has the form

$$\Delta = s\hat{\Delta} \ , \tag{4.14}$$

where $\hat{\Delta}$ is an integrated local field polynomial of dimension 4, i.e. of the dimension of the action. Redefining the action by adding $-\hat{\Delta}$ as a counterterm:

$$\Sigma \longrightarrow \Sigma' = \Sigma - \hbar^n \hat{\Delta} \ , \tag{4.15}$$

leads to a modification of the vertex functional:

$$\Gamma \longrightarrow \Gamma' = \Gamma - \hbar^n \hat{\Delta} + O(\hbar^{n+1}) \ ,$$

such that the Slavnov-Taylor identity now holds to the next order:

$$s\Gamma = O(\hbar^{n+1}) \ . \tag{4.16}$$

This ends the inductive proof of the renormalizability of the pure super-Yang-Mills theory.

Remark. The proof given here has been very sketchy. Beyond the fact that we have not written the Slavnov-Taylor in a proper way, using external fields coupled to the BRS variations of the quantum fields, we have not checked the stability of the gauge fixing, which turns out to hold unchanged to all orders. A complete demonstration may be found in [20].

Anomaly:

The result above holds for the theory without matter (described by chiral supermultiplets). In presence of matter, the right-hand side of (4.14) may contain a supplementary term which is not the BRS variation of an integrated local field polynomial, and thus cannot be reabsorbed as a counterterm in the action. This term is the *gauge anomaly*, which, if present, would definitively break BRS invariance. It has the form

$$\Delta = r\int d^4x\ \left(\varepsilon^{[\mu\nu\rho\sigma]}d_{(abc)}\left(\partial_\mu c^a - 2\lambda^a\sigma_\mu\bar\varepsilon - 2\varepsilon\sigma_\mu\bar\lambda^a\right)A^b_\nu\partial_\rho A^c_\sigma + \cdots\right)\ , \tag{4.17}$$

where $\varepsilon^{[\mu\nu\rho\sigma]}$ is the completely antisymmetric Levi-Civita tensor, and $d_{(abc)}$ a completely symmetric invariant tensor of the gauge group. The dots represent contributions of higher order in the number of fields[18].

Whereas the form of the anomaly is independent of the details of the theory, in particular of its matter content, the coefficient r depends from the latter. Its value at first order in $\hbar$ indeed is given by the calculation of the 1-loop fermion triangle graph of the Adler-Bardeen type – exactly like in usual gauge theories, and is proportional to $d_{(abc)}\mathrm{Tr}\,(T^aT^bT^c)$, where the matrices T^a are the generators of the gauge group in the matter field representation. As in the usual gauge theories also, it is possible to make the coefficient r vanish at first order by suitably choosing this representation, and also here (see [20]) a nonrenormalization theorem assures then its vanishing to all higher orders.

5 Broken Supersymmetry

As it is well known, nature is not supersymmetric, at least in the domain of energies accessible to present experiments. In particular, no boson-fermion particle doublet with degenerate mass has been observed as yet. Therefore supersymmetry must be broken. One may consider two ways of doing it, either *explicitly*, by adding e.g. nonsupersymmetric mass terms in the action, or *spontaneously*. The difficulty with the latter option is that it necessarily implies the existence of a massless Goldstone particle, which must be a fermion due to the fermionic nature of the supersymmetry generators. The only known candidates are the neutrinos. But low energy theorems – analogous to those for the pions in the framework of the chiral effective theories [29] – would suppress low energy phenomena such as β decay in an unacceptable way. One is thus left with the other term of the alternative, i.e. with the explicit supersymmetry breakdown by mass terms. I shall however also present a model with spontaneous breakdown of supersymmetry, the O'Raifeartaigh model [30] for its own theoretical interest: it shows indeed a mechanism through which a particle which is massless in the classical theory can acquire a (calculable) mass through the radiative corrections.

[18] See [20] for the derivation and the complete expression of this anomaly.

5.1 Explicit Supersymmetry Breaking

The lack of supersymmetry is most apparent in the fact that supermultiplets of particles, if they exist at all, must contain states of different masses. Indeed, the fact that no supersymmetric partner of any known particle has yet been observed implies, for these hypothetical particles, masses of the order of 10^2 GeV at least. A rather obvious way to achieve this goal is to introduce mass terms for each of them in the action, thus breaking supersymmetry explicitly.

Such breakings are soft in the sense that they don't influence the high energy-momentum behaviour of the theory, leaving in particular supersymmetry intact in the asymptotic region. But one would also like that the breakings don't hurt a property which singles out supersymmetric theories: the absence of quadratic divergences [1, 6]. This property assures the stability of the huge differences of scales which one encounters in the models of Grand Unification [24].

Girardello and Grisaru [31] have shown that a certain class of mass terms – which they also called "soft breakings" – do break supersymmetry, but preserve the absence of quadratic divergences. The "soft breakings" relevant for the supersymmetric gauge theories described in Section 3, and which may be added to the action (3.25), are listed in Table 6 together with interactions terms sharing the same properties.

Type	Breaking terms	
"Gaugino" mass	$m_\lambda \lambda^\alpha \lambda_a$	
Scalar matter field masses	$m_{ij} A^i A^j$ + conjugate ,	$m'{}_i{}^j A^i \bar{A}_j$
Scalar field interactions	$h_{ijk} A^i A^j A^k$ + conjugate	

Table 6: The possible "soft breaking" terms.

The renormalization of the super-Yang-Mills theory including all possible soft breakings has been carried out in [21], starting from the action (3.25), i.e. in the Wess-Zumino gauge.

Remark. It has been shown [24] that precisely the "soft" breaking terms listed in Table 6 appear in the low energy[19] effective theory resulting from a supergravity theory where *local* supersymmetry is spontaneously broken. There, the "soft" breaking terms are induced by a super-Higgs mechanism. Thus, although a spontaneous breakdown of rigid supersymmetry seems to be ruled out because of the apparition of a Goldstone fermion, the breakdown of local supersymmetry does not present this undesirable feature and may lead to phenomenologically interesting consequences.

[19] I.e. at energies well below of the Planck mass.

5.2 Spontaneously Broken Supersymmetry

Bibliography: [30, 32, 33, 6]

The model proposed by O'Raifeartaigh [30] consists of three chiral supermultiplets

$$A_k = \{A_k, \psi_k, F_k\} \ , \quad k = 0, 1, 2 \ , \tag{5.1}$$

and is described by the suptab-soft-brersymmetric action

$$\begin{aligned}
\Sigma &= \int d^4x \, \left(\sum_{k=0}^{2} \bar{A}_k A_k \Big|_D - \left(m \ A_1 A_2|_F + \frac{g}{8} \ A_0 {A_1}^2 \Big|_F + \lambda F_0 + \text{ conj.} \right) \right) \\
&= \int d^4x \, \left(\sum_{k=0}^{2} \left(\partial^\mu A_k \partial_\mu \bar{A}_k + \frac{i}{2} \psi_k \sigma^\mu \partial_\mu \bar{\psi}_k + F_k \bar{F}_k \right) \right. \\
&\qquad - m \left(A_1 F_2 + A_2 F_1 - \frac{1}{2} \psi_1 \psi_2 + \text{ conj.} \right) \\
&\qquad - \frac{g}{8} \left({A_1}^2 F_0 + 2 A_0 A_1 F_1 - \frac{1}{2} A_0 \psi_1 \psi_1 - A_1 \psi_0 \psi_1 + \text{ conj.} \right) \\
&\qquad - \lambda \left(F_0 + \bar{F}_0 \right) \ .
\end{aligned} \tag{5.2}$$

The three parameters m, g and λ are in general complex numbers. However they will be assumed to be real. This can indeed always be achieved through a redefinition of the three supermultiplets by multiplication with suitable phase factors. Explicit expressions for the F and D-components of the composite superfields appearing in the first line have been calculated using the rules given in Eqs. (3.8) and (3.9). Recall that the same symbol represents a supermultiplet as well as its first component.

The action (5.2) is uniquely defined by its invariances under supersymmetry, under the discrete symmetry

$$A'_0 = A_0 \ , \quad A'_1 = -A_1 \ , \quad A'_2 = -A_2 \ , \tag{5.3}$$

and under the R-symmetry (3.5), with particular weights n_k:

$$\begin{aligned}
&\delta_R A_k = i n_k A_k \ , \\
&\delta_R \psi_k = i(n_k + 1) \psi_k \ , \qquad n_0 = n_2 = -2 \ , \ n_1 = 0 \ , \\
&\delta_R F_k = i(n_k + 2) F_k \ .
\end{aligned} \tag{5.4}$$

The auxiliary fields F_k may be eliminated using their equations of motion

$$\begin{aligned}
F_0(\bar{A}) &= \lambda + \frac{g}{8} \bar{A}_1^2 \ , \\
F_1(\bar{A}) &= m \bar{A}_2 + \frac{g}{4} \bar{A}_0 \bar{A}_1 \ , \\
F_2(\bar{A}) &= m \bar{A}_1 \ ,
\end{aligned} \tag{5.5}$$

which leads to the potential term (c.f. (3.13))

$$V(A,\bar{A}) = \sum_{k=0}^{2} \left| F(\bar{A}) \right|^2 \ . \tag{5.6}$$

Let us assume that the values of the parameters of the model are lying in the range defined by

$$m^2 > \frac{1}{8}|\lambda g|$$

Then the minimum of the potential takes a positive, nonvanishing value:

$$\mathrm{Min}(V) = \lambda^2 \ , \quad \text{at:} \quad A_1 = A_2 = 0,\ A_0 = a = \ \text{arbitrary complex number.} \tag{5.7}$$

At the points which minimize the potential, defining the equilibrium state of the classical theory, the auxiliary field F_0 takes the nonvanishing value

$$F_0 = \lambda \ . \tag{5.8}$$

This characterizes a *spontaneous breakdown of the supersymmetry.* Indeed, if supersymmetry had remained intact, the equilibrium value of any supermultiplet $\{A,\psi,F\}$, which must be of the form $\{a,0,f\}$, with a and f arbitrary numbers, due to Poincaré invariance, would reduce to the value $\{a,0,0\}$, with a an arbitrary number, by consistency with the supersymmetry tranformation law (3.1) for ψ. But this would be in contradiction with the result (5.8). We thus have a nonsupersymmetric equilibrium state, although the dynamics defined by the action is symmetric: this is just the situation of a spontaneous breakdown.

In order to see the physical consequences, let us perform the shift

$$F_0 \longrightarrow F_0 + \lambda \ , \tag{5.9}$$

such that the new field variable F_0 takes the value 0 in the equilibrium state. This modifies the action, in particular, through the appearance of an additional mass term

$$-\frac{g\lambda}{8}\int d^4x \ \left({A_1}^2 + \bar{A}_1^2 \right) \ ,$$

which implies a mass splitting within the supermultiplet A_1, making thus obvious the breakdown of supersymmetry. On the other hand, the spinor field ψ_0 remains massless: it describes the Goldstone fermion associated with the spontaneous breakdown of supersymmetry – whose generator is a fermion as we know. It is precisely this field which transforms under an infinitesimal supersymmetry transformation into the field F_0 with nonvanishing equilibrium value, namely (after the shift (5.9)):

$$\delta_\alpha \psi_0^\beta = 2\delta_\alpha^\beta (F_0 + \lambda) \ .$$

It is this fact which characterizes ψ_0 as the Goldstone fermion field.

Renormalization and radiative mass generation [33, 6]:

Using the scheme that we have already applied to the renormalization of the super-Yang-Mills theories, in Section 4, let us first write the BRS algebra corresponding to the supersymmetry transformations and to the translations – leaving aside the R-transformations, which do not present any particular problem [6]:

$$\begin{aligned}
&sA_k = \varepsilon^\alpha \psi_{0\alpha} - i\xi^\mu \partial_\mu A_k, \\
&s\psi_{k\alpha} = -2\varepsilon_\alpha \left(F_k + \lambda\delta_{0k}\right) + 2i\partial_\mu A_k \sigma^\mu_{\alpha\dot\alpha}\bar\varepsilon^{\dot\alpha} - i\xi^\mu\partial_\mu\psi_{k\alpha}, \\
&sF_k = i\partial_\mu \psi^\alpha_k \sigma^\mu_{\alpha\dot\alpha}\bar\varepsilon^{\dot\alpha} - i\xi^\mu\partial_\mu F_k \ , \\
&s\varepsilon^\alpha = s\bar\varepsilon^{\dot\alpha} = 0 \ , \quad s\xi^m = -2\varepsilon^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\varepsilon^{\dot\alpha} \ , \\
&k = 0, 1, 2 \ ,
\end{aligned} \tag{5.10}$$

with

$$s^2 = 0 \ .$$

We have to show if we can implement the corresponding Slavnov-Taylor identity[20]

$$\sigma\Gamma \equiv \int d^4x \ \sum_\varphi s\varphi \frac{\delta}{\delta\varphi}\Gamma + s\xi^\mu \frac{\partial}{\partial\xi^\mu} = 0 \ , \tag{5.11}$$

to all orders in $\hbar$, where

$$\Gamma = \Sigma + O(\hbar)$$

is the vertex functional or generating functional of the one-particle irreducible Green functions (see Subsection 4.2).

We assume (5.11) to hold up to order $n-1$, thus writing

$$s\Gamma = \hbar^n \Delta + O(\hbar^{n+1}) \ ,$$

where Δ is an integrated local field polynomial of dimension ≤ 4 according to the quantum action principle. The nilpotency of s again leads to the consistency condition

$$s\Delta = 0 \ .$$

One finds that its most general solution is of the trivial form

$$\Delta = s\hat\Delta \ , \tag{5.12}$$

where $\hat\Delta$ has the dimension and the quantum numbers of the action, is R-invariant and could thus be absorbed as a counterterm, thus enforcing the Slavnov-Taylor identity to the next order. (C.f. Eqs. (4.15)–(4.16))

[20]The transformations being linear, here, we can write explicitly the form of the functional operator s without introducing external fields.

However, although this works well with almost all the terms which may contribute to $\hat{\Delta}$, there is one single term which poses a serious problem, namely

$$\hat{\Delta} = \mu^2 \int d^4x \; A_0 \bar{A}_0 \; , \tag{5.13}$$

where μ^2 is a coefficient calculable as a function of the parameters of the theory. It turns out to be nonvanishing already in the approximation of the one-loop graphs:

$$\mu^2 = \hbar\kappa \frac{g^4\lambda^2}{m^2} \; , \tag{5.14}$$

where κ is some numerical constant [32, 6], and where we have put explicitly the factor $\hbar$ in order to emphasize the quantum character of the breaking. As we already did in the case of the supersymmetric gauge theories for breakings of the form (5.12), we could try to absorb the expression $\hat{\Delta}$ as a counterterm, thus restablishing the Slavnov-Taylor identity. But we cannot do so as naively, since the insertion of this counterterm – which is bilinear in the massless field A_0 – would cut the propagator of A_0

$$\Delta_{A_0\bar{A}_0}(k) = \frac{i}{k^2} \tag{5.15}$$

in two pieces, thus giving rise to a nonintegrable infrared – i.e. at zero-momentum – singularity $1/(k^2)^2$. It seems therefore that, not being able to absorb $\hat{\Delta}$, we are confronted with the impossibility of implementing the Slavnov-Taylor identity, i.e. supersymmetry, for the quantum theory. This obstruction has been named an infrared anomaly.

However, one observes[21] that (5.13) has just the form of a mass term for the field A_0 [33, 6]. This means that, although one cannot absorb it as a counterterm, one can nevertheles absorb it as a mass parameter – but proportional to $\hbar$! – in the originally massless propagator (5.15):

$$\Delta^{\mathrm{new}}_{A_0\bar{A}_0}(k) = \frac{i}{k^2 - \mu^2} \; . \tag{5.16}$$

Starting the theory with this new, massive, propagator, we arrive at a Slavnov-Taylor identity which will hold exactly in the one-loop approximation.

It is now possible to go on to the next orders, since the breaking (5.12) with (5.13), which will not keep to appear, can now be safely absorbed as a counterterm since the propagator of A_0 has become massive. There is however a pecularity, characteristic of this phenomenon of "mass generation": the denominator of the propagator containing a mass proportional to $\hbar$, terms in $\log\hbar$ will appear in the computation of the Feynman graphs. This means that the usual perturbation series, which is a series in the powers of $\hbar$, will be replaced by a series in the powers of $\hbar$ and $\log\hbar$ [35, 33, 6].

[21] See [34] for an early reference to that matter.

Appendix: Some Notations and Conventions

The notations and conventions are those of [6, 10]. A more complete set of formulas may be found in the first appendix of [10].

Weyl Spinors and Pauli Matrices

Units: $\hbar = c = 1$

Space-time metric: $(g_{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1)$, $(\mu, \nu, \cdots = 0, 1, 2, 3)$

Fourier transform:

$$f(x) = \frac{1}{2\pi}\int dk\ e^{ikx}\tilde{f}(k) \ , \quad \tilde{f}(k) = \int dx\ e^{-ikx} f(k) \ , \qquad (\ \partial_\mu \leftrightarrow ik_\mu\ ) \ .$$

Weyl spinor: $\{\psi_\alpha\ ,\ \alpha = 1, 2\}\ \in$ repr. $(\frac{1}{2}, 0)$ of the Lorentz group.
The spinor components are Grassmann variables: $\psi_\alpha \psi'_\beta = -\psi'_\beta \psi_\alpha$

Complex conjugate spinor: $\left\{\bar{\psi}_{\dot{\alpha}} = (\psi_\alpha)^*,\ \dot{\alpha} = 1, 2\right\}\ \in$ repr. $(0, \frac{1}{2})$

Raising and lowering of spinor indices:

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta \ , \quad \psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta \ ,$$

with $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$, $\varepsilon^{12} = 1$, $\varepsilon_{\alpha\beta} = -\varepsilon^{a\beta}$, $\varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma} = \delta^\alpha_\gamma$,

(the same for dotted indices).

Derivative with respect to a spinor component:

$$\frac{\partial}{\partial\psi^\alpha}\psi^\beta = \delta^\beta_\alpha \ , \quad \frac{\partial}{\partial\psi_\alpha} = \varepsilon^{\alpha\beta}\frac{\partial}{\partial\psi^\beta} \ ,$$

(the same for dotted indices)

Pauli matrices:

$$
\begin{aligned}
&\left(\sigma^{\mu}_{\alpha\dot\beta}\right) = \left(\sigma^0_{\alpha\dot\beta},\, \sigma^1_{\alpha\dot\beta},\, \sigma^2_{\alpha\dot\beta},\, \sigma^3_{\alpha\dot\beta}\right)\\
&\bar\sigma^{\dot\alpha\beta}_{\mu} = \sigma^{\beta\dot\alpha}_{\mu} = \varepsilon^{\beta\alpha}\varepsilon^{\dot\alpha\dot\beta}\sigma_{\mu\,\alpha\dot\beta}\ ,\\
&(\sigma^{\mu\nu})_{\alpha}{}^{\beta} = \frac{i}{2}[\sigma^{\mu}\bar\sigma^{\nu} - \sigma^{\nu}\bar\sigma^{\mu}]_{\alpha}{}^{\beta}\ , \quad (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} = \frac{i}{2}[\bar\sigma^{\mu}\sigma^{\nu} - \bar\sigma^{\nu}\sigma^{\mu}]^{\dot\alpha}{}_{\dot\beta}\ ,
\end{aligned}
$$

with

$$
\begin{aligned}
&\sigma^0 = \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\ , \quad \sigma^1 = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\ , \quad \sigma^2 = \begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix}\ , \quad \sigma^3 = \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}\ ,\\
&\bar\sigma^0 = \sigma^0\ , \quad \bar\sigma^i = -\sigma^i = \sigma_i\ , \quad \sigma^{0i} = -\bar\sigma^{0i} = -i\sigma^i\ , \quad \sigma^{ij} = \bar\sigma^{ij} = \varepsilon^{ijk}\sigma^k\ ,\\
&\quad i,j,k = 1,2,3\ .
\end{aligned}
$$

Summation conventions and complex conjugation: Let ψ and χ be two Weyl spinors.

$$
\begin{aligned}
&\psi\chi = \psi^{\alpha}\chi_{\alpha} = -\chi_{\alpha}\psi^{\alpha} = \chi^{\alpha}\psi_{\alpha} = \chi\psi\ ,\\
&\bar\psi\bar\chi = \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} = -\bar\chi^{\dot\alpha}\bar\psi_{\dot\alpha} = \bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha} = \bar\chi\bar\psi\ ,\\
&\psi\sigma^{\mu}\bar\chi = \psi^{\alpha}\sigma^{\mu}_{\alpha\dot\alpha}\bar\chi^{\dot\alpha}\ , \quad \bar\psi\bar\sigma_{\mu}\chi = \bar\psi_{\dot\alpha}\bar\sigma^{\dot\alpha\alpha}_{\mu}\chi_{\alpha}\ ,\\
&(\psi\chi)^* = \bar\chi\bar\psi = \bar\psi\bar\chi\ ,\\
&(\psi\sigma^{\mu}\bar\chi)^* = \chi\sigma^{\mu}\bar\psi = -\bar\psi\bar\sigma^{\mu}\chi\ ,\\
&(\psi\sigma^{\mu\nu}\chi)^* = \bar\chi\bar\sigma^{\mu\nu}\bar\psi\ .
\end{aligned}
$$

Infinitesimal Lorentz transformations of the Weyl spinors:

$$
\begin{aligned}
&\text{if}\quad \delta^{\mathrm{L}}x^{\mu} = \omega^{\mu}{}_{\nu}x^{\nu}\ , \quad \text{with } \omega^{\mu\nu} = -\omega^{\nu\mu} \qquad (\omega^{\mu\nu} = g^{\nu\rho}\omega^{\mu}{}_{\rho})\ , \quad \text{then:}\\
&\delta^{\mathrm{L}}\psi_{\alpha}(x) = \frac{1}{2}\omega^{\mu\nu}\left((x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})\psi_{\alpha}(x) - \frac{i}{2}(\sigma_{\mu\nu})_{\alpha}{}^{\beta}\psi_{\beta}\right)\ ,\\
&\delta^{\mathrm{L}}\bar\psi^{\dot\alpha}(x) = \frac{1}{2}\omega^{\mu\nu}\left((x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})\bar\psi^{\dot\alpha}(x) + \frac{i}{2}(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\bar\psi^{\dot\beta}\right)\ .
\end{aligned}
$$

	θ^{α}	D_{α}	ϕ	A	c_+	c_-	B	ϕ^*	A^*	c_+^*
d	$-\frac{1}{2}$	$\frac{1}{2}$	0	1	0	1	1	2	2	3
n	-1	1	0	$-\frac{2}{3}$	0	-2	-2	0	$-\frac{4}{3}$	-2
$\Phi\Pi$	0	0	0	0	1	-1	0	-1	-1	-2

Table 7: Dimensions d, R-weights n and ghost numbers $\Phi\Pi$.

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