LECTURES ON INVARIANT THEORY

IGOR V. DOLGACHEV

February 12, 2003

Preface

This book is based on one-semester graduate courses I gave at Michigan in 1994 and 1998, and at Harvard in 1999. A part of the book is borrowed from an earlier version of my lecture notes which were published by the Seoul National University [22]. The main changes consist of including several chapters on algebraic invariant theory, simplifying and correcting proofs, and adding more examples from classical algebraic geometry. The last Lecture of [22] which contains some applications to construction of moduli spaces has been omitted. The book is literally intended to be a first course in the subject to motivate a beginner to study more. A new edition of D. Mumford's book *Geometric Invariant Theory* with appendices by J. Fogarty and F. Kirwan [75] as well as a survey article of V. Popov and E. Vinberg [91] will help the reader to navigate in this broad and old subject of mathematics. Most of the results and their proofs discussed in the present book can be found in the literature. We include some of the extensive bibliography of the subject (with no claim for completeness). The main purpose of this book is to give a short and self-contained exposition of the main ideas of the theory. The sole novelty is including many examples illustrating the dependence of the quotient on a linearization of the action as well as including some basic constructions in toric geometry as examples of torus actions on affine space. We also give many examples related to classical algebraic geometry. Each chapter ends with a set of exercises and bibliographical notes. We assume only minimal prerequisites for students: a basic knowledge of algebraic geometry covered in the first two chapters of Shafarevich's book [104] and/or Hartshorne's book [46], a good knowledge of multilinear algebra and some rudiments of the theory of linear representations of groups. Although we often use some of the theory of affine algebraic groups, the knowledge of the group GL_n is enough for our purpose.

I am grateful to some of my students and colleagues for critical remarks and catching numerous mistakes in my lecture notes. Special thanks go to Ana-Maria Castravet, Mihnea Popa, Janis Stipins and Ivan Arzhantsev.

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Introduction

Geometric invariant theory arises in an attempt to construct a quotient of an algebraic variety X by an algebraic action of a linear algebraic group G. In many applications X is the parametrizing space of certain geometric objects (algebraic curves, vector bundles, etc.) and the equivalence relation on the objects is defined by a group action. The main problem here is that the quotient space X/G may not exist in the category of algebraic varieties. The reason is rather simple. Since one expects that the canonical projection $f : X \to X/G$ is a regular map of algebraic varieties and so has closed fibres, all orbits must be closed subsets in the Zariski topology of X. This rarely happens when G is not a finite group. A possible solution to this problem is to restrict the action to an invariant open Zariski subset U, as large as possible, so that $U \to U/G$ exists. The geometric invariant theory (GIT) suggests a method for choosing such a set so that the quotient is a quasi-projective algebraic variety. The idea goes back to David Hilbert. Suppose $X = V$ is a linear space and G is a linear algebraic group acting on V via its linear representation. The set of polynomial functions on V invariant with respect to this action is a commutative algebra A over the ground field. Hilbert proves that A is finitely generated if $G = \mathrm{SL}_n$ or GL_n and any set of generators $f_1, \ldots, f_N$ of A defines an invariant regular map from X to some affine algebraic variety Y contained in affine space $\mathbb{A}^N$ whose ring of polynomial functions is isomorphic to A. By a theorem of Nagata the same is true for any reductive linear algebraic group. The map $f : X \to Y$ has a universal property for G-invariant maps of X and is called the categorical quotient. The inverse image of the origin is the closed subvariety defined by all invariant homogeneous polynomials of positive degree. It is called the null-cone. Its points cannot be distinguished by invariant functions; they are called unstable points. The remaining points are called semi-stable points. When we pass to the projective space $\mathbb{P}(V)$ associated to V, the images of semi-stable points form an invariant open subset $\mathbb{P}(V)^{\mathrm{ss}}$ and the map f induces a regular map $\bar{f} : \mathbb{P}(V)^{\mathrm{ss}} \to \bar{Y}$, where $\bar{Y}$ (denoted by $\mathbb{P}(V)^{\mathrm{ss}}/\!\!/G$) is a projective algebraic variety with the projective coordinate algebra isomorphic to A. In applications considered by Hilbert, $\mathbb{P}(V)$ parametrizes projective hypersurfaces of certain degree and dimension, and the projective algebraic variety $\bar{Y}$ is the "moduli space" of these hypersurfaces. The hypersurfaces represented by unstable points are left out from the moduli space; they are "too degenerate". A nonsingular hypersurface is always represented by a semi-stable point. Since $\bar{Y}$ is a projective variety, it is considered as a "compactification" of the moduli space of nonsingular hypersurfaces. The fibres of the map $\mathbb{P}(V)^{\mathrm{ss}} \to \mathbb{P}(V)^{\mathrm{ss}}/\!\!/G$ are not orbits in general; however, each fibre contains a unique closed orbit so that $\mathbb{P}(V)^{\mathrm{ss}}/\!\!/G$ parametrizes closed orbits in the set of semi-stable points.

Since the equations of the null-cone are hard to find without computing explicitly the ring of invariant polynomials, one uses another approach. This approach is to describe the set of semi-stable points by using the Hilbert–Mumford numerical criterion of stability. In many cases it allows one to determine the set $\mathbb{P}(V)^{\mathrm{ss}}$ very explicitly and to distinguish stable points among semi-stable ones. These are the points whose orbits are closed in $\mathbb{P}(V)^{\mathrm{ss}}$ and whose stabilizer subgroups are finite. The restriction of the map $\mathbb{P}(V)^{\mathrm{ss}} \to \mathbb{P}(V)^{\mathrm{ss}}/\!\!/G$ to the set of stable points $\mathbb{P}(V)^{\mathrm{s}}$ is an orbit map $\mathbb{P}(V)^{\mathrm{s}} \to \mathbb{P}(V)^{\mathrm{s}}/G$. It is called a geometric quotient.

More generally, if G is a reductive algebraic group acting on a projective algebraic variety X, the GIT approach to constructing the quotient consists of the following steps. First one chooses a linearization of the action, a G-equivariant embedding of X into a projective space $\mathbb{P}(V)$ with a linear action of G as above. The choice of a linearization is a parameter of the construction; it is defined by a G-linearized ample line bundle on X. Then one sets $X^{\mathrm{ss}} = X \cap \mathbb{P}(V)^{\mathrm{ss}}$ and defines the categorical quotient $X^{\mathrm{ss}} \to X^{\mathrm{ss}}/\!\!/G$ as the restriction of the categorical quotient $\mathbb{P}(V)^{\mathrm{ss}} \to \mathbb{P}(V)^{\mathrm{ss}}/\!\!/G$. The image variety $X^{\mathrm{ss}}/\!\!/G$ is a closed subvariety of $\mathbb{P}(V)^{\mathrm{ss}}/\!\!/G$.

Let us give a brief comment on the content of the book.

In Chapters 1 and 2 we consider the classical example of invariant theory in which the general linear group $\mathrm{GL}(V)$ of a vector space V of dimension n over a field k acts naturally on the space of homogeneneous polynomials $\mathrm{Pol}_d(V)$ of some degree d. We explain the classical symbolic method which allows one to identify an invariant polynomial function of degree m on this space with an element of the projective coordinate algebra $k[\mathrm{Gr}(n,m)]$ on the Grassmann variety $\mathrm{Gr}(n,m)$ of n-dimensional linear subspaces in k^m in its Plücker embedding. This interpretation is based on the First Fundamental Theorem of Invariant Theory. The proof of this theorem uses a rather technical algebraic tool, the so-called Clebsch omega-operator. We choose this less conceptual approach to show the flavor of the invariant theory of the nineteenth century. More detailed expositions of the classical invariant theory ([65], [123]) give a conceptual explanation of this operator via representation theory. The Second Fundamental Theorem of Invariant Theory is just a statement about the relations between the Plücker coordinates known in algebraic geometry as the Plücker equations. We use the available computations of invariants in later chapters to give an explicit description of some of the GIT quotients arising in classical algebraic geometry.

In Chapter 3 we discuss the problem of finite generatedness of the algebra of invariant polynomials on the space of a linear rational representation of an algebraic group. We begin with the Gordan–Hilbert theorem and explain the "unitary trick" due to Adolf Hurwitz and Hermann Weyl which allows one to prove the finite generatedness in the case of a semisimple or, more generally, reductive complex algebraic group. Then we introduce the notion of a geometrically reductive algebraic group and prove Nagata's theorem on finite generatedness of the algebra of invariant polynomials on the space of a linear rational representation of a reductive algebraic group.

In Chapter 4 we discuss the case of a linear rational representation of a nonreductive algebraic group. We prove a lemma due to Grosshans which allows one to prove finite generatedness for the restriction of a representation of a reductive algebraic group G to a subgroup H provided the algebra of regular functions on the homogeneous space G/H is finitely generated. A corollary of this result is a classical theorem of Weitzenböck about invariants of the additive group. The central part of this chapter is Nagata's counterexample to Hilbert's Fourteenth Problem. It asks about finite generatedness of the algebra of invariants for an arbitrary algebraic group of linear transformations. We follow the original construction of Nagata with some simplifications due to R. Steinberg.

Chapter 5 is devoted to covariants of an action. A covariant of an affine algebraic group G acting on an algebraic variety X is a G-equivariant regular map from X to an affine space on which the group acts via its linear representation. The covariants form an algebra and the main result of the theory is that this algebra is finitely generated if G is reductive. The proof depends heavily on the theory of linear representations of reductive algebraic groups which we review in this chapter. As an application of this theory we prove the classical Cayley-Sylvester formula for the dimension of the spaces of covariants and also the Hermite reciprocity.

In Chapter 6 we discuss categorical and geometric quotients of an algebraic variety under a regular action of an algebraic group. The material is fairly standard and follows Mumford's book.

Chapter 7 is devoted to linearizations of actions. The main result is that any algebraic action of a linear algebraic group on a normal quasi-projective algebraic variety X is isomorphic to the restriction of a linear action on a projective space in which X is equivariantly embedded. The proof follows the exposition of the theory of linearizations from [66].

Chapter 8 is devoted to the concept of stability of algebraic actions and the construction of categorical and geometric quotients. The material of this chapter is rather standard and can be found in Mumford's book as well as in many other books. We include many examples illustrating the dependence of the quotients on the linearization.

Chapter 9 contains the proof of Hilbert–Mumford's numerical criterion of stability. The only novelty here is that we also include Kempf's notion of stability and give an example of its application to the theory of moduli of abelian varieties.

The remaining Chapters 10–12 are devoted to some examples where the complete description of stable points is available. In Chapter 10 we discuss the case of hypersurfaces in projective space. We give explicit descriptions of the moduli spaces of binary forms of degree ≤ 5, plane curves of degree 3 and cubic surfaces. In Chapter 11 we discuss moduli spaces of ordered collections of linear subspaces in projective space, in particular of points in $\mathbb{P}^n$ or of lines in $\mathbb{P}^3$. The examples discussed in this chapter are related to some of the beautiful constructions of classical algebraic geometry. In Chapter 12 we introduce toric varieties as GIT quotients of an open subset of affine space. Some of the constructions discussed in the preceding chapters admit a nice interpretation in terms of the geometry of toric varieties. This approach to toric varieties is based on some recent work of D. Cox ([16]) and M. Audin ([3]).

We will be working over an algebraically closed field k sometimes assumed to be of characteristic zero.

Chapter 1

The symbolic method

1.1 First examples

The notion of an invariant is one of the most general concepts of mathematics. Whenever a group G acts on a set S we look for elements $s \in S$ which do not change under the action, i.e., which satisfy $g \cdot s = s$ for any $g \in G$. For example, if S is a set of functions from a set X to a set Y, and G acts on S via its action on X and its action on Y by the formula

$$(g \cdot f)(x) = g \cdot f(g^{-1} \cdot x),$$

then an *equivariant function* is a function $f : X \to Y$ satisfying $g \cdot f = f$, i.e.,

$$f(g \cdot x) = g \cdot f(x), \quad \forall g \in G, \forall x \in X.$$

In the case when G acts trivially on Y, an equivariant function is called an *invariant function*. It satisfies

$$f(g \cdot x) = f(x), \quad \forall g \in G, \forall x \in X.$$

Among all invariant functions there exists a universal function, the projection map $p : X \to X/G$ from the set X to the set of orbits X/G. It satisfies the property that for any invariant function $f : X \to Y$ there exists a unique map $\bar{f} : X/G \to Y$ such that $f = \bar{f} \circ p$. So if we know the set of orbits X/G, we know all invariant functions on X. We will be concerned with invariants arising in algebra and algebraic geometry. Our sets and our group G will be algebraic varieties and our invariant functions will be regular maps.

Let us start with some examples.

Example 1.1. Let A be a finitely generated algebra over a field k and let G be a group of its automorphisms. The subset

$$A^G = \{a \in A : g(a) = a, \forall g \in G\} \tag{1.1}$$

is a k-subalgebra of A. It is called the *algebra of invariants*. This definition fits the general setting if we let $X = \mathrm{Spm}(A)$ be the affine algebraic variety over k with coordinate ring equal to A, and let $Y = \mathbb{A}^1_k$ be the affine line over k. Then elements of A can be viewed as regular functions $a : X \to \mathbb{A}^1_k$ between algebraic varieties. A more general invariant function is an invariant map $f : X \to Y$ between algebraic varieties. If Y is affine with coordinate ring B, such a map is defined by a homomorphism of k-algebras $f^* : B \to A$ satisfying $g(f^*(b)) = f^*(b)$ for any $g \in G, b \in B$. It is clear that such a homomorphism is equal to the composition of a homomorphism $B \to A^G$ and the natural inclusion map $A^G \to A$. Thus if we take $Z = \mathrm{Spm}(A^G)$ we obtain that the map $X \to Z$ defined by the inclusion $A^G \hookrightarrow A$ plays the role of the universal function. So it is natural to assume that A^G is the coordinate ring of the orbit space X/G. However, we shall quickly convince ourselves that there must be some problems here. The first one is that the algebra A^G may not be finitely generated over k and so does not define an algebraic variety. This problem can be easily resolved by extending the category of algebraic varieties to the category of schemes. For any (not necessarily finitely generated) algebra A over k, we may still consider the *subring of invariants* A^G and view any homomorphism of rings $B \to A$ as a morphism of affine schemes $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$. Then the morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(A^G)$ is the universal invariant function. However, it is preferable to deal with algebraic varieties rather than to deal with arbitrary schemes, and we will later show that A^G is always finitely generated if the group G is a reductive algebraic group which acts algebraically on $\mathrm{Spm}(A)$. The second problem is more serious. The affine algebraic variety $\mathrm{Spm}(A^G)$ rarely coincides with the set of orbits (unless G is a finite group). For example, the standard action of the general linear group $\mathrm{GL}_n(k)$ on the space k^n has two orbits but no invariant nonconstant functions.

The following is a more interesting example.

Example 1.2. Let $G = \mathrm{GL}_n(k)$ act by automorphisms on the polynomial algebra $A = k[X_{11}, \ldots, X_{nn}]$ in n^2 variables $X_{ij}, i, j = 1, \ldots, n$, as follows. For any $g = (a_{ij}) \in G$ the polynomial $g(X_{ij})$ is equal to the ijth entry of the matrix

$$Y = g^{-1} \cdot X \cdot g, \tag{1.2}$$

where $X = (X_{ij})$ is the matrix with the entries X_{ij}. Then, the affine variety $\mathrm{Spm}(A)$ is the affine space Mat_n of dimension n^2. Its k-points can be interpreted as $n \times n$ matrices with entries in k and we can view elements of A as polynomial functions on the space of matrices. We know from linear algebra that any such matrix can be reduced to its Jordan form by means of a transformation (1.2) for an appropriate g. Thus any invariant function is uniquely determined by its values on Jordan matrices. Let D be the subspace of diagonal matrices identified with linear space k^n and let $k[\Lambda_1, \ldots, \Lambda_n]$ be the algebra of polynomial functions on D. Since the set of matrices with diagonal Jordan form is a Zariski dense subset in the set of all matrices, we see that an invariant function is uniquely determined by its values on diagonal matrices. Therefore the restriction homomorphism $A^G \to k[\Lambda_1, \ldots, \Lambda_n]$ is injective. Since two diagonal matrices with permuted diagonal entries are equivalent, an invariant function must be a symmetric polynomial in Λ_i. By the Fundamental Theorem on Symmetric Functions, such a function can be written uniquely as a polynomial in elementary symmetric functions s_i in the variables $\Lambda_1, \ldots, \Lambda_n$. On the other hand, let c_i be the coefficients of the characteristic polynomial

$$\det(X - tI_n) = (-1)^n t^n + c_1(-t)^{n-1} + \cdots + c_n$$

considered as polynomial functions on Mat_n, i.e., elements of the ring A. Clearly, the restriction of c_i to D is equal to the ith elementary symmetric function s_i. So we see that the image of A^G in $k[\Lambda_1, \ldots, \Lambda_n]$ coincides with the polynomial subalgebra $k[s_1, \ldots, s_n]$. This implies that A^G is freely generated by the functions c_i. So we can identify $\mathrm{Spm}(A^G)$ with affine space k^n. Now consider the universal map $\mathrm{Spm}(A) \to \mathrm{Spm}(A^G)$. Its fibre over the point $(0, \ldots, 0)$ defined by the maximal ideal $(c_1, \ldots, c_n)$ is equal to the set of matrices M with characteristic polynomial $\det(M - tI_n) = (-t)^n$. Clearly, this set does not consist of one orbit, any Jordan matrix with zero diagonal values belongs to this set. Thus $\mathrm{Spm}(A^G)$ is not the orbit set $\mathrm{Spm}(A)/G$.

We shall discuss later how to remedy the problem of the construction of the space of orbits in the category of algebraic varieties. This is the subject of the geometric invariant theory (GIT) with which we will be dealing later. Now we shall discuss some examples where the algebra of invariants can be found explicitly.

Let E be a finite-dimensional vector space over a field k and let

$$\rho : G \to \mathrm{GL}(E)$$

be a linear representation of a group G in E. We consider the associated action of G on the space $\mathrm{Pol}_m(E)$ of degree m homogeneous polynomial functions on E. This action is obviously linear. The value of $f \in \mathrm{Pol}_m(E)$ at a vector v is given, in terms of the coordinates $(t_1, \dots, t_r)$ of v with respect to some basis $(\xi_1, \dots, \xi_r)$, by the following expression:

$$f(t_1, \dots, t_r) = \sum_{\substack{i_1, \dots, i_r \geq 0 \\ i_1 + \dots + i_r = m}} a_{i_1 \dots i_r} t_1^{i_1} \cdots t_r^{i_r},$$

or in the vector notation,

$$f(\mathbf{t}) = \sum_{\substack{\mathbf{i} \in \mathbb{Z}^r_{\geq 0} \\ |\mathbf{i}| = m}} a_{\mathbf{i}} \mathbf{t}^{\mathbf{i}}. \tag{1.3}$$

The direct sum of the vector spaces $\text{Pol}_m(E)$ is equal to the graded algebra of polynomial functions $\text{Pol}(E)$. Since k is infinite (we assumed it to be algebraically closed), $\text{Pol}(E)$ is isomorphic to the polynomial algebra $k[T_1, \dots, T_r]$. In more sophisticated language, $\text{Pol}_m(E)$ is naturally isomorphic to the mth symmetric product $S^m(E^*)$ of the dual vector space E^* and $\text{Pol}(E)$ is isomorphic to the symmetric algebra $S(E^*)$.

We will consider the case when $E = \text{Pol}_d(V)$ and $G = \text{SL}(V)$ be the special linear group with its linear action on E described above. Let $A = \text{Pol}(\text{Pol}_d(V))$. We can take for coordinates on the space $\text{Pol}_d(V)$ the functions $A_{\mathbf{i}}$ which assign to a homogeneous form (1.3) its coefficient $a_{\mathbf{i}}$. So any element from A is a polynomial in the $A_{\mathbf{i}}$. We want to describe the subalgebra of invariants A^G.

The problem of finding A^G is almost two centuries old. Many famous mathematicians of the nineteenth century made a contribution to this problem. Complete results, however, were obtained only in a few cases. The most complete results are known in the case $\dim V = 2$, the case where E consists of *binary forms* of degree d. We write a binary form as

$$p(t_0, t_1) = a_0 t_0^d + a_1 t_0^{d-1} t_1 + \cdots + a_d t_1^d.$$

In this case we have $d + 1$ coefficients, and hence elements of A are polynomials $P(A_0, \dots, A_d)$ in $d + 1$ variables.

1.2 Polarization and restitution

To describe the ring $\text{Pol}(\text{Pol}_d(V))^{\text{SL}(V)}$ one uses the symbolic expression of a polynomial, which we now explain. We assume that $\text{char}(k) = 0$.

A homogeneous polynomial of degree 2 on a vector space E is a quadratic form. Recall its coordinate-free definition: a map $Q : E \to k$ is a quadratic form if the following two properties are satisfied:

(i) $Q(tv) = t^2 Q(v)$, for any $v \in E$ and any $t \in k$;

(ii) the map $\tilde{Q} : E \times E \to k$ defined by the formula

$$\tilde{Q}(v, w) = Q(v + w) - Q(v) - Q(w)$$

is bilinear.

A homogeneous polynomial $P \in \mathrm{Pol}_m(E)$ of degree m can be defined in a similar way by the following properties:

(i) $P(tv) = t^m P(v)$, for any $v \in E$ and any $t \in k$;

(ii) the map $\mathrm{pol}(P) : E^m \to k$ defined by the formula

$$\mathrm{pol}(P)(v_1, \dots, v_m) = \sum_{I \subset [m]} (-1)^{m - \# I} P\Big(\sum_{i \in I} v_i\Big)$$

is multilinear.

Here and throughout we use $[m]$ to denote the set $\{1, \dots, m\}$.

As in the case of quadratic forms, we immediately see that the map $\mathrm{pol}(P)$ is a symmetric multilinear form and also that P can be reconstructed from $\mathrm{pol}(P)$ by the formula

$$m! P(v) = \mathrm{pol}(P)(v, \dots, v).$$

The symmetric multilinear form $\mathrm{pol}(P)$ is called the *polarization* of P. For any symmetric multilinear from $F : E^m \to k$ the function $\mathrm{res}(F) : E \to k$ defined by

$$\mathrm{res}(F)(v) = F(v, \dots, v)$$

is called the *restitution* of F. It is immediately checked that $\mathrm{res}(F) \in \mathrm{Pol}_m(V)$ and

$$\mathrm{pol}(\mathrm{res}(F)) = m! F.$$

Since we assumed that $\mathrm{char}(k) = 0$, we obtain that each $P \in \mathrm{Pol}_m(E)$ is equal to the restitution of a unique symmetric m-multilinear form, namely $\frac{1}{m!}\mathrm{pol}(P)$.

Assume that P is equal to the product of linear forms $P = L_1 \cdots L_m$. We have

$$\mathrm{pol}(P)(v_1, \dots, v_m) = \sum_{I \subset [m]} (-1)^{m - \# I} L_1 \cdots L_m\Big(\sum_{i \in I} v_i\Big)$$

$$\begin{aligned}
&= \sum_{I\subset[m]} (-1)^{m-\#I} L_1\Big(\sum_{i\in I} v_i\Big)\cdots L_m\Big(\sum_{i\in I} v_i\Big) \qquad (1.4)\\
&= \sum_{I\subset[m]} (-1)^{m-\#I} \Big(\sum_{i\in I} L_1(v_i)\Big)\cdots \Big(\sum_{i\in I} L_m(v_i)\Big)\\
&= \sum_{\sigma\in\Sigma_m} L_1(v_{\sigma(1)}\cdots L_m(v_{\sigma(m)}) = \sum_{\sigma\in\Sigma_m} L_{\sigma(1)}(v_1)\cdots L_{\sigma(m)}(v_m)
\end{aligned}$$

Here Σ_m denotes the permutation group on m letters.

Let $(\xi_1,\ldots,\xi_n)$ be a basis of E and $(t_1,\ldots,t_n)$ be the dual basis of E^*. Any $v\in E$ can be written in a unique way as $v=\sum_{i=1}^n t_i(v)\xi_i$. Let $\mathrm{Sym}_m(E)$ be the vector space of symmetric m-multilinear forms on E^m. For any $v_1,\ldots,v_m\in E$ and any $F\in \mathrm{Sym}_m(E)$, we have

$$\begin{aligned}
F(v_1,\ldots,v_m) &= F\Big(\sum_{i=1}^n t_i(v_1)\xi_i,\ldots,\sum_{i=1}^n t_i(v_m)\xi_i\Big)\\
&= \sum_{i_1,\ldots,i_m=1}^n t_{i_1}(v_1)\cdots t_{i_m}(v_m)F(\xi_{i_1},\ldots,\xi_{i_m}).
\end{aligned}$$

Taking $v_1=\cdots=v_m=v$, we obtain that

$$\begin{aligned}
\mathrm{res}(F)(v) &= \sum_{i_1,\ldots,i_m=1}^n t_{i_1}(v)\cdots t_{i_m}(v)F(\xi_{i_1},\ldots,\xi_{i_m})\\
&= \Big(\sum_{i_1,\ldots,i_m=1}^n a_{i_1\ldots i_m} t_{i_1}\cdots t_{i_m}\Big)(v).
\end{aligned}$$

Thus any polynomial $P\in \mathrm{Pol}_m(E)$ can be written uniquely as a sum of monomials $t_{i_1}\cdots t_{i_m}$. This is the coordinate-dependent definition of a homogeneous polynomial. Since the *polarization map*

$$\mathrm{pol}:\mathrm{Pol}_m(E)\to \mathrm{Sym}_m(E)$$

is obviously linear, we obtain that $\mathrm{Sym}_m(E)$ has a basis formed by the polarizations of monomials $t_{i_1}\cdots t_{i_m}$. Applying (1.4), we have

$$\mathrm{pol}(t_{i_1}\cdots t_{i_m})(v_1,\ldots,v_m) = \sum_{\sigma\in\Sigma_m} t_{\sigma(1)}(v_1)\cdots t_{\sigma(m)}(v_m).$$

If we denote by $(t_1^{(j)}, \ldots, t_n^{(j)})$ a jth copy of the basis $(t_1, \ldots, t_n)$ in E^*, we can rewrite the previous expression as

$$\mathrm{pol}(t_{i_1} \cdots t_{i_m})(v_1, \ldots, v_m) = \sum_{\sigma \in \Sigma_m} t_{\sigma(1)}^{(1)} \cdots t_{\sigma(m)}^{(m)}(v_1, \ldots, v_m).$$

Here, we consider the product of m linear forms on V as an m-multilinear form on E^m. We have

$$\mathrm{pol}(t_{i_1} \cdots t_{i_m})(\xi_{j_1}, \ldots, \xi_{j_m}) = \#\{\sigma \in \Sigma_m : (j_1, \ldots, j_m) = (i_{\sigma(1)}, \ldots, i_{\sigma(m)})\}. \tag{1.5}$$

If we write $t_{i_1} \cdots t_{i_m} = t_1^{k_1} \cdots t_n^{k_n}$, then the right-hand side is equal to $k_1! \cdots k_n!$ if $\{i_1, \ldots, i_m\} = \{j_1, \ldots, j_m\}$ and zero otherwise.

Note that the polarization allows us to identify $\mathrm{Pol}_m(E)$ with the dual to the space $\mathrm{Pol}_m(E^*)$. To see this, choose a basis of $\mathrm{Pol}_m(E^*)$ formed by the monomials $\xi_{i_1} \cdots \xi_{i_m}$. For any $F \in \mathrm{Sym}_m(E)$ we can set

$$F(\xi_{i_1} \cdots \xi_{i_m}) = F(\xi_{i_1}, \ldots, \xi_{i_m})$$

and then extend the domain of F to all homogeneous degree m polynomials by linearity. Applying (1.5), we get

$$\mathrm{pol}(t_1^{k_1} \cdots t_n^{k_m})(\xi_1^{l_1} \cdots \xi_n^{l_n}) = \begin{cases} k_1! \cdots k_n! & \text{if } (k_1, \ldots, k_n) = (l_1, \ldots, l_n) \\ 0 & \text{otherwise.} \end{cases}$$

This shows that the map from $\mathrm{Pol}_m(E) \times \mathrm{Pol}_m(E^*)$ to k defined by

$$(P, Q) = \frac{1}{m!}\mathrm{pol}(P)(Q) \tag{1.6}$$

is a perfect duality, i.e., it defines isomorphisms

$$\mathrm{Pol}_m(E)^* \cong \mathrm{Pol}_m(E^*), \quad \mathrm{Pol}_m(E^*)^* \cong \mathrm{Pol}_m(E). \tag{1.7}$$

Moreover, the monomial basis $(\boldsymbol{\xi}^{\mathbf{k}}) = (\xi_1^{k_1} \cdots \xi_n^{k_n})$ of $\mathrm{Pol}_m(E^*)$ is dual to the basis $(\frac{m!}{k_1! \cdots k_n!} t_1^{k_1} \cdots t_n^{k_n}) = (\frac{m!}{\mathbf{k}!}\mathbf{t}^{\mathbf{k}})$.

Remark 1.1. Note that the coefficients $a_{\mathbf{k}}$ of a polynomial

$$P = \sum_{|\mathbf{k}|=m} \frac{m!}{\mathbf{k}!} a_{\mathbf{k}} \mathbf{t}^{\mathbf{k}} \in \mathrm{Pol}_m(E) \tag{1.8}$$

are equal to the value of $A_{\mathbf{k}} = \xi^{\mathbf{k}} = \xi_1^{k_1} \cdots \xi_n^{k_n}$ on P. We can view the expression $P_{\text{general}} = \sum_{|\mathbf{k}|=m} \frac{m!}{\mathbf{k}!} A_{\mathbf{k}} \mathbf{t}^{\mathbf{k}}$ as a "general" homogeneous polynomial of degree m. Thus we get a strange formula

$$P_{\text{general}} = \sum_{|\mathbf{k}|=m} \frac{m!}{\mathbf{k}!} A_{\mathbf{k}} \mathbf{t}^{\mathbf{k}} = \sum_{|\mathbf{k}|=m} \frac{m!}{\mathbf{k}!} \xi^{\mathbf{k}} \mathbf{t}^{\mathbf{k}} = \Big(\sum_{i=1}^{n} \xi_i t_i\Big)^m.$$

This explains the classical notation of a homogeneous polynomial as a power of a linear polynomial.

Remark 1.2. One can view a basis vector ξ_i as a linear differential operator on Pol(E) which acts on linear functions by $\xi_i(t_j) = \delta_{ij}$. It acts on any polynomial $P = \sum_{\mathbf{k}} a_{\mathbf{k}} \mathbf{t}^{\mathbf{k}}$ as the partial derivative $\partial_i = \frac{\partial}{\partial t_i}$. Thus we can identify any polynomial $D(t_1, \ldots, t_n) \in$ Pol(E^*) with the differential operator $\tilde{D}(\partial_1, \ldots, \partial_n)$ by replacing the variable ξ_i with ∂_i. In this way the duality Pol$_m(E^*) \times$Pol$_m(E) \to k$ is defined by the formula

$$(D, P) = \frac{1}{m!} \tilde{D}(P).$$

Remark 1.3. For the reader with a deeper knowledge of multilinear algebra, we recall that there is a natural isomorphism between the linear space Pol$_m(E)$ and the mth symmetric power $S^m(E^*)$ of the dual space E^*. The polarization map is a linear map from $S^m(E^*)$ to $S^m(E)^*$ which is bijective when (char$(k), m!) = 1$. The universal property of tensor product allows one to identify the spaces $S^m(E)^*$ and Sym$_m(E)$.

Let us now consider the case when $E =$ Pol$_d(V)$, where $\dim V = r$.

First recall that a *multihomogeneous function* of *multi-degree* $(d_1, \ldots, d_m)$ on V is a function on V^m which is a homogeneous polynomial function of degree d_i in each variable; when each $d_i = 1$, we get the usual definition of a multilinear function. We denote the linear space of multihomogeneous functions of multi-degree $(d_1, \ldots, d_m)$ by Pol$_{d_1,\ldots,d_m}(V)$. The symmetric group Σ_m acts naturally on the space Pol$_{d,\ldots,d}(V)$ by permuting the variables. The subspace of invariant (symmetric) functions will be denoted by Sym$_{d,\ldots,d}(V)$. In particular,

$$\text{Sym}_{1,\ldots,1}(V) = \text{Sym}_m(V).$$

Lemma 1.1. *We have a natural isomorphism of linear spaces*

$$\text{symb} : \text{Pol}_m(\text{Pol}_d(V)) \to \text{Sym}_{d,\ldots,d}(V^*).$$

Proof. The polarization map defines an isomorphism

$$\mathrm{Pol}_m(\mathrm{Pol}_d(V)) \cong \mathrm{Sym}_m(\mathrm{Pol}_d(V)).$$

Using the polarization again we obtain an isomorphism $\mathrm{Pol}_d(V)^* \cong \mathrm{Pol}_d(V^*)$. Thus any linear function on $\mathrm{Pol}_d(V)$ is a homogeneous polynomial function of degree d on V^*. Thus a multilinear function on $\mathrm{Pol}_d(V)$ can be identified with a multihomogeneous function on V^* of multi-degree $(d, \dots, d)$. $\square$

Let us make the isomorphism from the preceding lemma more explicit by using a basis $(\xi_1, \dots, \xi_r)$ in V and its dual basis $(t_1, \dots, t_r)$ in V^*. Let $A_{\mathbf{k}}, |\mathbf{k}| = d$, be the coordinate functions on $\mathrm{Pol}_d(V)$, where we write each $P \in \mathrm{Pol}_d(V)$ as in (1.8) with m replaced by d, so that $A_{\mathbf{k}}(P) = a_{\mathbf{k}}$. Any $F \in \mathrm{Pol}_m(\mathrm{Pol}_d(V))$ is a polynomial expression in the $A_{\mathbf{k}}$ of degree m. Let $(A_{\mathbf{k}}^{(1)}), \dots, (A_{\mathbf{k}}^{(m)})$ be the coordinate functions in each copy of $\mathrm{Pol}_d(V)$. The polarization $\mathrm{pol}(F)$ is a multilinear expression in the $A_{\mathbf{k}}^j$. Now, if we replace $A_{\mathbf{k}}^{(j)}$ with the monomial $\boldsymbol{\xi}^{(j)\mathbf{k}}$ in a basis $(\xi_1^{(j)}, \dots, \xi_r^{(j)})$ of the jth copy of V, we obtain the *symbolic expression* of F

$$\mathrm{symb}(F)(\boldsymbol{\xi}^{(1)}, \dots, \boldsymbol{\xi}^{(m)}) \in \mathrm{Pol}_{d,\dots,d}(V^*).$$

Remark 1.4. The mathematicians of the nineteenth century did not like superscripts and preferred to use different letters for vectors in different copies of the same space. Thus they would write a general polynomial $P = \sum_{\mathbf{k}} \frac{m!}{\mathbf{k}!} A_{\mathbf{k}} \mathbf{t}^{\mathbf{k}}$ of degree d as

$$P = \Big(\sum_i \alpha_i t_i\Big)^d = \Big(\sum_i \beta_i t_i\Big)^d = \cdots,$$

and the symbolic expression of a function $F(\dots, A_{\mathbf{k}}, \dots)$ as an expression in $\alpha_i, \beta_i, \dots$.

Example 1.3. Let $r = 2, d = 2$. In this case $\mathrm{Pol}_2(V)$ consists of quadratic forms in two variables $P = a_0 x_0^2 + 2a_1 x_0 x_1 + a_2 x_1^2$. The discriminant $D = A_{20}A_{02} - A_{11}^2$ is an obvious invariant of $\mathrm{SL}_2(k)$. We have

$$\mathrm{pol}(D) = A_{20}B_{02} + A_{02}B_{20} - 2A_{11}B_{11},$$

$$\mathrm{symb}(D) = \alpha_0^2\beta_1^2 + \alpha_1^2\beta_0^2 - 2\alpha_0\alpha_1\beta_0\beta_1 = (\alpha_0\beta_1 - \alpha_1\beta_0)^2 = (\alpha, \beta)^2,$$

where

$$(\alpha, \beta) = \det \begin{pmatrix} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \end{pmatrix}.$$

Example 1.4. Let $r = 2, d = 4$. The determinant (called the *Hankel determinant*)

$$\det\begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{pmatrix}$$

in coefficients of a binary quartic

$$f = a_0x_0^4 + 4a_1x_0^3x_1 + 6a_2x_0^2x_1^2 + 4a_3x_0x_1^3 + a_4x_1^2$$

defines a function $C \in \mathrm{Pol}_3(\mathrm{Pol}_4(k^2))$ on the space of binary quartics. It is called the *catalecticant*. We leave as an exercise to verify that its symbolic expression is equal to

$$\mathrm{symb}(C) = (\alpha, \beta)^2(\alpha, \gamma)^2(\beta, \gamma)^2.$$

It is immediate to see that the group $\mathrm{GL}_2(k)$ acts on $k[a_0, \ldots, a_4]$ via its action on α, β, γ by

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}, \ldots. \tag{1.9}$$

This implies that the catalecticant is invariant with respect to the group $\mathrm{SL}_2(k)$.

1.3 Bracket functions

It is convenient to organize the variables $\xi_1^{(1)}, \ldots, \xi_r^{(1)}, \ldots, \xi_1^{(m)}, \ldots, \xi_r^{(m)}$ as a matrix of size $r \times m$:

$$A = \begin{pmatrix} \xi_1^{(1)} & \ldots & \xi_1^{(m)} \\ \vdots & \ddots & \vdots \\ \xi_r^{(1)} & \ldots & \xi_r^{(m)} \end{pmatrix}.$$

First, we identify the space $\mathrm{Pol}_{d,\ldots,d}(V^*)$ with the subspace of the polynomial algebra $k[\xi_1^{(1)}, \ldots, \xi_r^{(1)}; \ldots; \xi_1^{(m)}, \ldots, \xi_r^{(m)}]$ consisting of polynomials which are homogeneous of degree d in each set of variables $\xi_1^{(j)}, \ldots, \xi_r^{(j)}$. Next, we identify the algebra $k[\xi_1^{(1)}, \ldots, \xi_r^{(1)}; \ldots; \xi_1^{(m)}, \ldots, \xi_r^{(m)}]$ with the algebra $\mathrm{Pol}(\mathrm{Mat}_{r,m})$ of polynomial functions on the space of matrices $\mathrm{Mat}_{r,m}$. The value of a variable $\xi_i^{(j)}$ at a matrix A is the (ij)-entry of the matrix. The group $(k^*)^m$ acts naturally on the space $\mathrm{Mat}_{r,m}$ by

$$(\lambda_1, \ldots, \lambda_m) \cdot [C_1, \ldots, C_m] = [\lambda_1 C_1, \ldots, \lambda_m C_m],$$

where we write a matrix A as a collection of its columns. In a similar way the group $(k^*)^r$ acts on $\mathrm{Mat}_{r,m}$ by row multiplication. We say that a polynomial $P \in \mathrm{Pol}(\mathrm{Mat}_{r,m})$ is *multihomogeneous* of *multi-degree* $(d_1,\ldots,d_m)$ if for any $\lambda \in k^*$, and any $A = [C_1,\ldots,C_m] \in \mathrm{Mat}_{r,m}$,

$$P([C_1,\ldots,C_{j-1},\lambda C_j,C_{j+1},\ldots,C_m]) = \lambda^{d_j} P([C_1,\ldots,C_j,\ldots,C_m]).$$

We say that P is *multiisobaric* of *multi-weight* $(w_1,\ldots,w_r)$ if the polynomial function $A \to P(A^t)$ on the space $\mathrm{Mat}_{r,m}$ is multihomogeneous of multi-degree $(w_1,\ldots,w_r)$. Let $\mathrm{Pol}(\mathrm{Mat}_{r,m})_{d_1,\ldots,d_m;w_1,\ldots,w_r}$ denote the linear space of polynomial functions on the space $\mathrm{Mat}_{r,m}$ which are multihomogeneous of multi-degree $(d_1,\ldots,d_m)$ and multiisobaric of multi-weight $(w_1,\ldots,w_r)$. If $d_1 = \cdots = d_m = d$ we write $d^m = (d_1,\ldots,d_m)$; we use similar notation for the weights.

It follows from the definition that the symbolic expression of any invariant polynomial from $\mathrm{Pol}_m(\mathrm{Pol}_d(V))$ is multilinear. Let us show that it is also multiisobaric:

Proposition 1.1.

$$\mathrm{symb}(\mathrm{Pol}_m(\mathrm{Pol}_d(V))^{\mathrm{SL}(V)}) \subset \mathrm{Pol}(\mathrm{Mat}_{r,m})_{d^m;w^r},$$

where

$$rw = md.$$

Proof. We shall consider any $F \in \mathrm{Pol}_m(\mathrm{Pol}_d(V))$ as a polynomial in coefficients $A_{\mathbf{i}}$ of the general polynomial $\sum_{\mathbf{i}} \binom{d}{\mathbf{i}} A_{\mathbf{i}} \mathbf{t}^{\mathbf{i}}$ from $\mathrm{Pol}_d(V)$. For any $g \in \mathrm{GL}_r(k)$ we can write

$$g^r = (\det g)\tilde{g},$$

where $\tilde{g} \in \mathrm{SL}_r(k)$. It is clear that the scalar matrix λI_r acts on each element ξ_i of the basis of V by multiplying it by λ. Hence it acts on the coordinate function t_i by multiplying it by λ^{-1} and on $\mathrm{Pol}_d(V)$ via multiplication by λ^{-d}. Hence it acts on $\mathrm{Pol}_m(\mathrm{Pol}_d(V))$ by multiplication by λ^{md} (recall that $(g \cdot F)(P) = F(g^{-1} \cdot P)$). Therefore we get

$$g^r \cdot F = (\det g)^{md} \tilde{g} \cdot F = (\det g)^{md} F.$$

Since any $g' \in \mathrm{GL}_r(k)$ can be written as an rth power, we obtain that $g \cdot F = \chi(g)F$ for some homomorphism $\chi : \mathrm{GL}_r(k) \to k^*$. Notice that when we fix F and $P \in \mathrm{Pol}_d(V)$, the function $g \to g \cdot F(P)$ is a polynomial function in entries of the matrix g which is homogeneous of degree md. Also, we know that $\chi(g)^r = (\det g)^{md}$. Since $\det g$ is an irreducible polynomial of degree r in entries of the matrix, we obtain that $\chi(g)$ is a nonnegative power of $\det g$. Comparing the degrees we get, for any $g \in \mathrm{GL}_r(k)$,

$$g \cdot F = (\det g)^w F.$$

Since the map symb : $\mathrm{Pol}_m(\mathrm{Pol}_d(V)) \to \mathrm{Pol}(\mathrm{Mat}_{r,m})$ is $\mathrm{GL}_r(k)$-equivariant, we see that

$$g \cdot \mathrm{symb}(F) = (\det g)^w F, \quad \forall g \in \mathrm{GL}_r(k).$$

If we take g to be the diagonal matrix of the form diag$[1, \ldots, 1, \lambda, 1, \ldots, 1]$ we immediately obtain that symb(F) is multiisobaric of multi-weight w^r. Also, by definition of the symbolic expression, symb(F) is multihomogeneous of multi-degree d^m. This proves the assertion. □

Corollary 1.1. *Assume $r \nmid md$. Then, for $m > 0$,*

$$\mathrm{Pol}_m(\mathrm{Pol}_d(V))^{\mathrm{SL}(V)} = \{0\}.$$

An example of a function from $\mathrm{Pol}(\mathrm{Mat}_{r,r})_{1^r,1^r}$ is the determinant function $\mathcal{D}_r : A \mapsto \det A$. More generally we define the *bracket function* $\det_J$ on $\mathrm{Mat}_{r,m}$ whose value on a matrix A is equal to the maximal minor formed by the columns from a subset J of $[m] := \{1, \ldots, m\}$. If $J = \{j_0, \ldots, j_n\}$ we will often use its classical notation for the minors

$$\det_J = (j_0 \ldots j_n) = [j_0, \ldots, j_n].$$

It is isobaric of weight 1 but not multihomogeneous if $m > r$. Using these functions one can construct functions from $\mathrm{Pol}(\mathrm{Mat}_{r,m})_{d^m,w^r}$ whenever $md = rw$. This is done as follows.

Definition. A (rectangular) *tableau* on the set $[m] = \{1, 2, \ldots, m\}$ of size $w \times r$ is a matrix

$$\begin{bmatrix} \tau_{11} & \ldots & \tau_{1r} \\ \vdots & \ddots & \vdots \\ \tau_{w1} & \ldots & \tau_{wr} \end{bmatrix} \tag{1.10}$$

with entries in $[m]$ satisfying the inequalities $\tau_{ij} < \tau_{ij+1}$. We say that the tableau is *homogeneous* of degree d if each i, $1 \leq i \leq m$, occurs exactly d times; clearly d must satisfy the relation $md = wr$.

An example of a tableau on the set $[4]$ of size 2×2 and degree 2 is

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 1 & 4 \end{bmatrix}.$$

For each tableau τ as above we define the *tableau function* μ_τ on $\mathrm{Mat}_{r,m}$ by

$$\mu_\tau = \prod_{i=1}^{w} [\tau_{i1}, \ldots, \tau_{ir}].$$

We say that μ_τ is *homogeneous* of degree d if τ is of degree d. It is clear that any such function belongs to $\mathrm{Pol}(\mathrm{Mat}_{r,m})_{d^m, w^r}$. For example, the symbolic expression of the determinant of a binary quadratic form from Example 1.3 is equal to $[12]^2$. The symbolic expression $(12)^2(23)^2(13)^2$ of the catalecticant corresponds to the function μ_τ, where

$$\tau = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 2 & 3 \\ 2 & 3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix}.$$

Notice the way a tableau function μ_τ changes when we apply a transformation $g \in \mathrm{GL}_r(k)$: each bracket function $[i_1, \ldots, i_r]$ is multiplied by $\det g$. So for each tableau τ on the set $[m]$ of size $w \times r$ the function μ_τ is multiplied by $\det(g)^w$. In particular, each such function is an invariant for the group $G = \mathrm{SL}_r(k)$ of matrices with determinant equal to 1. Taking linear combinations of homogeneous degree d tableau functions that are invariant with respect to permutation of columns, we get a lot of examples of elements in $\mathrm{Pol}(\mathrm{Pol}_d(V))^{\mathrm{SL}(V)}$. In the next chapter we will prove that any element from this ring is obtained in this way.

Bibliographical notes

The symbolic method for expression of invariants goes back to the earlier days of theory of algebraic invariants, which originates in the work of A. Cayley of 1846. It can be found in many classical books on invariant theory ([28], [38], [39], [47],

[98]). A modern exposition of the symbolic method can be found in [18], [65], [85]. The theory of polarization of homogeneous forms is a basis of many constructions in projective algebraic geometry; see for example [14], [39], [99], [100]. For a modern treatment of some of the geometric applications we refer to [24], [54].

Exercises

1.1 Show that $\mathrm{Pol}(\mathrm{Mat}_{r,m})_{d_1,\ldots,d_m;w_1,\ldots,w_r} = \{0\}$ unless $\sum_{i=1}^m d_i = \sum_{j=1}^r w_j$.

1.2 Let $W = \mathrm{Pol}_2(V)$ be the space of quadratic forms on a vector space V of dimension r.

(i) Assume that $\mathrm{char}(k) \neq 2$ or r is odd. Show that $\mathrm{Pol}(E)^{\mathrm{SL}(V)}$ is generated (as a k-algebra) by the *discriminant* function whose value at a quadratic form is equal to the determinant of the matrix defining its polar bilinear form.

(ii) Which level sets of the discriminant function are orbits of $\mathrm{SL}(V)$ in W?

1.3 Let $F \in \mathrm{Pol}_d(V)$. For any $w \in V$ and $t \in k^*$ consider the function on $V \times k^*$ defined by $(v,t) \to t^{-1}(F(v+tw) - F(v))$. Show that this function extends to $V \times k$ and let $P_w(F)$ denote the restriction of the extended function to $V \times \{0\}$.

(i) Show that $P_w(F) \in \mathrm{Pol}_{d-1}(V)$ and the pairing

$$V \times \mathrm{Pol}_d(V) \to \mathrm{Pol}_{d-1}(V), \quad (w,F) \mapsto P_w(F),$$

is bilinear.

(ii) Assume $d! \neq 0$ in k. Let $P_w : \mathrm{Pol}_d(V) \to \mathrm{Pol}_{d-1}(V)$ be the linear map $F \mapsto P_w(F)$. Show that the function $V^d \to k$ defined by

$$(w_1,\ldots,w_m) \mapsto \frac{1}{d!}(P_{w_1} \circ \cdots \circ P_{w_d})(F)$$

coincides with $\mathrm{pol}(F)$.

(iii) Show that $P_w(F) = \sum_{i=1}^r a_i \frac{\partial F}{\partial t_i}$, where $(a_1,\ldots,a_r)$ are the coordinates of w with respect to some basis $(\xi_1,\ldots,\xi_r)$.

1.4 Let $\mathbb{P}(V)$ be the projective space associated to a vector space V of dimension r. We consider each nonzero $v \in V$ as a point $\bar{v}$ in $\mathbb{P}(V)$. The hypersurface $P_{\bar{v}} : P_v(F) = 0$ in $\mathbb{P}(V)$ is called the *polar hypersurface* of the hypersurface $H_F : F = 0$ with respect to the point $\bar{v}$. Show that for any $x \in H_F \cap P_{\bar{v}}$ the tangent hyperplane of H_F at x contains the point $\bar{v}$.

1.5 Consider the bilinear pairing between $\mathrm{Pol}_m(V)$ and $\mathrm{Pol}_m(V^*)$ defined as in (1.6). For any $F \in \mathrm{Pol}_m(V), \Phi \in \mathrm{Pol}_s(V^*)$ denote the value of this pairing at (F, Φ) by $P_\Phi(F)$. Show that

(i) for fixed F the assignment $\Phi \mapsto P_\Phi(F)$ defines a linear map

$$\mathrm{ap}_s : \mathrm{Pol}_s(V^*) \to \mathrm{Pol}_{m-s}(V),$$

(ii) for any $\Phi' \in \mathrm{Pol}_{s'}(V^*)$, $P_{\Phi\Phi'}(F) = P_\Phi(P_{\Phi'}(F))$,

(iii) $P_\Phi(F) = P_{v_s} \circ \cdots \circ P_{v_1}(F)$ if Φ is the product of linear polynomials $v_1, \cdots, v_s \in V = (V^*)^*$.

1.6 In the notation of the preceding exercise, $\Phi \in \mathrm{Pol}_s(V^*)$ is called *apolar* to a homogeneous form $F \in \mathrm{Pol}_m(V)$ if $P_\Phi(F) = 0$. Show that

(i) $(\sum_{i=1}^r a_i\xi_i)^m$ is apolar to F if and only if $F(a_1, \ldots, a_r) = 0$,

(ii) $(\sum_{i=1}^r a_i\xi_i)^{m-1}$ is apolar to F if and only if all partial derivatives of F vanish at $a = (a_1, \ldots, a_r)$.

1.7 Consider the linear map ap_s defined in Exercise 1.5. The matrix of this map with respect to the basis in $\mathrm{Pol}_m(V^*)$ defined by the monomials $\xi^{\mathbf{i}}$ and the basis in $\mathrm{Pol}_m(V)$ defined by the monomials $\mathbf{t}^{\mathbf{j}}$ is called the *catalecticant matrix*. Show that

(i) Show that if $m = 2\dim V$ the determinant of the catalecticant matrix is an invariant on the space $\mathrm{Pol}_m(V)$ (it is called the *catalecticant invariant*) .

(ii) Show that, if $\dim V = 2$ and $m = 4$, the catalecticant invariant coincides with the one defined in Example 1.4.

(iii) Find the degree of the catalecticant invariant.

(iv) Show that the catalecticant invariant on the space $\mathrm{Pol}_2(V)$ coincides with the discriminant invariant.

(v) Compute the catalecticant matrix in the case $\dim V = 3, m = 4, s = 2$.

1.8 Let $P \in \mathrm{Pol}_m(V)$. For any $v_1, \ldots, v_m \in V$ and any $\lambda_1, \ldots, \lambda_m \in k$ write

$$P(\lambda_1 v_1 + \cdots + \lambda_m v_m) = \sum_{d_1+\cdots+d_m=m} \lambda_1^{d_1} \ldots \lambda_m^{d_m} P_{d_1,\ldots,d_m}(v_1, \ldots, v_m).$$

(i) Show that the function $P_{d_1,\ldots,d_m} : (v_1, \ldots, v_m) \mapsto P_{d_1,\ldots,d_m}(v_1, \ldots, v_m)$ is multihomogeneous of multi-degree $(d_1, \ldots, d_m)$.

(ii) Show that $P_{1,\ldots,1} = \mathrm{pol}(P)$.

1.9 Find the symbolic expression for the polynomial $F = a_0a_4 - 4a_1a_3 + 3a_2^2$ on the space of binary quartics $\mathrm{Pol}_4(k^2)$. Show that it is an invariant for the group $\mathrm{SL}_2(k)$.

1.10 Find the polarization of the determinant polynomial $\mathcal{D}_r$.

1.11 Let $\chi : \mathrm{GL}_r(k) \to k^*$ be a homomorphism of groups. Assume that χ is given by a polynomial in the entries of $g \in \mathrm{GL}_r(k)$. Prove that there exists a nonnegative integer t such that, for all $g \in \mathrm{GL}_r(k)$, $\chi(g) = (\det g)^t$.

Chapter 2

The First Fundamental Theorem

2.1 The omega-operator

We saw in the preceding chapter that the symbolic expressions of the discriminant of a binary quadratic form and of the catalecticant of a binary quartic are polynomials in the bracket functions. The theorem from the title of this chapter shows that this is the general case for invariants of homogeneous forms of any degree and in any number of variables. In fact we will show more: the bracket functions generate the algebra $\mathrm{Pol}(\mathrm{Mat}_{r,m})^{\mathrm{SL}_r(k)}$. Recall that the group $\mathrm{SL}_r(k)$ acts on this ring via its action on matrices by left multiplication.

We start with some technical lemmas.

For any polynomial $P(X_1, \ldots, X_N)$ let $\tilde{P}$ denote the (differential) operator on $k[X_1, \ldots, X_N]$ obtained by replacing each unknown X_i with the partial derivative operator $\frac{\partial}{\partial X_i}$ (cf. Remark 1.2).

In this section we will use only a special operator of this sort. We take $N = r^2$ with unknowns $X_{ij}, i, j = 1, \ldots, r$, and let P be the determinant function $\mathcal{D}_r$ of the matrix with entries X_{ij}. We denote the corresponding operator $\tilde{P}$ by Ω. It is called the *omega-operator* or the *Cayley operator*.

Lemma 2.1.

$$\Omega(\mathcal{D}_r^s) = s(s+1)\ldots(s+r-1)\mathcal{D}_r^{s-1}.$$

Proof. First observe that for any permutation $\sigma \in \Sigma_r$ we have

$$\frac{\partial^r}{\partial X_{1\sigma(1)} \ldots \partial X_{r\sigma(r)}}(\mathcal{D}_r) = \epsilon(\sigma), \tag{2.1}$$

where $\epsilon(\sigma)$ is the sign of the permutation σ. This immediately gives that $\Omega(\mathcal{D}_r) = r!$. For any subset $J = \{j_1, \ldots, j_k\}$ of $[r]$ set

$$\begin{aligned}\Omega(J,\sigma) &= \frac{\partial^k}{\partial X_{j_1\sigma(j_1)} \cdots \partial X_{j_k\sigma(j_k)}},\\ \Delta(J,\sigma) &= \Omega(J,\sigma)(\mathcal{D}_r).\end{aligned}$$

Analogously to (2.1) we get

$$\Delta(J,\sigma)(\mathcal{D}_r) = \epsilon(J,\sigma) M_{\bar{J}\overline{\sigma(J)}}, \tag{2.2}$$

where for any two subsets K, L of $[r]$ of the same cardinality we denote by $M_{K,H}$ the minor of the matrix (X_{ij}) formed by the rows corresponding to the set K and the columns corresponding to the set L. The bar denotes the complementary set and

$$\epsilon(J,\sigma) = \text{sign}\left(\prod_{\substack{a,b\in J\\ a<b}} (\sigma(a) - \sigma(b))\right).$$

Now applying the chain rule we get

$$\Omega([r],\sigma)(\mathcal{D}_r^s) = \Omega([r-1],\sigma)\frac{\partial \mathcal{D}_r^s}{\partial X_{r\sigma(r)}}$$

$$\begin{aligned}&= \Omega([r-1],\sigma)(s\mathcal{D}_r^{s-1}\Delta(\{r\},\sigma)) = \Omega([r-2],\sigma)\frac{s\partial \mathcal{D}_r^{s-1}\Delta(\{r\},\sigma)}{\partial X_{r-1\sigma(r-1)}}\\ &= \Omega([r-2],\sigma)\Big(s(s-1)\mathcal{D}_r^{s-2}\Delta(\{r\},\sigma)\Delta(\{r-1\},\sigma) + s\mathcal{D}_r^{s-1}\Delta(\{r-1,r\},\sigma)\Big)\\ &= \sum_{k=1}^{r} s(s-1)(s-k+1)\mathcal{D}_r^{s-k}\Big(\sum_{J_1\sqcup\cdots\sqcup J_k=[r]} \Delta(J_1,\sigma)\ldots\Delta(J_k,\sigma)\Big).\end{aligned}$$

Now recall a well-known formula from multilinear algebra which relates the minors of a matrix A and the minors of its adjoint (also called *adjugate* in classic literature) matrix $\tilde{A} = \text{adj}(A)$ (see [8], Chapter 3, §11, exercise 10):

$$\tilde{A}_{H,K} = \det(A)^{|H|-1} A_{\bar{H},\bar{K}}. \tag{2.3}$$

Applying (2.3) we obtain

$$\Delta(J_1,\sigma)\ldots\Delta(J_k,\sigma) = \mathcal{D}_r^{k-r}\prod_{i=1}^{k} \epsilon(J_i,\sigma)\tilde{M}_{J_i,\sigma(J_i)}.$$

Now recall the Laplace formula for the determinant of a square matrix A of size r:

$$\det(A) = \epsilon(J_1, \dots, J_k) \sum_{I_1 \sqcup \cdots \sqcup I_k = [r]} \epsilon(I_1, \dots, I_k) M_{J_1, I_1} \dots M_{J_k, I_k}, \tag{2.4}$$

where

$$[r] = J_1 \sqcup \cdots \sqcup J_k$$

is a fixed partition of the set of rows of A and $\epsilon(I_1, \dots, I_k)$ is equal to the sign of the permutation $(I_1 \dots I_k)$ where we assume that the elements of each set I_j are listed in the increasing order. Applying this formula to $\tilde{M}$ we find

$$\sum_{\sigma \in \Sigma_r} \epsilon(\sigma)\epsilon(J_1, \sigma) \dots \epsilon(J_k, \sigma) \tilde{A}_{J_1, \sigma(J_1)} \dots \tilde{A}_{J_k, \sigma(J_k)} = j_1! \dots j_k! \mathcal{D}_r^{r-1},$$

where $j_i = \# J_i, i = 1, \dots, k$. Thus, letting σ run through the set Σ_r, we sum up the expressions $\epsilon(\sigma)\Omega([r], \sigma)(\mathcal{D}_r^r)$ to get

$$\Omega(\mathcal{D}_r^s) = \sum_{k=0}^{n} s(s-1) \dots (s-k) p(r, k) \mathcal{D}_r^{s-1} = c(r, s) \mathcal{D}_r^{s-1},$$

where

$$p(r, k) = \sum_{J_1 \sqcup \ldots \sqcup J_k = [r]} j_1! \dots j_k!.$$

We leave to the reader as an exercise to verify that

$$c(r, s) = s(s+1) \dots (s+r-1).$$

The precise value of the nonzero constant $c(r, s)$ is irrelevant for what follows. $\square$

Lemma 2.2. *Let $F = P_1 \cdots P_r \in k[X_{11}, \dots, X_{rm}]$, where each P_i is equal to the product of m_i linear forms $L_i^{(j)} = \sum_{s=1}^{r} a_{is}^{(j)} X_{is}, j = 1, \dots, m_i$. Then*

$$\Omega(F) = \sum \det \begin{pmatrix} a_{11}^{(j_1)} & \dots & a_{1r}^{(j_r)} \\ \vdots & \ddots & \vdots \\ a_{r1}^{(j_1)} & \dots & a_{rr}^{(j_r)} \end{pmatrix} (P_1/L_1^{(j_1)}) \cdots (P_r/L_n^{(j_r)}),$$

where the sum is taken over the set $S = \{(j_1, \dots, j_r) : 1 \leq j_i \leq m_i\}$.

Proof. By the chain rule,

$$\frac{\partial^r F}{\partial X_{1i_1}\cdots\partial X_{ri_r}} = \sum_{(j_1,\ldots,j_r)\in S} a_{1i_1}^{(j_1)}\cdots a_{ri_r}^{(j_r)}(P_1/L_1^{(j_1)})\cdots(P_r/L_r^{(j_r)}).$$

After multiplying by the sign of the permutation $(i_1,\ldots,i_r)$ and summing up over the set of permutations, we get the desired formula from the assertion of the lemma. $\square$

2.2 The proof

Now we are ready to prove the First Fundamental Theorem of Invariant Theory:

Theorem 2.1. *The algebra of invariants* $\mathrm{Pol}(\mathrm{Mat}_{r,m})^{\mathrm{SL}_r(k)}$ *is generated by the bracket functions* $[j_1,\ldots,j_r]$.

Proof. Let $\mathrm{Pol}(\mathrm{Mat}_{r,m})_w$ be the subspace of polynomials which are multiisobaric of multi-weight w^r. It is clear that

$$\mathrm{Pol}(\mathrm{Mat}_{r,m})^{\mathrm{SL}_r(k)} = \bigoplus_{w\geq 0}\mathrm{Pol}(\mathrm{Mat}_{r,m})_w{}^{\mathrm{SL}_r(k)}.$$

So we may assume that an invariant polynomial $F\in\mathrm{Pol}(\mathrm{Mat}_{r,m})^{\mathrm{SL}_r(k)}$ belongs to $\mathrm{Pol}(\mathrm{Mat}_{r,m})_w$. Fix a matrix $A\in\mathrm{Mat}_{r,m}$ and consider the assignment $g\mapsto F(g\cdot A)$ as a function on $\mathrm{Mat}_{r,r}$. It follows from the proof of Proposition 1.1 that

$$F(g\cdot A) = \det(g)^w F(A).$$

Since F is multiisobaric, it is easy to see that $F(g\cdot A)$ can be written as a sum of products of linear polynomials as in Lemma 2.2, with $m_i = w$. Applying the omega-operator to the left-hand side of the identity w times we will be able to get rid of the variables g_{ij} and get a polynomial in bracket functions. On the other hand, by Lemma 2.1 we get a scalar multiple of F. This proves the theorem. $\square$

Let $\mathrm{Tab}_{r,m}(w)$ denote the subspace of $\mathrm{Pol}_w(\mathrm{Mat}_{r,m})$ spanned by tableau functions on $[m]$ of size $w\times r$ and let $\mathrm{Tab}_{r,m}(w)_{\mathrm{hom}}$ be its subspace spanned by homogeneous tableau functions of degree d. Recall that, as follows from the definition of a tableau, $rw = md$. The symmetric group Σ_m acts linearly on the space $\mathrm{Tab}_{r,m}(w)$ via its action on tableaux by permuting the elements of the set $[m]$. We denote by $\mathrm{Tab}_{r,m}(w)^{\Sigma_m}$ the subspace of invariant elements. Clearly,

$$\mathrm{Tab}_{r,m}(w)^{\Sigma_m}\subset\mathrm{Tab}_{r,m}(w)_{\mathrm{hom}}.$$

Corollary 2.1. *Let* $w = \frac{md}{r}$. *We have*

$$\mathrm{Pol}(\mathrm{Mat}_{r,m})^{\mathrm{SL}_r(k)}_{d^m,w^r} = \mathrm{Tab}_{r,m}(w)_{\mathrm{hom}}.$$

By Proposition 1.1, the symbolic expression of any invariant polynomial F from $\mathrm{Pol}_m(\mathrm{Pol}_d(V))^{\mathrm{SL}_r(k)}$ belongs to $\mathrm{Pol}(\mathrm{Mat}_{r,m})^{\mathrm{SL}_r(k)}_{d^m,w^r}$, and hence must be a linear combination of tableau functions from $\mathrm{Tab}_{r,m}(w)$. The group Σ_m acts naturally on $\mathrm{Mat}_{r,m}$ by permuting the columns and hence acts naturally on $\mathrm{Pol}(\mathrm{Mat}_{r,m})$ leaving the subspaces $\mathrm{Pol}(\mathrm{Mat}_{r,m})_{d^m,w^r}$ invariant. Applying Lemma 1.1, we get

Corollary 2.2.

$$\mathrm{symb}(\mathrm{Pol}_m(\mathrm{Pol}_d(V))^{\mathrm{SL}_r(k)}) = \mathrm{Tab}_{r,m}(w)^{\Sigma_m}_{\mathrm{hom}},$$

where $rw = md$.

2.3 Grassmann varieties

The ring $\mathrm{Pol}(\mathrm{Mat}_{r,m})^{\mathrm{SL}_r(k)}$ has a nice geometric interpretation. Let $\mathrm{Gr}(r,m)$ be the Grassmann variety of r-dimensional linear subspaces in k^m (or, equivalently, $(r-1)$-dimensional linear projective subspaces of $\mathbb{P}^{m-1}$). Using the Plücker map $L \to \bigwedge^r(L)$, we can embed $\mathrm{Gr}(r,m)$ in $\mathbb{P}(\bigwedge^r(k^m)) = \mathbb{P}^{\binom{m}{r}-1}$. The projective coordinates in this projective space are the Plücker coordinates $p_{i_1\ldots i_r}, 1 \le i_1 < \cdots < i_r \le m$. Consider the set $\Lambda(r,m)$ of ordered r-tuples in $[m]$. Let $k[\Lambda(r,m)]$ be the polynomial ring whose variables are the Plücker coordinates p_J indexed by elements of the set $\Lambda(r,m)$. We view it as the projective coordinate ring of $\mathbb{P}(\bigwedge^r(k^m))$. Consider the natural homomorphism

$$\phi : k[\Lambda(r,m)] \to \mathrm{Pol}(\mathrm{Mat}_{r,m})$$

which assigns to $p_{i_1\ldots i_r}$ the bracket polynomial $[i_1,\ldots,i_r]$. By Theorem 2.1, the image of this homomorphism is equal to the subring $\mathrm{Pol}(\mathrm{Mat}_{r,m})^{\mathrm{SL}_r(k)}$ of invariant polynomials.

Theorem 2.2. *The kernel* $I_{r,m}$ *of* ϕ *is equal to the homogeneous ideal of the Grassmann variety* $\mathrm{Gr}(r,m)$ *in its Plücker embedding.*

Proof. Let $\mathrm{Mat}'_{r,m}$ be the dense open subset of the affine space $\mathrm{Mat}_{r,m}$ formed by matrices of maximal rank r. Consider the map $f : \mathrm{Mat}'_{r,m} \to \mathbb{A}^{\binom{m}{r}} = \mathrm{Spec}(k[\Lambda(r,m)])$ given by assigning to $A \in \mathrm{Mat}'_{r,m}$ the values of the bracket functions $[i_1, \ldots, i_r]$ on A. Clearly, the corresponding map f^* of the rings of regular functions coincides with ϕ. Also it is clear that the image Z of f is contained in the affine cone $\tilde{\mathrm{Gr}}(r,m)$ over $\mathrm{Gr}(r,m)$. The composition of f and the canonical projection $\tilde{\mathrm{Gr}}(r,m) \setminus \{0\} \to \mathrm{Gr}(r,m)$ is surjective. Let F be a homogeneous polynomial from $\mathrm{Ker}(\phi)$. Then its restriction to Z is zero, and hence, since it is homogeneous, its restriction to the whole of $\tilde{\mathrm{Gr}}(r,m)$ is zero. Thus F belongs to $I_{r,m}$. Conversely, if F belongs to $I_{r,m}$, its restriction to Z is zero, and hence $f^*(F) = 0$ because $f : \mathrm{Mat}'_{r,m} \to Z$ is surjective. Since $\mathrm{Gr}(r,m)$ is a projective subvariety, $I_{r,m}$ is a homogeneous ideal (i.e. generated by homogeneous polynomials). Thus it was enough to assume that F is homogeneous. □

Corollary 2.3.

$$\mathrm{Pol}(\mathrm{Mat}_{r,m})^{\mathrm{SL}_r(k)} \cong k[\mathrm{Gr}(r,m)].$$

The symmetric group Σ_m acts naturally on $\mathrm{Gr}(r,m)$ by permuting the coordinates in the space k^m. This corresponds to the action of Σ_m on the columns of matrices from $\mathrm{Mat}_{r,m}$. Let T be the subgroup of diagonal matrices in $\mathrm{SL}_m(k)$. It acts naturally on $\mathrm{Gr}(r,m)$ by scalar multiplication of columns. Let $k[\mathrm{Gr}(r,m)]_w$ be the subspace generated by the cosets of homogeneous polynomials of degree w. Applying Corollary 2.1 and Corollary 2.2, we obtain

Corollary 2.4. *Let $rw = md$. Then*

$$\mathrm{Pol}_m(\mathrm{Pol}_d(k^r))^{\mathrm{SL}_r(k)} \cong k[\mathrm{Gr}(r,m)]_w^{\Sigma_m \times T}.$$

2.4 The straightening algorithm

We now describe a simple algorithm which allows one to construct a basis of the space $\mathrm{Tab}_{r,m}(w)$.

Definition. A tableau on the set $[m]$ of size $w \times s$

$$\tau = \begin{bmatrix} \tau_{11} & \ldots & \tau_{1r} \\ \vdots & \ddots & \vdots \\ \tau_{w1} & \ldots & \tau_{wr} \end{bmatrix}$$

is called *standard* if $\tau_{ij} \leq \tau_{(i+1)j}$ for every i and j.

For example,

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

is standard but

$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

is not.

Theorem 2.3. *The tableau functions μ_τ corresponding to standard tableaux form a basis of the space* $\mathrm{Tab}_{r,m}(w)$.

Proof. We will describe the *straightening law* due to A. Young. It is an algorithm which allows one to write any tableau function as a linear combination of tableau functions corresponding to standard tableaux.

We will use the following relation between the bracket functions:

$$\sum_{s=1}^{r+1}(-1)^s[i_1,\ldots,i_{r-1},j_s][j_1,\ldots,j_{s-1},j_{s+1},\ldots,j_{r+1}] = 0. \tag{2.5}$$

Here $(i_1,\ldots,i_{r-1})$ and $(j_1,\ldots,j_{r+1})$ are two fixed increasing sequences of numbers from the set $[m]$ and we assume that in the bracket function $[i_1,\ldots,i_{r-1},j_s]$, the sequence $(i_1,\ldots,i_{r-1},j_s)$ is rearranged to be in increasing order or equal to zero if two of the numbers are equal.

This relation follows from the observation that the left-hand side, considered as a function on the subspace $(k^r)^{r+1}$ of $\mathrm{Mat}_{r,m}$ formed by the columns with indices $j_1,\ldots,j_{r+1}$, is $(r+1)$-multilinear and alternating. Since the exterior power $\bigwedge^{r+1}(k^r)$ equals zero, the function must be equal to zero.

Suppose a tableau function μ_τ is not standard. By permuting the rows of τ we can assume that $\tau_{i1} \leq \tau_{(i+1)1}$ for all i. Let j be the smallest index such that $\tau_{ij} > \tau_{(i+1)j}$ for some i. We assume that $\tau_{kj} \leq \tau_{(k+1)j}$ for $k < i$. We call the pair (ij) with this property the *mark* of τ. Consider equation (2.5) corresponding to the sequences

$$\begin{aligned}
(i_1,\ldots,i_{r-1}) &= (\tau_{(i+1)1},\ldots,\tau_{(i+1)(j-1)},\tau_{(i+1)(j+1)},\ldots,\tau_{(i+1)r}),\\
(j_1,\ldots,j_{r+1}) &= (\tau_{i1},\ldots,\tau_{ij},\ldots,\tau_{ir},\tau_{(i+1)j}).
\end{aligned}$$

Here we assume that the second sequence is put in increasing order. It allows us to express $[\tau_{i1},\ldots,\tau_{ir}][\tau_{(i+1)1},\ldots,\tau_{(i+1)r}]$ as a sum of the products

$$[\tau_{i1},\ldots,\hat{\tau}_{is},\ldots,\tau_{ir},\tau_{(i+1)j}][\tau_{(i+1)1},\ldots,\hat{\tau}_{(i+1)j},\ldots,\tau_{(i+1)r},\tau_{is}].$$

Substituting this in the product μ_τ of the bracket functions corresponding to the rows of τ, we express μ_τ as a sum of the $\mu_{\tau'}$ such that the mark of each τ' is greater than the mark of τ (with respect to the lexicographic order). Continuing in this way we will be able to write μ_τ as a sum of standard tableau functions.

This shows that the standard tableau functions span the space $\mathrm{Tab}_{r,m}(w)$. We skip the proof of their linear independence (see, for example, [48], p. 381). $\square$

Corollary 2.5. *The homogeneous ideal $I_{r,m}$ defining $\mathrm{Gr}(r,m)$ in its Plücker embedding is generated by the quadratic polynomials*

$$P_{I,J} = \sum_{s=1}^{n+2} (-1)^s p_{i_1\ldots i_{r-1},j_s} p_{j_1\ldots,j_{s-1}j_{s+1},\ldots,j_{r+1}},$$

where $I = (i_1, \ldots, i_{r-1}), J = (j_1, \ldots, j_{r+1})$ are increasing sequences of numbers from the set $[m]$.

Proof. It is enough to show that any homogeneous polynomial F from $I_{r,m}$ can be expressed as a polynomial in the $P_{I,J}$. Let $I'_{r,m}$ be the ideal generated by the polynomials the $P_{I,J}$. It follows from the straightening algorithm that, modulo $I'_{r,m}$, the polynomial F is equal to a linear combination of monomials which are mapped to standard tableau functions in the ring $k[\mathrm{Mat}_{r,m}]$. Since the standard tableau functions are linearly independent, we obtain that $F \in I'_{r,m}$. $\square$

Remark 2.1. The equations $P_{I,J} = 0$ defining the Grassmannian $\mathrm{Gr}(r,m)$ are called the *Plücker equations*. Corollary 2.3 implies that the Plücker equations describe the basic relations between the bracket functions. This result is sometimes referred to as the *Second Fundamental Theorem* of Invariant Theory.

Now we are in business and finally can compute something. We start with the case $r = 2$. Let us write any degree d homogeneous standard tableau in the form

$$\tau = \begin{bmatrix} a_1^1 & a_2^2 \\ a_2^1 & a_3^2 \\ \vdots & \vdots \\ a_{m-1}^1 & a_m^2 \end{bmatrix},$$

where a_i^j denotes a column vector with coordinates equal to i. Let $|a_i^j|$ be the length of this vector. It is clear that

$$\begin{aligned} |a_1^1| &= |a_m^2| = d, \quad |a_i^1| + |a_i^2| = d, \quad 1 < i < m, \\ \sum_{i=2}^{m-1} |a_i^1| &= \sum_{i=2}^{m-1} |a_i^2| = w - d = (m-2)d/2. \end{aligned}$$

So if we set $|a_i^1| = \alpha_{i-1}, i = 2, \ldots, m-1$, then a standard tableau is determined by a point with integer coordinates inside of the convex polytope $\Pi(1, d, m)$ in $\mathbb{R}^{m-2}$ defined by the inequalities

$$0 \le \alpha_i \le d, \quad \sum_{i=1}^{m-2} \alpha_i = w - d.$$

Example 2.1. Let $d = 3$. We have

$$\Pi(1,3,m) = \Big\{(\alpha_1, \ldots, \alpha_{m-2}) \in \mathbb{R}^{m-2} : 0 \le \alpha_i \le 3, \sum_{i=1}^{m-2} \alpha_i = 3(m-2)/2\Big\}.$$

The first nontrivial case is $m = 2$. We have the unique solution $(0,0)$ for which the corresponding standard tableau is

$$\tau = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}.$$

The only nontrivial permutation of two letters changes μ_τ to $-\mu_\tau$. Thus

$$\mathrm{Pol}_2(\mathrm{Pol}_3(k^2))^{\mathrm{SL}_2(k)} = \{0\}.$$

Next is the case $m = 4$. We have the following solutions:

$$(\alpha_1, \alpha_2) = (0,3), (3,0), (1,2), (2,1).$$

The corresponding standard tableaux are

$$\tau_1 = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 3 & 4 \\ 3 & 4 \\ 3 & 4 \end{bmatrix}, \quad \tau_2 = \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 1 & 3 \\ 2 & 4 \\ 2 & 4 \\ 2 & 4 \end{bmatrix}, \quad \tau_3 = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 3 \\ 2 & 4 \\ 3 & 4 \\ 3 & 4 \end{bmatrix}, \quad \tau_4 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 3 \\ 2 & 4 \\ 2 & 4 \\ 3 & 4 \end{bmatrix}.$$

Let us see how the group Σ_4 acts on the space $\mathrm{Tab}_{2,4}(6)_{\mathrm{hom}}$. The group Σ_4 is generated by the transpositions $(23), (12), (14)$. We have

$$(23)\mu_{\tau_1} = \mu_{\tau_2}, \quad (23)\mu_{\tau_3} = \mu_{\tau_4} \tag{2.6}$$

By the straightening algorithm,

$$[23][14] = [13][24] - [12][34],$$

so that

$$\begin{aligned}
(12)\mu_{\tau_1} &= -\mu_{\tau_1}, \\
(12)\mu_{\tau_2} &= ([13][24] - [12][34])^3 \\
&= \mu_{\tau_2} - \mu_{\tau_1} - 3\mu_{\tau_4} + 3\mu_{\tau_3}, \\
(12)\mu_{\tau_3} &= [12]^2[23][14][34]^2 \\
= [12]^3[13][24][34]^2 - [12]^3[34]^3 &= \mu_{\tau_3} - \mu_{\tau_1} \\
(12)\mu_{\tau_4} &= -[12][23]^2[14]^2[34],
\end{aligned}$$

$$\begin{aligned}
&= -[12][34][13]^2[24]^2 + 2[12][34][13][24][12][34] - [12][34][12]^2[34]^2 \\
&= -\mu_{\tau_4} + 2\mu_{\tau_3} - \mu_{\tau_1}.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
(13)\mu_{\tau_1} &= -\mu_{\tau_2} + 3\mu_{\tau_4} - 3\mu_{\tau_3} + \mu_{\tau_1}, \\
(13)\mu_{\tau_2} &= -\mu_{\tau_2}, \\
(13)\mu_{\tau_3} &= -\mu_{\tau_2} + 2\mu_{\tau_4} - \mu_{\tau_3}.
\end{aligned}$$

This implies that any Σ_4-invariant combination of the standard tableau functions must be equal to $F = a\mu_{\tau_1} + b\mu_{\tau_2} + c\mu_{\tau_3} + d\mu_{\tau_4}$, where

$$a = b, \quad c = d, \quad 2c + 3a = 0.$$

This gives that $\mathrm{Tab}_{2,4}(6)^{\Sigma_4}$ is spanned by

$$F = -2\mu_{\tau_1} - 2\mu_{\tau_2} + 3\mu_{\tau_3} + 3\mu_{\tau_4}$$

$$= -2[12]^3[34]^3 - 2[13]^3[24]^3 + 3[12]^2[13][24][34]^2 + 3[12][13]^2[24]^2[34].$$

We leave to the reader to verify that this expression is equal to $\mathrm{symb}(D)$, where

$$D = 6a_0a_1a_2a_3 + 3a_1^2a_2^2 - 4a_1^3a_3 - 4a_0a_2^3 - a_0^2a_3^2. \tag{2.7}$$

This is the discriminant of the cubic polynomial

$$f = a_0x_0^3 + 3a_1x_0^2x_1 + 3a_2x_0x_1^2 + a_3x_1^3.$$

Bibliographical notes

Our proof of the First Fundamental Theorem based on the use of the omega-operator (the *Cayley* Ω*-process*) is borrowed from [110]. The Ω-process is also discussed in [7], [85], [115]. A proof based on the Capelli identity (see the exercises below) can be found in [65], [123]. Another proof using the theory of representations of the group $\mathrm{GL}(V)$ can be found in [18] and [65]. Theorem 2.1 is concerned with invariant polynomial functions on m-vectors in a vector space V with respect to the natural representation of $\mathrm{SL}(V)$ in $V^{\oplus m}$. One can generalize it by considering polynomial functions in m vectors in V and m' covectors, i.e. vectors in the dual space V^*. The First Fundamental Theorem asserts that the algebra of $\mathrm{SL}(V)$-invariant polynomials on $V^{\oplus m} \oplus (V^*)^{\oplus m'}$ is generated by the bracket functions on the space $V^{\oplus m}$, bracket functions on the space $(V^*)^{\oplus m'}$, and the functions $[i|j], 1 \leq i \leq m, 1 \leq j \leq m'$, whose value at $(v_1, \ldots, v_m; \phi_1, \ldots, \phi_{m'}) \in V^{\oplus m} \oplus (V^*)^{\oplus m'}$ is equal to $\phi_j(v_i)$. The proof can be found in [18], [65], [123]. One can also find there a generalization of Theorem 2.1 to invariants with respect to other subgroups of $\mathrm{GL}_n(k)$.

There is a vast amount of literature devoted to the straightening algorithm and its various generalizations (see, for example, [17]). We followed the exposition from [48]. It is not difficult to see that the Plücker equations define set theoretically the Grassmann varieties in their Plücker embedding (see, for example, [40]). Corollary 2.5 describes the homogeneous ideal of the Grassmannian. As far as I know the only textbook in algebraic geometry which contains a proof of this fact is [48]. We refer to [33] for another proof based on the representation theory.

Exercises

2.1 Prove that $\Omega_f \circ \Omega_g = \Omega_{fg}$ for any two polynomials $f, g \in k[X_1, \ldots, X_N]$.

2.2 Let Ω be the omega-operator in the polynomial ring $k[Mat_{r,r}]$. Prove that

(i)$\Omega(\mathcal{D}_r^s) = s(s+1)\ldots(s+r-1)\mathcal{D}_r^{s-1}$ for negative integers s,

(ii) $\Omega((1-\mathcal{D}_r)^{-1}) = r!(1-\mathcal{D}_r)^{-r-1}$,

(iii) the function $f = \sum_{i=0}^{\infty} \frac{\mathcal{D}_r^i}{1\cdot 2!\cdots(i+1)!}$ is a solution of the differential equation $\Omega f = f$ in the ring of formal power series $k[[(X_{ij})]]$.

2.3 For each $i, j \in [m]$ define the operator D_{ij} acting in $\mathrm{Pol}(\mathrm{Mat}_{r,m})$ by the formula $D_{ij} f = \sum_{s=1}^{r} X_{si} \frac{\partial f}{\partial X_{sj}}$.

(i) Prove that the operators D_{ij} commute with each other and commute with Ω if $i \neq j$.

(ii) Check the following identity (the *Capelli identity*):

$$\det \begin{pmatrix} D_{mm} + (m-1)\mathrm{id} & D_{m(m-1)} & \ldots & D_{m1} \\ D_{(m-1)m} & D_{(m-1)(m-1)} + (m-2)\mathrm{id} & \ldots & D_{m-11} \\ \vdots & \vdots & \ddots & \vdots \\ D_{2m} & \ldots & D_{22} + \mathrm{id} & D_{21} \\ D_{1m} & \ldots & D_{12} & D_{11} \end{pmatrix}$$

$$= \begin{cases} 0 & \text{if } m > r, \\ \mathcal{D}_r \Omega & \text{if } m = r. \end{cases}$$

2.4 Using the Capelli identity show that the operator $\tilde{\Omega} : \mathrm{Pol}_r(\mathrm{Pol}_d(V)) \to \mathrm{Pol}_r(\mathrm{Pol}_{d-1}(V))$ defined by $\Omega F = F'$, where $\mathrm{symb}(F') = \Omega(\mathrm{symb}(F))$, is well-defined and transforms an $\mathrm{SL}(V)$-invariant to an $\mathrm{SL}(V)$-invariant.

2.5 Show that $\mathrm{Pol}_3(\mathrm{Pol}_4(k^2))^{\mathrm{SL}_2(k)}$ is spanned by the catalecticant invariant from Example 1.4 in Chapter 1.

2.6 Show that $\mathrm{Pol}(\mathrm{Pol}_3(k^2))^{\mathrm{SL}_2(k)}$ is generated (as a k-algebra) by the discriminant invariant from Example 2.1.

2.7 Show that $\mathrm{Pol}(\mathrm{Pol}_2(V))^{\mathrm{SL}(V)}$ is equal to $k[D]$, where $D : \mathrm{Pol}_2(V) \to k$ is the discriminant of quadratic form. Find $\mathrm{symb}(D)$.

2.8 Let $G = \mathrm{O}_r(k)$ be the orthogonal group of the vector space k^r equipped with the standard inner product. Consider the action of G on $\mathrm{Mat}_{r,m}$ by left multiplication. Show that $\mathrm{Pol}(\mathrm{Mat}_{r,m})^{\mathrm{O}_r(k)}$ is generated by the functions $[ij]$ whose value on a matrix A is equal to the dot-product of the ith and jth columns.

2.9 With the notation from the preceding exercise let $\mathrm{O}_r^+(k) = \mathrm{O}_r(k) \cap \mathrm{SL}_r(k)$. Show that $\mathrm{Pol}(\mathrm{Mat}_{r,m})^{\mathrm{O}_r^+(k)}$ is generated by the functions $[ij]$ and the bracket functions.

2.10 Show that the field of fractions of the ring $\mathrm{Pol}(\mathrm{Mat}_{r,m})^{\mathrm{SL}_r(k)}$ is a purely transcendental extension of k of transcendence degree $r(m-r)+1$.

Chapter 3

Reductive algebraic groups

3.1 The Gordan–Hilbert Theorem

In this chapter we consider a class of linear group actions on a vector space E for which the algebra of invariant polynomials $\mathrm{Pol}(E)^G$ is finitely generated. We start with the case of finite group actions.

Theorem 3.1. *Let G be a finite group of automorphisms of a finitely generated k-algebra A. Then the subalgebra A^G is finitely generated over k.*

Proof. This follows easily from standard facts from commutative algebra. First we observe that A is integral over $B = A^G$. Let $x_1, \ldots, x_n$ be generators of A. Let B' be the subalgebra of A generated by the coefficients of the monic polynomials $p_i(t) \in B[t]$ such that $p_i(x_i) = 0$. Then $A = B'[x_1, \ldots, x_n]$ is a finite B'-module. Since B' is noetherian, B is also a finite B'-module. Since B' is finitely generated over k, B must be finitely generated over k. $\square$

Let us give another proof of this theorem in the special case when the order d of G is prime to the characteristic of k and G acts on $A = \mathrm{Pol}(E)$ via its linear action on E. In this case G leaves invariant the subspace of homogeneous polynomials of degree m so that

$$\mathrm{Pol}(E)^G = \bigoplus_{m=0}^{\infty} \mathrm{Pol}(E)^G_m.$$

Let I be the ideal in A generated by invariant polynomials vanishing at 0 (or, equivalently, by invariant homogeneous polynomials of positive degree). Applying the Hilbert Basis Theorem, we obtain that the ideal I is finitely generated by a finite set of polynomials $F_1, \ldots, F_n$ in A^G. We may assume that each F_i is homogeneous of degree $m_i > 0$. Then for any homogeneous $F \in A^G$ of degree m we can write

$$F = P_1F_1 + \cdots + P_nF_n \tag{3.1}$$

for some homogeneous polynomials P_i of degree $m - m_i$. Now consider the operator av : $A \to A$ defined by the formula

$$\text{av}(P) = \frac{1}{d}\sum_{g \in G} g(P).$$

Clearly,

$$\text{av}|A^G = \text{id}, \quad \text{av}(A) = A^G.$$

Applying the operator av to both sides of (3.1) we get

$$F = \text{av}(P_1)F_1 + \cdots + \text{av}(P_n)F_n.$$

By induction we can assume that each invariant homogeneous polynomial of degree $< m$ can be expressed as a polynomial in F_i's. Since $\text{av}(P_i)$ is homogeneous of degree $< m$, we are done.

Let us give another application of the Hilbert Basis Theorem (it was proven by Hilbert exactly for this purpose):

Theorem 3.2. *(Gordan–Hilbert) The algebra of invariants* $\text{Pol}(\text{Pol}_d(V))^{\text{SL}(V)}$ *is finitely generated over* k.

Proof. Let $E = \text{Pol}_d(V)$. The proof uses the same idea as the one used in the second proof of Theorem 3.1. Instead of the averaging operator av we use the omega-operator Ω. Let $F \in \text{Pol}_m(E)^{\text{SL}(V)}$. Write

$$F = P_1F_1 + \cdots + P_nF_n$$

for some $P_i \in \text{Pol}(E)_{m-m_i}$ and $F_i \in \text{Pol}_{m_i}(E)^{\text{SL}(V)}$. By the proof of Proposition 1.1 there exists an integer e such that, for any $v \in E$,

$$F(g \cdot v) = (\det g)^e F(v).$$

The number e is called the *weight* of F.

Now, for a general matrix g, we have the identity of functions on $\text{GL}(V)$:

$$F(g \cdot v) = (\det g)^e F(v) = \sum_{i=1}^n (\det g)^{e_i} P_i(g \cdot v) F_i(v).$$

Now let us apply the omega-operator Ω to both sides e times. We get

$$cF(v) = \sum_{i=1}^{n} \Omega^e((\det g)^{e_i} P_i(g \cdot v))F_i(v),$$

where c is a nonzero constant. Now the assertion follows by showing that the value of $\Omega^e((\det g)^{e_i} P_i(g \cdot v))$ at $g = 0$ is an invariant and using induction on the degree of the polynomial. $\square$

Lemma 3.1. *For any $P \in \mathrm{Pol}(E)$ let*

$$F(g, v) = \Omega^r((\det g)^q P(g \cdot v)).$$

Then $F(0, v)$ is either zero or an invariant of weight $r - q$.

Proof. This is nothing more than the change of variables in differentiation. Let t be a general square matrix of size N. We have

$$\begin{aligned}
F(g, t \cdot v) &= \Omega^r((\det g)^q P(gt \cdot v)) \\
&= (\det t)^{-q} \Omega^r(\det(gt)^q P(gt \cdot v)) \\
&= (\det t)^{-q} \det(t)^r \Omega^r_{\det(gt)}(\det(gt)^q P(gt \cdot v)) \\
&= (\det t)^{r-q} F(gt, v).
\end{aligned}$$

Here $\Omega_{\det(gt)}$ denotes the omega-operator in the ring $k[\ldots X_{ij}, \ldots, Y_{ij}, \ldots]$ corresponding to the determinant of the matrix (Z_{ij}) where $Z_{ij} = \sum_s X_{is} Y_{sj}$. We use the formula

$$\Omega(\Phi(Z)) = \det(Y_{ij}) \Omega_{\det(gt)}(\Phi(Z)) \tag{3.2}$$

for any polynomial $\Phi(Z)$ in the variables Z_{ij}. This easily follows from the differentiation rules and we leave its proof to the reader. Now plugging in $g = 0$ in (3.2) (although it is not in $\mathrm{GL}(V)$ the left-hand side extends to the whole polynomial ring in the matrix entries) we obtain

$$F(0, t \cdot v) = (\det t)^{r-q} F(0, v).$$

This proves the assertion. $\square$

Remark 3.1. In fact, the same proof applies to a more general situation when $\mathrm{GL}_n(k)$ acts on a vector space E by means of a rational linear representation (see the definition of a rational representation in the next section). We have to use that in this case $g \cdot F = \det(g)^e F$ for any $g \in \mathrm{GL}_n(k)$ and $F \in \mathrm{Pol}(E)^{\mathrm{SL}_n(k)}$.

Remark 3.2. The proof shows that the algebra of invariants $\text{Pol}(E)^{\text{SL}_n(k)}$ is generated by a finite generating set $F_1, \ldots, F_n$ of the ideal I generated by invariant homogeneous polynomials of positive degree. Let $Z = V(I) \subset E$ be the subset of common zeros of $F_1, \ldots, F_n$. Let J be the ideal in $\text{Pol}(E)^{\text{SL}_n(k)}$ of all polynomials vanishing on Z. By Hilbert's Nullstellensatz, for each $i = 1, \ldots, n$, there exists a positive integer ρ_i such that $F_i^{\rho_i} \in J$. Let $G_1, \ldots, G_N$ be homogeneous generators of J. Let d be the largest of the degrees of the F_i and r be the largest of the numbers ρ_i. Then it is easy to see that any invariant homogeneous polynomial of degree $\geq drn$ can be expressed as a polynomial in $G_1, \ldots, G_N$. This implies that the ring $\text{Pol}(E)^{\text{SL}_n(k)}$ is integral over the subring $k[G_1, \ldots, G_N]$ generated by $G_1, \ldots, G_N$. In fact, it can be shown that it coincides with the integral closure of $k[G_1, \ldots, G_N]$ in the field of fractions of $\text{Pol}(E)$ (see, for example, [115], Corollary 4.6.2). In Chapter 9 we will learn how to describe the set Z (it will be identified with the null-cone) without explicitly computing the ring of invariants. This gives a constructive approach to finding the algebra of invariants.

3.2 The unitary trick

Let us give another proof of the Gordan–Hilbert Theorem using another device replacing the averaging operator av due to A. Hurwitz (later called the "*unitary trick*" by H. Weyl). We assume that $k = \mathbb{C}$.

Let $G = \text{SL}_n(\mathbb{C})$ and $K = \text{SU}(n)$ be its subgroup of unitary matrices. Let G act on $\text{Pol}(E)$ via its linear representation $\rho : G \to \text{GL}(E)$.

Lemma 3.2. *(Unitary trick)*

$$\text{Pol}(E)^G = \text{Pol}(E)^K$$

Proof. Let $F \in \text{Pol}(E)$. For any $M \in \text{Mat}_n$ consider the function on $\mathbb{R} \times E$ defined by

$$\phi(t; v) = F(e^{tM} \cdot v).$$

Let $\langle M, F \rangle$ be the function on E defined by

$$\langle M, F \rangle(v) = \frac{\mathrm{d}\phi(t; v)}{\mathrm{d}t}(0).$$

Since $\phi(t + a; v) = \phi(t; e^{aM} \cdot v)$ we see that $\langle M, F \rangle(v) = 0$ for all $v \in E$ if and only if $\frac{\mathrm{d}\phi(t;v)}{\mathrm{d}t}(a) = 0$ for all $a \in \mathbb{R}$ and all $v \in E$. The latter is equivalent to the condition that $F(\mathrm{e}^{tM} \cdot v) = F(v)$ for all $t \in \mathbb{R}$ and all $v \in E$. Let denote the space of complex matrices of size $n \times n$ with zero trace. Since any $g \in \mathrm{SL}_n(\mathbb{C})$ can be written as $g = \mathrm{e}^M$ for some $M \in \mathfrak{sl}_n(\mathbb{C})$, we see that the condition

$$\langle M, F \rangle = 0, \quad \forall M \in \mathfrak{sl}_n, \tag{3.3}$$

is equivalent to F being invariant. Next we easily convince ourselves (by using the chain rule) that the map $M \to \langle M, F \rangle$ is linear, so it is enough to check (3.3) for the set of the M which spans $\mathfrak{sl}_n(\mathbb{C})$. Consider a basis of $\mathfrak{sl}_n(\mathbb{C})$ formed by the matrices

$$E_{ij} - E_{ji}, \quad \sqrt{-1}(E_{ij} + E_{ji}), \quad \sqrt{-1}(E_{ii} - E_{jj}),$$

where $1 \leq i < j \leq n$. Observe that the same matrices form a basis over $\mathbb{R}$ of the subspace $\mathfrak{su}(n)$ of $\mathfrak{sl}_n(\mathbb{C})$ formed by skew-hermitian matrices M (i.e. satisfying ${}^{\mathrm{t}}M = -\overline{M}$). Now we repeat the argument replacing G by $K = \mathrm{SU}(n)$. We use that any $g \in K$ can be written in the form e^M for some $M \in \mathfrak{su}(n)$. We find that $F \in \mathrm{Pol}(E)^K$ if and only if $\langle M, F \rangle = 0$ for all $M \in \mathfrak{su}(n)$. Since the properties $\langle M, F \rangle = 0$ for all $M \in \mathfrak{su}(n)$ and $\langle M, F \rangle = 0$ for all $M \in \mathfrak{sl}_n(\mathbb{C})$ are equivalent we are done. □

The group $K = \mathrm{SU}(n)$ is a compact smooth manifold. If $g = (g_{ij}) \in K$ and $g_{ij} = g'_{ij} + \sqrt{-1}g''_{ij}$, where g'_{ij}, g''_{ij} are real, then K is a closed and a bounded submanifold of $\mathbb{R}^{2n^2}$ defined by the equations

$$\sum_{j=1}^{n} g_{aj}\bar{g}_{bj} = \delta_{ab}, 1 \leq a \leq b \leq n, \quad \det(g) = 1,$$

where δ_{ab} is the Kronecker symbol. This allows one to integrate over it. We consider any polynomial complex valued function on K as a restriction of a polynomial function on $\mathrm{GL}_n(\mathbb{C})$. For each such function $\phi(g)$ set

$$\mathrm{av}(\phi) = \frac{\int_K \phi(g)\mathrm{d}g}{\int_K \mathrm{d}g},$$

where $\mathrm{d}g = \prod_{1 \leq i,j \leq N} \mathrm{d}g'_{ij}\mathrm{d}g''_{ij}$.

Lemma 3.3. *For any $F \in \mathrm{Pol}(E)$ the function $\widetilde{F}$ defined by*

$$\widetilde{F}(v) = \mathrm{av}(F(g \cdot v))$$

is K-invariant.

Proof. For any matrix $g = (g_{ij}) \in K$ let $g' = (g'_{ij})$ and $g'' = (g''_{ij})$. For any $s, g, u \in K$ with $u = g \cdot s$ we have

$$(u'\; u'') = (g'\; g'') \cdot \begin{pmatrix} s' & -s'' \\ s'' & s' \end{pmatrix}.$$

Here we use block-expressions of these matrices. It is easy to see that

$$S = \begin{pmatrix} s' & -s'' \\ s'' & s' \end{pmatrix}$$

is an orthogonal real matrix of size $2n \times 2n$. Thus the jacobian of the change of variables $g \mapsto u = g \cdot s$ is equal to $\det S = \pm 1$. Since K is known to be a connected manifold, the function $s \mapsto \det S$ is constant; it takes the value 1 at $s = I_n$, so $\det S \equiv 1$. Applying the formula for the change of variables in the integration we get

$$\int_K F(gs \cdot v)\mathrm{d}g = \int_K F(g \cdot (s \cdot v))\mathrm{d}(gs) = \int_K F(u \cdot v)\mathrm{d}u,$$

hence

$$\begin{aligned} \widetilde{F}(s \cdot v) &= \mathrm{av}(F(gs \cdot v)) \\ = \frac{\int_K F(gs \cdot v)\mathrm{d}g}{\int_K \mathrm{d}g} &= \frac{\int_K F(u \cdot v)\mathrm{d}u}{\int_K \mathrm{d}g} \\ = \mathrm{av}(F(u \cdot v)) &= \widetilde{F}(v). \end{aligned}$$

□

One can generalize the preceding proof to a larger class of groups of complex matrices. What is important in the proof is that such a group G contains a compact subgroup K such that the complex Lie algebra of G is isomorphic to the complexification of the real Lie algebra of K. Here are examples of such groups, their compact subgroups, and their corresponding Lie algebras:

$$\begin{array}{lll} 1) & G = \mathrm{GL}_n(\mathbb{C}), & \mathrm{Lie}(G) = \mathfrak{gl}_n(\mathbb{C}) = \mathrm{Mat}_n(\mathbb{C}), \\ & K = \mathrm{SU}(n), & \mathrm{Lie}(K) = \mathfrak{u}(n) \cap \mathfrak{sl}_n(\mathbb{C}). \\ 2) & G = \mathrm{O}_n(\mathbb{C}), & \mathrm{Lie}(G) = \{A \in \mathfrak{gl}_n(\mathbb{C}) : {}^{\mathrm{t}}A = -A\}, \\ & K = \mathrm{O}_n(\mathbb{R}), & \mathrm{Lie}(K) = \{A \in \mathfrak{gl}_n(\mathbb{R}) : {}^{\mathrm{t}}A = -A\}. \end{array}$$

These groups satisfy the following property

(LR) Let $\rho : G \to \mathrm{GL}(V)$ be a homomorphism of complex Lie groups, and $v \in V^G \setminus \{0\}$. Then there exists an invariant subspace W such that $V = \mathbb{C}v \oplus W$. Or, in other words, there exists a G-invariant linear function f on V such that $f(v) \neq 0$.

One checks this property by first replacing G with its compact subgroup K as above. Taking any linear function f with $f(v) > 0$ we average it by integration over K to find a nonzero K-invariant function with the same property. Then we apply Lemma 3.3 to ensure that f is G-invariant.

3.3 Affine algebraic groups

Next we observe that property (LR) from the preceding section can be stated over any algebraically closed field k. Instead of complex Lie groups, we will be dealing with affine algebraic groups G over k.

Definition. An *affine algebraic group* G over a field k is an affine algebraic variety over k with the structure of a group on its set of points such that the multiplication map $\mu : G \times G \to G$ and the inversion map $\beta : G \to G$ are regular maps.

Although we assume that the reader is familiar with some rudiments of algebraic geometry, we have to fix some terminology which may be slightly different from the standard textbooks (for example, [104]). We shall use an embedding-free definition of an affine algebraic variety over an algebraically closed field k . Namely, a set $X := \mathrm{Spm}(A)$ of homomorphisms of a finitely generated k-algebra A without zerodivisors to k. The algebra A is called the *coordinate algebra* of X and is denoted by $\mathcal{O}(X)$ (or $k[X]$). An element $a \in A$ can be considered as a k-valued function on X whose value at a point $x : A \to k$ is equal to $x(a)$. Functions on X of this form are called *regular functions*. A point x is uniquely determined by the maximal ideal $\mathfrak{m}_x$ of functions vanishing at x. A choice of generators $x_1, \ldots, x_n$ of $\mathcal{O}(X)$ defines a bijection from X to a subset of the affine space $\mathbb{A}^n = \mathrm{Spm}(k[T_1, \ldots, T_n])$ identified naturally with the set k^n. This subset is equal to the set of common zeros of the ideal of relations between the generators. A regular map (or morphism) $f : X \to Y$ of affine algebraic varieties is defined as a map given by composition with a homomorphism of the coordinate algebras $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$. This makes a category of affine algebraic varieties over k which is equivalent to the dual of the category of finitely generated domains over k. This latter category has direct products defined by the tensor product of

k-algebras. A subset $V(I)$ of X of homomorphisms vanishing on an ideal I of $\mathcal{O}(X)$ is called a closed subset. It can be identified with an affine algebraic variety Spm(A/J), where $J = \text{rad } I$ is the radical of I. A point $x \in X$ is a closed subset corresponding to the maximal ideal $\mathfrak{m}_x$ of A. Closed subsets define a topology on X, the Zariski topology. Open subsets $D(f) = X \setminus V((f)), f \in A$, form a basis of the topology. Each subset $D(f)$ can be identified with an affine algebraic variety Spm$(A[1/f])$.

A choice of n generators of the k-algebra $\mathcal{O}(X)$ defines an isomorphism from X to a closed subset of the affine space $\mathbb{A}^n$. A morphism of affine varieties Spm$(A) \to$ Spm(B) corresponding to a surjective homomorphism $B \to A$ of k-algebras defines an isomorphism from Spm(B) to a closed subset of Spm(A). It is called a closed embedding.

The multiplication and the inversion morphisms μ, β defining an affine algebraic group G can equivalently be given by homomorphisms of k-algebras

$$\mu^* : \mathcal{O}(G) \to \mathcal{O}(G) \otimes_k \mathcal{O}(G), \quad \beta^* : \mathcal{O}(G) \to \mathcal{O}(G),$$

which are called the *comultiplication* and the *coinverse*.

For any k-algebra K we define the set $X(K)$ of K*-points* of X to be the set of homomorphisms of k-algebras $\mathcal{O}(X) \to K$. In particular, if $K = \mathcal{O}(Y)$ for some affine algebraic variety Y, the set $X(K)$ can be identified naturally with the set of morphisms from Y to X.

Here are some examples of affine algebraic groups which we will be using in the book.

(a) $\mathrm{GL}_{n,k} =$ Spm$(k[\ldots, X_{ij}, \ldots][\det((X_{ij}))^{-1}])$ (a *general linear group* over k):

$$\mathrm{GL}_{n,k}(K) = \mathrm{GL}(n, K), \quad \mu^*(X_{ij}) = \sum_{s=1}^{n} X_{is}X_{sj}, \quad \beta^*(X_{ij}) = X^{ij},$$

where X^{ij} is equal to the (ij)th entry of the inverse of the matrix (X_{ij}).

(b) $\mathbb{G}_{m,k} = \mathrm{GL}_{1,k} =$ Spm$(k[T, T^{-1}])$ (the *multiplicative group* over k):

$$\mathbb{G}_{m,k}(K) = K^*, \quad \mu^*(T) = T \otimes T, \quad \beta^*(T) = T^{-1}.$$

(c) $\mathbb{G}_{a,k} =$ Spm$(k[T])$ (the *additive group* over k):

$$\mathbb{G}_{a,k}(K) = K^+, \quad \mu^*(T) = T \otimes 1 + 1 \otimes T, \quad \beta(T) = -T.$$

Other examples of affine algebraic groups can be realized by taking direct products or by taking a closed subvariety which is an affine algebraic group with respect to the restriction of the multiplication and the inverse morphisms (a *closed subgroup*). For example, we have

(d) $T_k^n = \mathbb{G}_{m,k}^n$ (an *affine torus* over k),

(e) $\mathrm{SL}_{n,k}$ (a *special linear group* over k).

Affine algebraic groups over k form a category. Its morphisms are morphisms of affine algebraic varieties which induce homomorphisms of the corresponding group structures. One can prove that any affine algebraic group G admits a morphism to the group $\mathrm{GL}_{n,k}$ such that it is a closed embedding. In other words, G is isomorphic to a *linear algebraic group*, i.e., a closed subvariety of $\mathrm{GL}_{n,k}$ whose K-points for any k-algebra K form a subgroup of $\mathrm{GL}_n(K)$. If no confusion arises, we will also drop the subscript k in the notation of groups $\mathrm{GL}_{n,k}, \mathbb{G}_{m,k}$, and so on.

From now on all of our groups will be linear algebraic groups and all of our maps will be morphisms of algebraic varieties.

We define an action of G on a variety X to be a regular map $\alpha : G \times X \to X$ satisfying the usual axioms of an action (which can be expressed by the commutativity of some natural diagrams). We call such an action a *rational action* or, better, a *regular action*. In particular, a linear representation $\rho : G \to \mathrm{GL}(V) \cong \mathrm{GL}_n(k)$ will be assumed to be given by regular functions on the affine algebraic variety G. Such linear representations are called *rational representations*.

Let an affine algebraic group G act on an affine variety $X = \mathrm{Spm}(A)$. This action can be described in terms of the *coaction homomorphism*

$$\alpha^* : A \to \mathcal{O}(G) \otimes A,$$

where $\mathcal{O}(G)$ is the coordinate ring of G. It satisfies a bunch of axioms which are "dual" to the usual axioms of an action; we leave their statements to the reader. For any $a \in A$ we have

$$\alpha^*(a) = \sum_i f_i \otimes a_i,$$

where $f_i \in \mathcal{O}(G), a_i \in A$. An element $g \in G$ is a homomorphism $\mathcal{O}(G) \to k, f \mapsto f(g)$, and we set

$$g(a) := (g \otimes 1) \circ \alpha^*(a) = \sum f_i(g) a_i. \tag{3.4}$$

A homomorphism $\alpha : G \to \mathrm{Aut}(A)$ arising in this way is called a *rational action* of G on a k-algebra A. We will continue to denote the subalgebra of invariant elements by A^G.

An important property of a rational action is the following.

Lemma 3.4. *For any $a \in A$, the linear subspace of A spanned by the "translates" $g(a), g \in G,$ is finite-dimensional.*

Proof. This follows immediately from equation (3.4). The set of elements a_i is a spanning set. □

Note that not every homomorphism of groups $G \to \mathrm{Aut}(A)$ arises from a rational action of G on X.

Example 3.1. Let $G = \mathbb{G}_m$ act on an affine algebraic variety $X = \mathrm{Spm}(A)$. Let $\alpha^* : A \to \mathcal{O}(G) \otimes A = k[T, T^{-1}] \otimes A$ be the corresponding coaction homomorphism. For any $a \in A$ we can write

$$\alpha^*(a) = \sum_{i \in \mathbb{Z}} T^i \otimes a_i. \tag{3.5}$$

It is easy to see, using the axioms of an action, that the maps $p_i : A \to A, a \mapsto a_i$ are the projection operators, i.e., $p_i(a_i) = a_i$. Denoting the image $p_i(A)$ by A_i we have $A_iA_j \subset A_{i+j}$ and

$$A = \bigoplus_{i \in \mathbb{Z}} A_i. \tag{3.6}$$

This defines a grading on A. Conversely, given a grading of A, we define α^* by $\alpha^*(a) = \sum_{i \in \mathbb{Z}} T^i \otimes a_i$, where a_i is the ith graded part of a. This gives a geometric interpretation of a grading of a commutative k-algebra.

Assume now that grading (3.5) on A satisfies $A_i = \{0\}$ for $i < 0$ and $A_0 = k$. Such a grading is called a *geometric grading* and the corresponding action is called a *good* $\mathbb{G}_m$*-action*. In this case, the ideal $\mathfrak{m}_\circ = \sum_{i>0} A_i$ is a maximal ideal of A and hence defines a point p_0 of X, called the *vertex*. We set

$$X^* = \mathrm{Spm}^*(A) = \mathrm{Spm}(A) \setminus \{p_0\}.$$

The group $\mathbb{G}_m$ acts on the open set X^*; the quotient set is denoted by $\mathrm{Projm}(A)$ and is called the *projective spectrum* of A. Assume that A is a finitely generated k-algebra with a geometric grading. Choose a set of its homogeneous generators $\{x_0, \ldots, x_n\}$. If $x_i \in A_{q_i}$ for some $q_i > 0$, then any $t \in \mathbb{G}_m$ acts on A by sending x_i to $t^{q_i}x_i$. Use the generators to identify X with a closed subset of $\mathbb{A}^{n+1}$ defined by the homogeneous ideal I of relations between $x_0, \ldots, x_n$. The vertex of X becomes the origin 0 in $\mathbb{A}^{n+1}$. We obtain a natural bijection from $\mathrm{Projm}(A)$ to the set $\{(a_0, \ldots, a_n) \in \mathrm{Spm}(A) \setminus \{0\}\}/k^*$, where k^* acts by

$$t \cdot (a_0, \ldots, a_n) = (t^{q_0}a_0, \ldots, t^{q_n}a_n). \tag{3.7}$$

In the special case when $x_0, \dots, x_n$ are algebraically independent (i.e., $I = \{0\}$), so that $A \cong k[T_0, \dots, T_n]$ with grading defined by $T_i \in A_{q_i}$, the set

$$\mathbb{P}(q_0, \dots, q_n) = \text{Projm}(A) = \Big(\mathbb{A}^{n+1} \setminus \{0\}\Big)/k^*$$

is called the *weighted projective space* with weights $q_0, \dots, q_n$. When all the q_i are equal to 1, we obtain the usual definition of the n-dimensional projective space $\mathbb{P}^n(k)$.

Let μ_n be the closed subgroup of $\mathbb{G}_m = \text{Spm}(k[T, T^{-1}])$ defined by the ideal $(T^n - 1)$. As an abstract group it is isomorphic to the group of nth roots of 1 in k. Let A be a graded k-algebra and $\mathbb{G}_m \to \text{Aut}(A)$ be the corresponding action. It follows from the definition that

$$A^{\mu_n} = A^{(n)} := \sum_{i \in \mathbb{Z}} A_{in}.$$

The inclusion $A^{(n)} \subset A$ defines a natural map $\text{Spm}^*(A) \to \text{Spm}^*(A^{(n)})$ which coincides with the quotient map for the action of μ_n on $\text{Spm}^*(A)$ (use that $x^n \in A^{(n)}$ for any $x \in A$). Let $\mathbb{G}_m$ act on $\text{Spm}^*(A^{(n)}$ with respect to the grading defined by

$$A_i^{(n)} = A_{in}. \tag{3.8}$$

Then

$$\begin{aligned} \text{Projm}(A) &= \text{Spm}^*(A)/k^* = (\text{Spm}^*(A)/\mu_n)/k^* \\ &= \text{Spm}^*(A^{(n)})/k^* = \text{Projm}(A^{(n)}). \end{aligned}$$

It is known that for any finitely generated geometrically graded k-algebra A there exists a number n such that $A^{(n)}$ is generated by elements of degree 1 with respect to the grading defined by (3.8) (see [9], Chap. III, §1). This implies that $\text{Projm}(A)$ is bijective to a subset of some $\mathbb{P}^N(k)$ equal to the set of common zeros of a homogeneous ideal in the ring of polynomials $k[T_0, \dots, T_N]$ with the standard grading.

One can make this statement more precise by defining the category of projective varieties. First of all we notice that for any nonzero homogeneous element $f \notin \mathfrak{m}_0$, the subset $D(f)$ of $\text{Spm}(A)$ of all points not vanishing on f does not contain the vertex and is invariant with respect to the action of $\mathbb{G}_m$ defining the grading. Since any ideal in A is contained in a homogeneous ideal of A, the union of the sets $D(f)$ is equal to $\text{Spm}^*(A)$. So $\text{Projm}(A)$ is equal to the union of the subsets $D(f)^+ = D(f)/k^*$. If we identify $D(f)$ with $\mathrm{Spm}(A[1/f])$, the action of $\mathbb{G}_m$ on $D(f)$ corresponds to the (not necessarily geometric) grading defined by

$$A[1/f]_i = \{a/f^s : a \in A_{i+s\deg(f)}\}.$$

Let $A_{(f)} = A[1/f]_0 = A[1/f]^{\mathbb{G}_m}$. It is called the *homogeneous localization* of the graded ring A with respect to f. Any element of $A[1/f]_i^{(\deg(f))}$ can be written uniquely in the form $f^i A_{(f)}$. This implies that the image of any point $x \in D(f)$ in $D(f)^+$ is determined by its restriction to $A[1/f]_0$. Thus, any point in $D(f)^+$ is uniquely determined by a homomorphism $A_{(f)} \to k$. This shows that we can identify $D(f)^+$ with $\mathrm{Spm}(A_{(f)})$. Since the union of sets of the form $D(f)^+$ is the whole set $\mathrm{Projm}(A)$, we can define a topology on $\mathrm{Projm}(A)$ in which an open set is a set whose intersection with any set $D(f)^+$ is an open set in its Zariski topology. The open subsets $D(f)^+$ form a basis of the topology.

A *quasi-projective algebraic variety* over k is defined to be a locally closed subset (i.e., the intersection of an open subset with a closed subset) of some $\mathrm{Projm}(A)$. A closed subset is called a *projective variety* over k. For any open subset U of $\mathrm{Projm}(A)$ we define a regular function on U as a function $f : U \to k$ such that its restriction to any subset $D(f)^+ \subset U$ is a regular function. Regular functions on U form a k-algebra which we will denote by $\mathcal{O}(U)$. Let $X \subset \mathrm{Projm}(A)$ and $Y \subset \mathrm{Projm}(B)$ be two quasi-projective algebraic varieties over k. A morphism $\Phi : X \to Y$ is defined to be a continuous map from X to Y (with respect to the induced Zariski topologies) such that for any open subset $U \subset Y$ and any $\phi \in \mathcal{O}(U)$, the composition $\phi \circ \Phi$ is a regular function on $f^{-1}(U)$.

For example, any surjective homomorphism of graded algebras $\alpha : A \to B$ preserving the grading (the latter will be always assumed) defines a closed embedding $\Phi : \mathrm{Spm}(B) \subset \mathrm{Spm}(A)$ whose restriction to any subset $D(f)$ is a closed embedding of affine varieties. It corresponds to the homomorphism $\alpha : A[1/f] \to B[1/\alpha(f)]$. This defines a closed embedding from $D(f)^+$ to $D(\alpha(f))^+$ and a morphism $\Phi : \mathrm{Projm}(B) \to \mathrm{Projm}(A)$. In particular, a choice of homogeneous generators of degrees $q_0, \ldots, q_n$ of A defines a morphism $\mathrm{Projm}(A) \to \mathbb{P}(q_0, \ldots, q_n)$ which is a *closed embedding* (i.e., an isomorphism onto a closed subset of the target space).

One can show (see Exercise 3.6) that any projective algebraic variety is isomorphic to some $\mathrm{Projm}(A)$. Any affine algebraic variety is isomorphic to a quasi-projective algebraic variety because the affine space $\mathbb{A}^n$ is isomorphic to an open subset U_i of $\mathbb{P}^n = \mathrm{Projm}(k[T_0, \ldots, T_n])$ whose complement is the closed subset defined by the ideal (T_i). Thus any locally closed subset of an affine variety is a quasi-projective algebraic variety. We will employ topological terminology dealing with the Zariski topology of a quasi-projective variety. For example, we can speak about irreducible, connected quasi-projective algebraic varieties. We refer the reader to textbooks in algebraic geometry for the notion of a nonsingular quasi-projective variety.

Note that an algebraic group is irreducible if and only if it is connected; this follows from Exercise 3.2.

Even when we study rational actions of an algebraic group on an affine algebraic varieties we have to deal with nonaffine quasi-projective algebraic varieties.

Example 3.2. Let $\alpha : G \times X \to X$ be a rational action of an affine algebraic group G on an affine algebraic variety X. For any point $x \in X$, we have a regular map $\alpha_x : G \to X$ defined by $\alpha_x(g) = \alpha(g, x)$. The fibre of this map over the point x is a closed subgroup of G, called the ***stabilizer subgroup*** of x. It is an affine algebraic group. The image $\mathrm{O}(x)$ of this map is a subset of X, called the ***orbit*** of x, which is not necessarily closed. However, if G is irreducible, the orbit $\mathrm{O}(x)$ is a locally closed subset of X, and hence is a quasi-projective algebraic variety. It follows from the Chevalley Theorem (see [46], p. 94), that the image of a regular map is a disjoint finite union of locally closed subsets. However, since G is irreducible, the image is irreducible and hence must be a locally closed subset, i.e., a quasi-projective variety. Of course, the image of an affine variety is not always affine.

Example 3.3. Let H be a closed subgroup of an algebraic group G. Consider the subspace V of $\mathcal{O}(G)$ spanned by the G-translates of generators of the ideal I defining H. By Lemma 3.4 V is finite-dimensional of some dimension N. Let $W = V \cap I$ and $n = \dim W$. Then G acts rationally on the Grassmannian variety $\mathrm{Gr}(n, N)$ of n-dimensional subspaces of W. One can show that H is the subgroup of G which fixes $W \in \mathrm{Gr}(n, N)$. Thus we can identify the quasi-projective algebraic variety $\mathrm{O}(W) \subset \mathrm{Gr}(n, N)$ with the set of conjugacy classes G/H.

3.4 Nagata's Theorem

Our goal is to prove the following theorem of M. Nagata

Theorem 3.3. *Let G be a geometrically reductive group which acts rationally on an affine variety* $\mathrm{Spm}(A)$. *Then A^G is a finitely generated k-algebra.*

Let us first explain the notion of a geometrically reductive group.

Definition. A linear algebraic group G is called *linearly reductive* if for any rational representation $\rho : G \to \mathrm{GL}(V)$ and any nonzero invariant vector v there exists a linear G-invariant function f on V such that $f(v) \neq 0$.

The unitary trick shows that GL_n and SL_n and their products are linearly reductive groups over $\mathbb{C}$. This is not true anymore for the same groups defined over a field of characteristic $p > 0$. In fact, even a finite group is not linearly reductive if its order is not coprime to the characteristic. However, it turns out (*Haboush's Theorem*, [44]) that all these groups are geometrically reductive in the following sense.

Definition. A linear algebraic group G is called *geometrically reductive* if for any rational representation $\rho : G \to \mathrm{GL}(V)$ and any nonzero invariant vector v there exists a homogeneous G-invariant polynomial f on V such that $f(v) \neq 0$.

In fact, one can define the notion of a *reductive algebraic group* over any field which will include the groups $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{O}_n$ and their products and Haboush's Theorem asserts that any reductive group is geometrically reductive. We are not going into the proof of Haboush's Theorem, but let us give the definition of a reductive affine algebraic group (over an algebraically closed field) without going into details.

A linear algebraic group T is called an *algebraic torus* (or simply a torus) if it is isomorphic to $\mathbb{G}_m^n$. An algebraic group is called *solvable* if it admits a composition series of closed normal subgroups whose successive quotients are abelian groups. Each algebraic group G contains a maximal connected solvable normal subgroup. It is called the *radical* of G. A group G is called *reductive* if its radical is a torus. A connected linear algebraic group G is called *semisimple* if its radical is trivial.

Each semisimple group is *isogeneous* (i.e., there exists a surjective homomorphism from one to another with a finite kernel) to the direct product of *simple* algebraic groups. A simple algebraic group is a non-commutative algebraic group characterized by the property that it does not contain proper closed normal subgroups of positive dimension.

There is a complete classification of semisimple affine algebraic groups. Examples of simple groups are the classical groups

$$\mathrm{SL}_{n+1} \text{ (type } A_n), \mathrm{SO}_{2n+1} \text{ (type } B_n), \mathrm{Sp}_{2n} \text{ (type, } C_n), \mathrm{SO}_{2n}, n > 2, \text{ (type } D_n).$$

There are also some simple groups of *exceptional type* of types F_4, G_2, E_6, E_7, E_8. Every simple algebraic group is isogeneous to one of these groups .

We shall start the proof of Nagata's Theorem with the following.

Lemma 3.5. *Let a geometrically reductive algebraic group G act rationally on a k-algebra A leaving an ideal I invariant. Consider $A^G/I \cap A^G$ as a subalgebra of $(A/I)^G$ by means of the injective homomorphism induced by the inclusion $A^G \subset A$. For any $a \in (A/I)^G$ there exists $d > 0$ such that $a^d \in A^G/I \cap A^G$. If G is linearly reductive then d can be taken to be 1.*

Proof. Let $\bar{a}$ be a nonzero element from $(A/I)^G$, let a be its representative in A and let $\mu^*(a) = \sum_i \alpha_i \otimes a_i$. Let V be the G-invariant subspace of A spanned by the G-translates of a. By Lemma 3.4 V is finite-dimensional and is contained in the subspace spanned by the a_i's. Let $v = g'(a) \in V$. We have $g(v) = g(g'(a)) = gg'(a) = a + w$, where $w \in W = I \cap V$. This shows that any $v \in V$ can be written in the form

$$v = \lambda a + w$$

for some $\lambda \in k$ and $w \in W$. Let $l : V \to k$ be the linear map defined by $v \mapsto \lambda$. We have

$$g(v) = g(l(v)a + w) = l(v)g(a) + g(w) = l(v)a + w' = l(g(v))a + w''$$

for some $w, w', w'' \in W$. This implies that $l(g(v)) = l(v), w' = w''$, and, in particular, the linear map $l : V \to k$ is G-invariant. Consider it as an element of the dual space V^*. The group G acts linearly on V^* and l is a G-invariant element. Choose a basis $(v_1, \ldots, v_n)$ of V with $v_1 = a$, and $v_i \in W$ for $i \geq 2$. Then we can identify V^* with the affine space $\mathbb{A}^n$ by using the dual basis, so that $l = (1, 0, \ldots, 0)$. By definition of geometrical reductiveness, we can find a G-invariant homogeneous polynomial $F(Z_1, \ldots, Z_n)$ of degree d such that $F(1, 0, \ldots, 0) \neq 0$. We may assume that $F = Z_1^d + \cdots$. Now we can identify v_i with the linear polynomial Z_i, hence $(F - Z_1^d)(v_1, \ldots, v_n) = F(v_1, \ldots, v_n) - a^d$ belongs to the ideal J of A generated by $v_2, \ldots, v_n$. Since each generator of J belongs to $W \subset I$, we see that $a^d \equiv F(v_1, \ldots, v_n)$ modulo I. Since $F(v_1, \ldots, v_n) \in A^G$ (because F is G-invariant), we are done. $\square$

Now we are ready to finish the proof of Nagata's Theorem. To begin, by noetherian induction, we may assume that for any nontrivial G-invariant ideal I the algebra $(A/I)^G$ is finitely generated.

Assume first that $A = \sum_{n \geq 0} A_n$ is a geometrically graded k-algebra (i.e., $A_0 = k$) and that the action of G preserves the grading. For example, A could be a polynomial algebra on which G acts linearly. The subalgebra A^G inherits the grading. Suppose A^G is an integral domain. Take a homogeneous element $f \in A^G$ of positive degree. We have $fA \cap A^G = fA^G$ since, for any $x \in A$, $g(xf) - xf = f(g(x) - x) = 0$ implies that $x \in A^G$. Since $(A/fA)^G$ is finitely generated and integral over $A^G/fA^G = A^G/fA \cap A^G$ (Lemma 3.5), we obtain that A^G/fA^G is finitely generated. Hence its maximal ideal $(A^G/fA^G)_+$ generated by elements of positive degree is finitely generated. If we take the set of representatives of its generators and add f to this set, we obtain a set of generators of the ideal $(A^G)_+$ in A^G. But now, using the same inductive (on degree) argument as in the second proof of Theorem 3.1, we obtain that A^G is a finitely generated algebra.

Now assume that A^G contains a zero-divisor f. Then fA and the annihilator ideal $R = (0 : f) := \{a \in A : fa = 0\}$ are nonzero G-invariant ideals. As above, $A^G/fA \cap A^G$ and $A^G/R \cap A^G$ are finitely generated. Let B be the subring of A^G generated by representatives of generators of both algebras. It is mapped surjectively to $A^G/fA \cap A^G$ and $A^G/R \cap A^G$. Let $c_1, \ldots, c_n$ be representatives in A of generators of $(A/R)^G$ as a $B/R \cap B$-module. Since $g(c_i) - c_i \in R$ for all $g \in G$, we get $f(g(c_i) - c_i) = 0$, i.e., $fc_i \in A^G$. Let us show that $A^G = B[fc_1, \ldots, fc_n]$. Then we will be done. If $a \in A^G$, we can find $b \in B$ such that $a - b \in fA$ (since B is mapped surjectively to $A^G/fA \cap A^G$). Then $a - b = fr$ is G-invariant implies that $r \in (A/R)^G$. Thus $r \in \Sigma_i Bc_i$. This implies $a = b + fr = b + fc \in B[fc_1, \ldots, fc_n]$ as we wanted.

So we are done in the graded case.

Now let us consider the general case. Let $t_1, \ldots, t_n$ be generators of A. Consider the k-vector space $V \subset A$ spanned by G-translates of the t_i. It follows from Lemma 3.4 that V is finite-dimensional. Without loss of generality we may assume now that $(t_1, \ldots, t_n)$ is a basis of this space. Let $\phi : S = k[T_1, \ldots, T_n] \to A$ be the surjective homomorphism defined by $T_i \mapsto t_i$. The group G acts on S linearly by $g(T_i) = \sum \alpha_{ij} T_j$, where $g(t_i) = \sum \alpha_{ij} t_j$. Let I be the kernel of ϕ. It is obviously G-invariant. We obtain that $A^G = (S/I)^G$. By Lemma 3.5, A^G is integral over $S^G/I \cap S^G$. Since we have shown already that S^G is finitely generated, we are almost done (certainly done in the case when G is linearly reductive). By a previous case we may assume that A^G has no zerodivisors. A result from commutative algebra (see, for example, [26], Corollary 13.3) gives that the integral closure R of $S^G/I \cap S^G$ in the field of fractions $Q(A^G)$ of A^G is a finitely generated k-algebra provided that $Q(A^G)$ is a finite extension of the field of fractions of $S^G/I \cap S^G$. Since R is integral over A^G this would imply that A^G is finitely generated (see [26], Exercise 4.3.2). Thus it is enough to show that the field $Q(A)$ is a finite extension of the field of fractions of $S^G/I \cap S^G$. Since A^G is integral over this ring, it is enough to show that $Q(A^G)$ is finitely generated as a field. If A is a domain this is obvious (a subfield of a finitely generated field is finitely generated). In the general case we use the total ring of fractions of A, the localization A_T with respect to the set T of nonzerodivisors. For any maximal ideal $\mathfrak{m}$ of A_T we have $\mathfrak{m} \cap A^G = 0$ since A^G is a domain. This shows that the field of fractions of A^G is a subfield of $A_T/\mathfrak{m}$. But the latter is a finitely generated field equal to the field of fractions of $A/\mathfrak{m} \cap A$. The proof is now complete.

In the next chapter we will give an example (due to M. Nagata) of a rational linear representation $\rho : G \to \mathrm{GL}(V)$ of a linear algebraic group such that $\mathrm{Pol}(V)^G$ is not finitely generated.

The algebra of invariants A^G, where G is a reductive algebraic group and A is a finitely generated algebra, inherits many algebraic properties of A. We shall not go into this interesting area of algebraic invariant theory; however, we mention the following simple but important result.

Proposition 3.1. *Let G be a reductive algebraic group acting algebraically on a normal finitely generated k-algebra A. Then A^G is a normal finitely generated algebra.*

Proof. Recall that a *normal ring* is a domain integrally closed in its field of fractions. Let K be the field of fractions of A. It is clear that the field of fractions L of A^G is contained in the field K^G of G-invariant elements of K. We have to check that the ring A^G is integrally closed in L. Suppose $x \in L$ satisfies a monic equation

$$x^n + a_1 x^{n-1} + \cdots + a_0 = 0$$

with coefficients $a_i \in A^G$. Since A is normal, $x \in A \cap K^G = A^G$ and the assertion is verified. $\square$

Bibliographical notes

The proof of the Gordan–Hilbert Theorem follows the original proof of Hilbert (see [47]). The proof using the unitary trick can be found in [64], [110], and [123]. The original proof of Nagata's Theorem can be found in [79]. Our proof is rather close to the original one. It can be found in [31], [75], [82], and [111] as well. Haboush's Theorem was a culmination of efforts of many people. There are other proofs of Haboush's Theorem with more constraints on a group (see a survey of these results in [75], p. 191).

A good introduction to Lie groups and Lie algebras can be found in [34] or [86] and [6]; [112], [53] are excellent first courses in algebraic groups.

We refer to [91], §3.9 for a survey of results in the spirit of Proposition 3.1. An interesting question is when the algebra $\mathrm{Pol}(V)^G$, where V is a rational linear representation of a reductive group G, is isomorphic to a polynomial algebra. When G is a finite group, a theorem of Chevalley [11] asserts that this happens if and only if the representation of G in V is equivalent to a unitary representation where G acts as a group generated by unitary reflections. The classification of such unitary representations is due to Shephard and Todd ([107]). The classification of pairs (G, V) with this property when G is a connected linear algebraic group group is known when G is simple, or when G is semisimple and V is its irreducible representation. We refer to [91], §8.7 for the survey of the corresponding results.

Exercises

3.1 For any abstract finite group G construct an affine algebraic k-group such that its group of K-points is equal to G for any K/k.

3.2. Prove that any affine algebraic group is a nonsingular algebraic variety.

3.3 Show that there are no nontrivial homomorphisms from $\mathbb{G}_{m,k}$ to $\mathbb{G}_{a,k}$, or in the other direction.

3.4 Prove that a finite group G over a field characteristic $p > 0$ is linearly reductive if and only if its order is prime to p. Show that such G is always geometrically reductive.

3.5 Give an example of a nonrational action of an affine algebraic group on an affine space.

3.6 Prove that any closed subset of $\mathrm{Projm}(A)$ is isomorphic to $\mathrm{Projm}(A/I)$, where I is a homogeneous ideal of A.

3.7 Let GL_n act on $\mathrm{Pol}(E)$ via its linear representation in E. A polynomial $F \in \mathrm{Pol}(E)$ is called a *projective invariant* of *weight* $w \geq 0$ if, for any $g \in G$ and any $v \in E$, $F(g \cdot v) = (\det g)^w F(v)$. Let $\mathrm{Pol}(E)^G_w$ be the space of projective invariants of weight w. Show that the graded ring

$$\bigoplus_{w \geq 0}^{\infty} \mathrm{Pol}(E)^G_w$$

is finitely generated.

Chapter 4

Hilbert's Fourteenth Problem

4.1 The problem

The assertions about finite generatedness of algebras of invariants are all related to one of the Hilbert Problems. The precise statement of this problem (number 14 in Hilbert's list) is as follows.

Problem 1. *Let k be a field, and let $k(t_1, \ldots, t_n)$ be its purely transcendental extension, and let K/k be a field extension contained in $k(t_1, \ldots, t_n)$. Is the k-algebra $K \cap k[t_1, \ldots, t_n]$ finitely generated?*

Hilbert himself gave an affirmative answer to this question in the situation when $K = k(t_1, \ldots, t_n)^{\mathrm{SL}_n(k)}$ where SL_n acts linearly on $k[t_1, \ldots, t_n]$ (Theorem 3.2 from Chapter 3). The subalgebra $K \cap k[t_1, \ldots, t_n]$ is of course the subalgebra of invariant polynomials $k[t_1, \ldots, t_n]^{\mathrm{SL}_n(k)}$. A special case of his problem asks whether the same is true for an arbitrary group G acting linearly on the ring of polynomials. A first counterexample was given by M. Nagata in 1959; we shall explain it in this chapter. For the reader with a deeper knowledge of algebraic geometry, which we assume in this book, we give a geometric interpretation of Hilbert's Fourteenth Problem due to O. Zariski.

For any subfield $K \subset k(t_1, \ldots, t_n)$ we can find a normal irreducible algebraic variety X over k with field of rational functions $k(X)$ isomorphic to K. The inclusion of the fields gives rise to a rational map

$$f : \mathbb{P}^n - - \to X.$$

Let $Z \subset \mathbb{P}^n \times X$ be the closure of the graph of the regular map of the largest open subset of $\mathbb{P}^n$ on which f is defined. Let H be the hyperplane at infinity in $\mathbb{P}^n$ and

$D' = \mathrm{pr}_2(\mathrm{pr}_1^{-1}(H))$. This is a closed subset of X. By blowing up, if necessary, we may assume that D' is the union of codimension 1 irreducible subvarieties D_i. Let D be the Weil divisor on X equal to the sum of components D_i such that $\mathrm{pr}_1(\mathrm{pr}_2^{-1}(D_i)) \subset H$; note that D could be the zero divisor. Thus for any rational function $\phi \in k(X), f^*(\phi)$ is regular on $\mathbb{P}^n \setminus H$ if and only if ϕ has poles only along the irreducible components of D. Let $L(mD)$ be the linear subspace of $k(X)$ which consists of rational functions such that $\mathrm{div}(f) + mD \geq 0$. After identifying $k(X)$ with K and $\mathcal{O}(\mathbb{P}^n \setminus H)$ with $k[t_1, \ldots, t_n]$ (by means of f^*), we see that $K \cap k[t_1, \ldots, t_n]$ is isomorphic to the subalgebra

$$R(D) = \sum_{m=0}^{\infty} L(mD)$$

of $k(X)$. So the problem is reduced to the problem of finite generatedness of the algebras $R(D)$ where D is any positive Weil divisor on a normal algebraic variety X.

Assume now that X is nonsingular. Then each Weil divisor is a Cartier divisor and hence can be given locally by an equation $\phi_U = 0$ for some rational function ϕ_U on X regular on some open subset $U \subset X$. These functions must satisfy $\phi_U = g_{UV}\phi_V$ on $U \cap V$ for some $g_{UV} \in \mathcal{O}(U \cap V)^*$. We can take them to be the transition functions of a line bundle L_D. Rational functions R with poles along D must satisfy $a_U = R\phi_U^n \in \mathcal{O}(U)$ for some $n \geq 0$. This implies that the functions a_U satisfy $a_U = g_{UV}^n a_V$, hence form a section of the line bundle $L_D^{\otimes n}$. This shows that the algebra $R(D)$ is equal to the union of the linear subspaces $\Gamma(X, L_D^{\otimes n})$ of the field $k(X)$. Let

$$R^*(D) = \bigoplus_{n \geq 0} \Gamma(X, L_D^{\otimes n}).$$

Recall that we can view $\Gamma(X, L_D^{\otimes n})$ as the space of regular functions on the line bundle L_D^{-1} whose restrictions to fibres are monomials of degree n. This allows one to identify the algebra $R^*(D)$ with the algebra $\mathcal{O}(L_D^{-1})$. Let P be the variety obtained from L_D^{-1} by adding the point at infinity in each fibre of L_D^{-1}. More precisely, let $\mathcal{O}_X$ be the trivial line bundle. Then the variety P can be constructed as the quotient of the rank 2 vector bundle $\mathbb{V}(L_D^{-1} \oplus \mathcal{O}_X)$ with the deleted zero section by the group $\mathbb{G}_m$ acting diagonally on fibres; here the direct sum means that the transition functions of the vector bundle are chosen to be diagonal matrices

$$\begin{pmatrix} g_{UV} & 0 \\ 0 & 1 \end{pmatrix}.$$

Then we obtain that $R^*(D)$ is equal to the ring $R(S)$ where S is the divisor at infinity in P. In this way we are led to the following.

Problem 2. *(O. Zariski) Let X be a nonsingular algebraic variety and let D be an effective divisor on X. When is the algebra $R^*(D)$ finitely generated?*

It can be shown that Nagata's counterexample to the Hilbert problem is of the form $R^*(D)$ (see Exercise 4.3). It turns out that the algebras $R^*(D)$ are often not finitely generated. However, if we impose certain conditions on D (for example, that the complete linear system defined by L_D has no base points) then $R^*(D)$ is finitely generated. One of the fundamental questions in algebraic geometry is the question of finite generatedness of the ring $R^*(D)$, where D is the canonical divisor of X. This is closely related to the theory of minimal models of algebraic varieties (see [70]).

4.2 The Weitzenböck Theorem

Let us first discuss the case of algebras of invariants of algebraic groups that are not necessarily reductive. We will later give an example of Nagata which shows that A^G is not finitely generated for some nonreductive group G. Notice that according to a result of V. Popov ([89]), if A^G is always finitely generated, then G must be reductive. In fact, the proof of this result relies on Nagata's counterexample.

Since any affine algebraic group H is a closed subgroup of a reductive group G, we may ask how the rings A^G and A^H are related. First of all we have the following (see [41], [91]).

Lemma 4.1. *Let an affine algebraic group G act on a finitely generated k-algebra A. Then*

$$A^H \cong (\mathcal{O}(G)^H \otimes A)^G.$$

Here H acts on G by left multiplication and G acts on itself by right multiplication.

Proof. Let $X = \mathrm{Spm}(A)$ be the affine algebraic variety with $\mathcal{O}(X) \cong A$. Let $f(g,x) \in \mathcal{O}(G \times X) = \mathcal{O}(G) \otimes A$. Assume $f \in (\mathcal{O}(G)^H \otimes A)^G$. This means that $f(hgg'^{-1}, g'x) = f(g,x)$ for any $g' \in G, h \in H$. Let $\phi(x) = f(1,x)$. Then

$$\phi(hx) = f(1,hx) = f(hh^{-1}, h \cdot x) = f(1,x) = \phi(x).$$

This shows that $\phi \in A^H$. Conversely, if $\phi \in A^H$, the function $f(g,x) = \phi(g \cdot x)$ satisfies

$$f(hgg'^{-1}, g' \cdot x) = \phi(hg \cdot x) = \phi(h \cdot (g \cdot x)) = \phi(g \cdot x) = f(g,x).$$

Thus $f \in (\mathcal{O}(G)^H \otimes A)^G$. We leave to the reader to check that the maps

$$(\mathcal{O}(G)^H \otimes A)^G \to A^H, \quad f(g,x) \mapsto f(1,x),$$

$$A^H \to (\mathcal{O}(G)^H \otimes A)^G, \quad \phi(x) \mapsto \phi(g \cdot x)$$

are inverse to each other. $\square$

Corollary 4.1. *Assume that a rational action of H on an affine variety X extends to an action of a geometrically reductive group G containing H and also assume that $\mathcal{O}(G)^H$ is finitely generated. Then $\mathcal{O}(X)^H$ is finitely generated.*

The algebra $\mathcal{O}(G)^H$ can be interpreted as the algebra of regular functions on the quasi-projective algebraic variety G/H (see Example 3.3). It could be affine, for example when H is a reductive subgroup of a reductive group G. It also could be a projective variety (for example, when $G = \mathrm{GL}_n$ and H contains the subgroup of upper triangular matrices, or more generally, when H is a *parabolic subgroup* of a reductive group G). A closed subgroup H of affine algebraic group G is called *observable* if G/H is *quasi-affine* (i.e., isomorphic to an open subvariety of an affine variety). An observable subgroup H is called a *Grosshans subgroup* if $\mathcal{O}(G)^H$ is finitely generated.

Theorem 4.1. *Let H be an observable subgroup of a connected affine algebraic group G. The following properties are equivalent:*

(i) G is a Grosshans subgroup;

(ii) there exist a rational linear representation of G in a vector space V of finite dimension and a vector $v \in V$ such that $H = G_v$ and the orbit $G \cdot v$ of v is of codimension ≥ 2 in its closure $\overline{G \cdot v}$.

Proof. (i) $\Rightarrow$ (ii) Let $A = \mathcal{O}(G)^H$ and let $X = \mathrm{Spm}(A)$. X is an irreducible algebraic variety on which G acts (via the action of G on A). Consider the canonical morphism $\phi : G/H \to X$ such that $\phi^* : \mathcal{O}(X) \to \mathcal{O}(G/H) = \mathcal{O}(G)^H$ is the identity. Since G/H is isomorphic to an open subset of an affine variety Y, the restriction map $\mathcal{O}(Y) \to \mathcal{O}(G/H) = \mathcal{O}(X)$ defines a morphism of affine varieties $f : X \to Y$ such that the composition $f \circ \phi : G/H \to X \to Y$ is the open embedding $G/H \hookrightarrow Y$. Since ϕ is dominant, this easily implies that ϕ is an open embedding. So we may assume that G/H is an open subset of X and that the restriction homomorphism $\mathcal{O}(X) \to \mathcal{O}(G/H)$ is bijective. Let $Z = X \setminus (G/H)$. This is a closed subset of X. Since G is a nonsingular irreducible algebraic variety, G is a normal affine variety, i.e., the ring $\mathcal{O}(G)$ is normal. By Proposition 3.1 the ring $\mathcal{O}(G)^H$ has the same property and hence X is a normal affine variety. In particular, A is a Krull domain ([9], Chapter VII, §1) and we can apply the theory of divisors. It follows from the approximation theorem (loc. cit., Proposition 9) that one can find a rational function R on X such that it has a pole only at one irreducible component of Z of codimension 1. Thus the rational function R is regular on G/H but not regular on X. This contradiction shows that each irreducible component of Z is of codimension ≥ 2. Now, by Lemma 3.5, we can embed X into affine space in such a way that G acts on X via a linear representation. The closure of the G-orbit of $\phi(eH)$ is a closed subset of X containing G/H, and hence the complement of the orbit in its closure is of codimension ≥ 2.

(ii) $\Rightarrow$ (i) Let X be the closure of the orbit $\mathrm{O} = \mathrm{O}(v)$. Replacing X by its normalization, we may assume that $\mathrm{O} \cong G/H$ is isomorphic to an open subset of a normal affine algebraic variety X with the complement of O of codimension ≥ 2. It remains to use that for each such open subset U the restriction map $\mathcal{O}(X) \to \mathcal{O}(U)$ is bijective (see [26]). □

Example 4.1. Let $G = \mathrm{SL}_2$ and H be the subgroup of upper triangular matrices with diagonal entries equal to 1. Obviously, $H \cong \mathbb{G}_a$. In the natural representation of G in the affine plane $\mathbb{A}^2$, the orbit of G of the vector $v = (1,0)$ is equal to $\mathbb{A}^2 \setminus \{0\}$ and the stabilizer subgroup G_v is equal to H. Thus H is a Grosshans subgroup of G. More generally, any maximal unipotent subgroup of an affine algebraic group G is a Grosshans subgroup (see [41], Thm. 5.6).

Let $G = \mathbb{G}_a$. We know that G is not geometrically reductive (Exercise 4.1). However, we have the following classical result.

Theorem 4.2. *(Weitzenböck's Theorem) Assume* $\mathrm{char}(k) = 0$. *Let* $\rho : \mathbb{G}_a \to \mathrm{GL}(V)$ *be a rational linear representation. Then the algebra* $\mathrm{Pol}(V)^{\mathbb{G}_a}$ *is finitely generated.*

Proof. To simplify the proof let us assume that $k = \mathbb{C}$. We shall also identify $\mathbb{G}_a$ with its image G in GL_n; which is isomorphic to k. This can be done since

k does not contain finite nontrivial subgroups in characteristic zero so ρ is either trivial or injective. Let $g \in G$ be a nonzero element. Since there are no nontrivial rational homomorphisms from k to k^*, all eigenvalues of g must be equal to 1. Since G is commutative, there is a common eigenvector e for all $g \in G$. Consider the induced action of G on k^n/ke. Let f be a common eigenvector for all $g \in G$ in this space. Then $g(f) = f + a_g e$ for all $g \in G$. Continuing in this way, we find a basis of V such that each $t \in k$ is represented by a unipotent matrix $A(t)$. Consider the differential of the homomorphism $\rho : G \to \mathrm{GL}_n(k)$ at the origin. It is defined by $a \mapsto aB$, where $B = \frac{\mathrm{d}A(t)}{\mathrm{d}t}(0)$. Clearly B is a nilpotent matrix. Since $A(t + t') = A(t)A(t')$, it is easy to see that $A(t)' = BA(t)$ and hence $A(t) = \exp(tB)$. By changing basis of V, we may assume that B is a Jordan matrix. Let $V = V_1 \oplus \cdots \oplus V_r$, where V_i corresponds to a Jordan block B_i of B of size n_i. It is easy to see that the representation of G in V_i defined by $t \mapsto \exp(tB_i)$ is isomorphic to the representation of G in $\mathrm{Pol}_{n_i}(k^2)$ obtained by restriction of the natural representation of $\mathrm{SL}_2(k)$ in $\mathrm{Pol}_{n_i}(k^2)$. Here we consider G as a subgroup U of upper triangular matrices in $\mathrm{SL}_2(k)$. Thus G acts on V by the restriction of the representation of $\mathrm{SL}_2(k)$ in the direct sum of linear representations in $\mathrm{Pol}_{n_i}(k^2)$. Now we can apply Lemma 4.1. Observe that any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(k)$ can be reduced after multiplication by some $u \in U$ to a matrix of the form

$$\begin{pmatrix} 0 & -c^{-1} \\ c & d \end{pmatrix} \quad (c \neq 0) \quad \text{or} \quad \begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix} \quad (c = 0).$$

Thus any U-invariant regular function on SL_2 is uniquely determined by its values on such matrices. Since the set of such matrices forms a subvariety of SL_2 isomorphic to $\mathbb{A}^2 \setminus \{0\}$, the restriction of functions defines an isomorphism

$$\mathcal{O}(\mathrm{SL}_2)^U \cong \mathcal{O}(\mathbb{A}^2 \setminus \{0\}).$$

Since $\mathcal{O}(\mathbb{A}^2 \setminus \{0\}) \cong \mathcal{O}(\mathbb{A}^2)$, we conclude that $\mathcal{O}(\mathrm{SL}_2)^U$ is finitely generated. So we can apply Lemma 4.1 to the pair (G, SL_2) and the representation of SL_2 on $V = \oplus_{i=1}^r \mathrm{Pol}_{n_i}(k^2)$ to obtain the assertion of the theorem. $\square$

4.3 Nagata's counterexample

Now we are ready to present Nagata's counterexample to the Fourteenth Hilbert Problem.

Let G' be the subgroup of $\mathbb{G}_a^n$ equal to the set of solutions $(t_1,\dots,t_n)$ of a system of linear equations

$$\sum_{j=1}^{n} a_{ij}x_j = 0, \quad i = 1,2,3. \tag{4.1}$$

We will specify the coefficients later. The group G' acts on the affine space $\mathbb{A}^{2n}$ by the formula

$$(t_1,\dots,t_n)\cdot(x_1,y_1,\dots,x_n,y_n) = (x_1+t_1y_1,y_1,\dots,x_n+t_ny_n,y_n).$$

Now let us consider the subgroup

$$C = \left\{(c_1,\dots,c_n)\in\mathbb{G}_m^n : \prod_{i=1}^{n} c_i = 1\right\}$$

of $\mathbb{G}_m^n$. It acts on $\mathbb{A}^{2n}$ by the formula

$$(c_1,\dots,c_n)\cdot(x_1,y_1,\dots,x_n,y_n) = (c_1x_1,c_1y_1,\dots,c_nx_n,c_ny_n)$$

Both of these groups are identified naturally with subgroups of SL_{2n} and we enlarge G' by considering the group $G = G'\cdot C$. The group G is contained in the subgroup of matrices of the form:

$$\begin{pmatrix}
c_1 & \alpha_1 & 0 & \dots & \dots & \dots & \dots & \dots & 0\\
0 & c_1 & 0 & \dots & \dots & \dots & \dots & \dots & 0\\
0 & 0 & c_2 & \alpha_2 & 0 & \dots & \dots & \dots & 0\\
0 & 0 & 0 & c_2 & 0 & \dots & \dots & \dots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & \dots & \dots & \dots & \dots & 0 & c_n & \alpha_n\\
0 & 0 & \dots & \dots & \dots & \dots & 0 & 0 & c_n
\end{pmatrix}. \tag{4.2}$$

Theorem 4.3. *For an appropriate choice of the system of linear equations* (4.1) *and the number n the algebra of invariants*

$$k[X_1,\dots,X_n,Y_1,\dots,Y_n]^G = k[X,Y]^G$$

is not finitely generated.

We start the proof with the following:

Lemma 4.2. *Assume that the determinant of the matrix $(a_{ij})_{1\le i,j\le 3}$ is not equal to zero. Then*

$$k(X,Y)^G = k(T, Z_1, Z_2, Z_3),$$

where

$$T = Y_1 \dots Y_n,\ Z_i = \sum_{j=1}^{n} a_{ij}\Big(\frac{X_j T}{Y_j}\Big), \quad i = 1,2,3.$$

Moreover, Z_1, Z_2, Z_3, T are algebraically independent over k.

Proof. Under the action of g, defined by the matrix (4.2) from above, we have

$$g^*\Big(\frac{X_j}{Y_j}\Big) = \frac{X_j}{Y_j} + \alpha_j, \quad g^*(T) = T,$$

and, since $\sum_{j=1}^{n} a_{ij}\alpha_j = 0$, we obtain that $g^*(Z_i) = Z_i, i = 1,2,3$. This shows that the right-hand side is contained in the left-hand side. Using the assumption on the coefficients a_{ij}, we can write $X_iT/Y_i, i = 1,2,3$, as a linear combination of Z_1, Z_2, Z_3 to obtain

$$\begin{aligned} k(X,Y) &= k(Z_1, Z_2, Z_3, X_4, \dots, X_n, Y_1, \dots, Y_n) \\ &= k(T, Z_1, Z_2, Z_3, X_4, \dots, X_n, Y_1, \dots, Y_{n-1}). \end{aligned}$$

The first equality shows that $Z_1, Z_2, Z_3, Y_1, \dots, Y_n$ are algebraically independent over k, hence Z_1, Z_2, Z_3, T are algebraically independent.

Let H be the subgroup of G defined by the conditions $\alpha_5 = \cdots = \alpha_n = 0, c_i = 1, i = 1, \dots, n$. Obviously it is isomorphic to $\mathbb{G}_a$. We see that

$$\begin{aligned} k(X,Y)^G &\subset k(T, Z_1, Z_2, Z_3, X_4, \dots, X_n, Y_1, \dots, Y_{n-1})^H \\ &= k(T, Z_1, Z_2, Z_3, X_5, \dots, X_n, Y_1, \dots, Y_{n-1}). \end{aligned}$$

Continuing in this way, we eliminate $X_5, \dots, X_n$ to obtain

$$k(X,Y)^G \subset k(T, Z_1, Z_2, Z_3, Y_1, \dots, Y_{n-1}).$$

Now we throw in the torus part C which acts on Y_i by multiplying it by c_i. It is clear that any C-invariant rational function in $Y_1, \dots, Y_{n-1}$ with coefficients in $k(T, Z_1, Z_2, Z_3)$ must be equal to a constant. This proves the lemma. $\square$

Consider now each column (a_{1j}, a_{2j}, a_{3j}) of the matrix (a_{ij}) as the homogeneous coordinates of a point P_j in the projective plane $\mathbb{P}^2$. Let $R(m)$ be the ideal in $k[Z_1, Z_2, Z_3]$ generated by homogeneous polynomials F with *multiplicity* $\geq m$ at each point P_j. If char$(k) = 0$, this means that all partials of F of order $< m$ vanish at P_j. In the general case, it means the following. By a linear change of variables we may assume that $P_j = (0, 0, 1)$. Then F has multiplicity $\geq m$ at P_j if considered as a polynomial in Z_3 all its nonzero coefficients are homogeneous polynomials in Z_1, Z_2 of degree $\geq m$.

Lemma 4.3.

$$k[X, Y]^G = \Big\{ \sum_{m=0}^{\infty} F_m(Z_1, Z_2, Z_3)T^{-m} : F_m \in R(m) \Big\}.$$

Proof. By the preceding lemma, $k[X, Y]^G = k[X, Y] \cap k(Z_1, Z_2, Z_3, T)$. First notice that, since $X_i = Z_iY_i/T$ for $i = 1, 2, 3$, we have

$$k[X_1, \ldots, X_n, Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}] = k[Z_1, Z_2, Z_3, X_4, \ldots, X_n, Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}].$$

The intersection of the right-hand side with the field $k(T, Z_1, Z_2, Z_3)$ is equal to $k[T, T^{-1}, Z_1, Z_2, Z_3]$. Thus

$$k[X, Y]^G = k[Z_1, Z_2, Z_3, T, T^{-1}].$$

Write any invariant homogeneous polynomial $F \in k[X, Y]_d^G$ as a sum of monomials $Z_1^{i_1} Z_2^{i_2} Z_3^{i_r} T^{-m}$, where $i_1, i_2, i_3 \geq 0$ and $m \in \mathbb{Z}$. Since each Z_i is homogeneous in X of degree 1 and in Y of degree $n-1$, and T is homogeneous of degree n in Y, we must have $(i_1+i_2+i_3)+(n-1)(i_1+i_2+i_3)-mn = n(i_1+i_2+i_3)-mn = d$. This implies that we can write F as a sum $\sum_m F_m(Z_1, Z_2, Z_3)T^{-m}$, where each F_m is homogeneous in Z_1, Z_2, Z_3 of degree $i_1 + i_2 + i_3 = m + \frac{d}{n}$. Now write F as a polynomial in X whose coefficients are polynomials in Y. Since the degree of F in X is equal to $i_1 + i_2 + i_3$, we obtain that each $F_m(Z_1, Z_2, Z_3)T^{-m}$ is the X-homogeneous component of F, and hence $F_m(Z_1, Z_2, Z_3)T^{-m}$ is a polynomial in X, Y.

It remains to show that $F_m(Z_1, Z_2, Z_3)T^{-m} \in k[X, Y]$ if and only if each $F_m \in R(m)$. Assume that none of a_{3j} is equal to zero. After a linear change of variables, we obtain that $F_m \in R(m)$ if and only if its coefficients as a polynomial in Z_3 are homogeneous polynomials in $z_j = a_{3j}Z_1 - a_{1j}Z_3,\ z_j' = a_{3j}Z_2 - a_{2j}Z_3$

of degree $\geq m$. Since z_j and z'_j are both divisible by Y_j in $k[X, Y]$, we see that, for any polynomial $F \in R(m)$, we have

$$FT^{-m} \in k[X, Y].$$

We leave to the reader to prove the converse. $\square$

Next, we need a lemma from algebraic geometry.

Let $C : F(T_0, T_1, T_2) = 0$ be an irreducible plane cubic curve in the projective plane $\mathbb{P}^2$ over an algebraically closed field k. It is known that the set C° of nonsingular points of C has the structure of an algebraic group (in the case when C is nonsingular this can be found for example in [104], Chapter 3, §3). If C is singular, this is easy to see. The normalization $\bar{C}$ of C is isomorphic to $\mathbb{P}^1$ and the projection map $\bar{C} \to C$ is an isomorphism outside one point (a cuspidal cubic) or two points (a nodal cubic). The complement of one point in $\mathbb{P}^1$ is isomorphic to the affine line, and hence has a structure of an algebraic group isomorphic to the additive group $\mathbb{G}_a$. The complement of two points is isomorphic to $\mathbb{A}^1 \setminus \{0\}$ and has a structure of an affine algebraic group isomorphic to the multiplicative group $\mathbb{G}_m$. For example, if char$(k) \neq 3$, any cuspidal cubic is isomorphic to the plane curve given by the equation

$$T_2^2 T_0 - T_1^3 = 0 \tag{4.3}$$

(see Chapter 10). Its singular point is $(1, 0, 0)$ and the set of nonsingular points is the subset of k^2 defined by the equation $X^3 - Y = 0$. The group law is given by the formula

$$(x, y) + (x', y') = (x + x', (x + x')^3).$$

Each irreducible plane cubic curve C has at least one nonsingular inflection point, i.e., a point where the tangent to the curve has multiplicity of intersection with the curve is equal to 3 (the only exception are certain cuspidal cubics in characteristic 3, see Chapter 10). Any of these points can be chosen as the zero point of the group law. In the example (4.3), the point $(0, 0, 1)$ is the unique nonsingular inflection point. We denote the sum of two points $p, q \in C$ with respect to the group law by $p \oplus q$.

Lemma 4.4. *Let C be an irreducible plane cubic curve with a nonsingular inflection point o taken to be the zero of the group law on the set C° of nonsingular points of C. Let $p_1, \ldots, p_9 \in C^\circ$. Then the order of the sum $\oplus_{i=1}^9 p_i$ in the group law on C° is equal to $m > 0$ if and only if there exists a homogeneous polynomial F of degree $3m$ not vanishing identically on C with multiplicity m at each point p_i.*

Proof. We assume that C is nonsingular; however, everything we say is valid in the singular case too. We use the following geometric interpretation of the group law. Given two nonsingular points p and q in C the line joining them intersects the curve at the point equal to $-(p \oplus q)$. Also, for any point p its negative $-p$ is the third point of intersection of the line joining p and o with the curve C. This immediately implies that the sum $p \oplus q$ is the unique point r such that there exists a rational function on C with divisor is equal to $p + q - r - o$. By induction, this implies that $p_1 \oplus \ldots \oplus p_n$ is the unique point r such that there exists a rational function f on C whose divisor $\operatorname{div}(f)$ is equal to $p_1 + \cdots + p_n - r - (n-1)o$. Conversely, suppose such f exists. Let $r' = p_1 \oplus \cdots \oplus p_n$. By the above there exists a rational function g such that $\operatorname{div}(g) = p_1 + \cdots + p_n - r' - (n-1)o$. But then $\operatorname{div}(f/g) = r' - r$. This implies that $r = r'$ (otherwise the rational map from C to $\mathbb{P}^1$ defined by the function f is an isomorphism).

In particular, we obtain that $p_1 \oplus \cdots \oplus p_n$ is an m-torsion element if and only if $m(p_1 + \cdots + p_n) - mno$ is the divisor of a rational function. Let us now take $n = 9$. Assume that there exists a polynomial G_{3m} as in the statement of the lemma. Let $L = 0$ be the equation of the inflection tangent at the point o. Then the restriction of the rational function G_{3m}/L^{3m} on $\mathbb{P}^2$ to the curve C defines a rational function f with $\operatorname{div}(f) = m(p_1 + \cdots + p_9) - 9mo$. Thus $p_1 \oplus \cdots \oplus p_m$ is an m-torsion element in the group law. Conversely, assume that the latter occurs. By the above there exists a rational function f with $\operatorname{div}(f) = m(p_1 + \cdots + p_9) - 9mo$. By changing the projective coordinates if necessary, we may assume that the equation of L is $T_0 = 0$ and that none of the points p_i is the point with projective coordinates $(1, 0, 0)$. Then the rational function f is regular on the affine curve $C \setminus \{T_0 = 0\}$. Hence it can be represented by a polynomial $G'(T_1/T_0, T_2/T_0)$ with nonzero constant term. Homogenizing this polynomial, we obtain a homogeneous polynomial G which is not divisible by T_0 such that the curve $G = 0$ cuts out the divisor $m(p_1 + \cdots + p_9)$. By Bézout's Theorem, the degree of G is equal to $3m$. Note that G is not defined uniquely since we can always add to it a polynomial of the form $F \cdot H$, where H is a homogeneous polynomial of degree $3m-3$. The rational function $(G + F \cdot H)/T_0^{2m}$ cuts out the same divisor on C. Now we have to show that H can be chosen in such a way that G has multiplicity m at each point p_i. Let $\mathcal{O}_i$ be the local ring of $\mathbb{P}^2$ at the point p_i and let $\mathfrak{m}_i$ be its maximal ideal. Since C was assumed to be nonsingular, one can find a system of generators x, y of $\mathfrak{m}_i$ such that $x = 0$ is a local equation of C at p_i. We shall identify the formal completion $\hat{\mathcal{O}}_i$ of $\mathcal{O}_i$ with the ring of formal power series $k[[X, Y]]$ in such a way that under the inclusion $\mathcal{O}_i \subset \hat{\mathcal{O}}_i$ the image of x is equal to X and the image of y is equal to Y. Let

$$g = \sum_{n=0}^{\infty} g_n(X,Y)$$

be the Taylor expansion of the rational function G/T_0^{3m}, where $g_n(X,Y)$ is a homogeneous form of degree n in X,Y. We denote by $[g]_s$ the sth Taylor polynomial $\sum_{n=0}^{s} g_n(X,Y)$. The polynomial G has multiplicity $\geq m$ at p_i if and only if $[g]_{m-1} = 0$. The local ring $\mathcal{O}_i/(x)$ is isomorphic to the local ring $\mathcal{O}_{C,p_i}$ of C at p_i, and its completion $\hat{\mathcal{O}}_{C,p_i}$ is isomorphic to $k[[Y]]$. The image $\bar{g}$ of g in $\hat{\mathcal{O}}_{C,p_i}$ is equal to $\sum_{n=0}^{\infty} g_n(0,Y)$ and the fact that the order of the restriction of G/T_0^{3m} to C at p_i is equal to m gives that $g_n(0,Y) = 0, n < m$. This implies that

$$[g]_{m-1} = Xh_i(X,Y)$$

for some polynomial $h_i(X,Y)$ of degree $\leq m-2$. Now consider the k-linear map

$$\phi : k[T_0,T_1,T_2]_{3m-3} \to \bigoplus_{i=1}^{9} k[X,Y]_{\leq m-2}$$

which assigns to a homogeneous polynomial H of degree $3m-3$ the element $(u_1,\ldots,u_9)$, where u_i is the $(m-2)$th Taylor polynomial of the rational function H/T_0^{3m-3} at the point p_i. We claim that this map is surjective. Computing the dimensions of both spaces we find that

$$\dim k[T_0,T_1,T_2]_{3m-3} - \dim \bigoplus_{i=1}^{9} k[X,Y]_{\leq m-2} = \binom{3m-1}{2} - 9\binom{m}{2} = 1.$$

Thus it suffices to show that the kernel of the map is one-dimensional. An element in the kernel defines a homogeneous polynomial H of degree $3m-3$ which has multiplicity $\geq m-1$ at each point p_i. Since we assume that the order of the sum of the points is exactly m, the polynomial H must vanish on C. Dividing H by F and continuing the argument, we see that $H = cF^{m-1}$ for some $c \in k$. This proves the surjectivity. Now, it remains to choose H in such a way that its image under ϕ is equal to $(h_1,\ldots,h_9)$. Then the $(m-1)$th Taylor expansion of $(G-FH)/T_0^{3m}$ at p_i is equal to $[g]_{m-1} - Xh_i = 0$. Thus $G-FH$ has multiplicity m at each point p_i. $\square$

Remark 4.1. Let $G_{3m} = 0$ be the equation of the curve D cutting out the divisor $m(p_1 + \cdots + p_9)$. Let $F = 0$ be the equation of C. For any $\lambda, \mu \in k$, the polynomial $\lambda G_{3m} + \mu F^m$ defines a curve $D(\lambda, \mu)$ which cuts out the same divisor $m(p_1 + \cdots + p_9)$ on C. When m is equal to the order of the point $p_1 \oplus \cdots \oplus p_9$, the "pencil" of curves $D(\lambda, \mu)$ is called the *Halphen pencil* of index m (see [15], Chapter 5). One can show that its general member is an irreducible curve with m-tuple points at $p_1, \ldots, p_9$. The genus of its normalization is equal to 1.

Lemma 4.5. *Let $p_1, \ldots, p_9$ be nine distinct nonsingular points on an irreducible plane cubic $C : F = 0$. Assume that their sum in the group law is not a torsion element.*

(i) A homogeneous polynomial G of degree $\leq 3m$ which has multiplicity $\geq m$ at each point p_i is divisible by F^m.

(ii) The dimension of the space V_d of homogeneous polynomials of degree $d \geq 3m$ which have multiplicity $\geq m$ at each p_i is equal to $\binom{d+2}{2} - 9\binom{m+1}{2}$.

Proof. Assume G is not divisible by F. By Bézout's Theorem, $\deg G = 3m$. Now this contradicts Lemma 4.4, so we may write $G = FG'$ for some homogeneous polynomial of degree $3m - 3$. Clearly, the multiplicity of G' at each p_i is equal to $m - 1$. Applying the lemma again, we find that the sum of the p_i in the group law is a torsion element unless F divides G'. Continuing in this way we find that F^m divides G. This proves the first assertion.

Let us prove the second one. We may assume that all the points p_i lie in the affine part $T_0 \neq 0$. Consider the linear functions $\phi_i^j, i = 1, \ldots, 9, j = 1, \ldots, \binom{m+1}{2}$, on the space of homogeneous polynomials $k[T_0, T_1, T_2]_d$ of degree d which assign to a polynomial P the partial derivatives of order $\leq m$ of the dehomogenized polynomial P/T_0^d at the point p_i, $i = 1, \ldots, 9$. Obviously, V_d is the space of common zeros of the functions ϕ_i^j. To check assertion (ii) it suffices to show that the functions ϕ_i^j are linearly independent. The subspace of common zeros of the restriction of these functions to the space V_d' formed by the polynomials $T_0^{d-3m}G$, where $G \in k[T_0, T_1, T_2]_{3m}$, is of dimension 1 (by (i) it consists of polynomials proportional to F^m, where $F = 0$ is the curve C). Since $\binom{3m+2}{2} - 9\binom{m+1}{2} = 1$, the restriction of the functions ϕ_i^j to V_d' is a linearly independent set. Therefore the functions ϕ_i^j are linearly independent. □

Now we are ready to prove Theorem 4.3.

Proof. We take $n = 9$ and in the equations (4.1) we take (a_{1i}, a_{2i}, a_{3i}) to be the coordinates of the points p_i which lie in the nonsingular part of an irreducible plane cubic C and which do not add up to an m-torsion point for any $m > 0$. Also, to satisfy Lemma 4.2, we assume that the first three points do not lie on a line. This can always be arranged unless $\text{char}(k) > 0$ and C is a cuspidal cubic. Assume that $k[X,Y]^G$ is finitely generated. By Lemma 4.3, we can find a generating set of the form $F_{n_j}/T^{m_j}, j = 1, \ldots, N$, where F_{n_j} is a polynomial of some degree n_j which has multiplicity m_j at the points $p_1, \ldots, p_9$. By Lemma 4.5(i), $n_j \geq 3m_j$. Choose m larger than every m_j and prime to $\text{char}(k)$. By Lemma 4.5(ii), the dimension of the space V_{3m+1} of polynomials of degree $3m+1$ which have multiplicity $\geq m$ at each p_i is equal to $\binom{3m+3}{2} - 9\binom{m+1}{2} = 3m + 3$. On the other hand the dimension of the subspace of polynomials in V_{3m+1} which vanish on C is equal to $\binom{3m}{2} - 9\binom{m}{2} = 3m$. Thus there exists a polynomial $F \in V_{3m+1}$ which does not vanish on the curve C. Let us show that F/T^m cannot be expressed as a polynomial in F_{n_j}/T^{m_j}. Consider any monomial $U_1^{d_1} \cdots U_N^{d_N}$. After we replace U_j with F_{n_j}/T^{m_j}, its degree in Z_1, Z_2, Z_3 is equal to $\sum n_j d_j$ and its degree in T is equal to $\sum m_j d_j$ (here we use that Z_1, Z_2, Z_3, T are algebraically independent). Suppose our monomial enters into a polynomial expression of F/T^m in the generators F_{n_j}/T^{m_j}. Then $3m + 1 = \sum n_j d_j, m = \sum m_j d_j$. Thus

$$\sum_j (n_j - 3m_j)d_j = 1.$$

Since F does not vanish on C, we may assume that $d_j = 0$ if $n_j = 3m_j$ (in this case $F_{n_j} = 0$ defines C). Thus $n_j > 3m_j$ for all j with $d_j \neq 0$, and we get that the only possible case is $d_j = 1$, $n_j = 3m_j + 1$ for one j and all other d_j are equal to zero. Thus $m = \sum m_j d_k = m_j$ for some j. This contradicts the choice of m. $\square$

Remark 4.2. If we take C to be the cuspidal cubic $T_2^2 T_0 - T_1^3 = 0$ over a field of zero characteristic, and the points $p_i = (a_i^3, a_i, 1)$ with the first three points not on a line, then the conditions on p_i will always be satisfied unless $\sum_{i=1}^9 a_i = 0$. In fact, the group law on C° has no nonzero torsion points.

Remark 4.3. If we restrict the action only to the group $G' \cong \mathbb{G}_a^6$ (not including the torus), the algebra of invariants is also not finitely generated. This follows from Nagata's Theorem since the torus is a reductive group. One may ask what is the smallest r such that there exists a rational action of $\mathbb{G}_a^r$ on a polynomial algebra for which the algebra of invariants is not finitely generated. Recall that by Weitzenböck's Theorem, $r > 1$. Examples with $r = 3$ and 4 were given recently by S. Mukai ([71]).

Finally we sketch Nagata's original proof of Theorem 4.3, which leads to a very interesting conjecture on plane algebraic curves. We keep the previous notations.

Lemma 4.6. *For any homogeneous ideal $I \subset k[Z_1, Z_2, Z_3]$ let $\deg(I)$ denote the smallest positive integer d such that $I \cap k[Z_1, Z_2, Z_3]_d \neq \{0\}$. Assume that n is chosen to be such that $\deg(R(m)) > m\sqrt{n}$ for all $m > 0$. Then for any natural number m there exists a natural number N such that $R(m)^N \neq R(mN)$.*

Proof. Let $R(m)_d = k[Z_1, Z_2, Z_3]_d \cap R(m)$ be the space of homogeneous polynomials of degree d in $R(m)$. As we explained in the proof of Lemma 4.5, the dimension of this space is greater than or equal to $(d+2)(d+1)/2 - n(m+1)m/2$. Thus we see that $\overline{\lim}_{m\to\infty}(\deg(R(m))/m) \leq \sqrt{n}$. In view of our assumption we must have $\overline{\lim}_{m\to\infty}(\deg(R(m))/m) = \sqrt{n}$. Since again by assumption $\deg(R(m))/m > \sqrt{n}$ we see that for sufficiently large N,

$$\deg R(mN) \leq mN\sqrt{n} < N\deg(R(m)) = \deg R(m)^N.$$

This implies that $R(mN)$ is strictly larger than $R(m)^N$. □

Lemma 4.7. *The assumptions of the previous lemma are satisfied when $n = s^2$ where $s \geq 4$ and the coordinates of the points p_i generate a field of sufficiently high transcendence degree over k.*

For the proof we refer to [80]. It is rather hard.

Let us show that the four preceding lemmas imply the assertion. Assume that the algebra $k[X, Y]^G$ is generated by finitely many polynomials $P_i(X, Y)$. We can write them in the form $P_i = \sum_m F_{i,m}T^{-m}$ as in Lemma 4.3. Let $r = \max_{i,m}\{\deg F_{i,m}\}$. By Lemma 4.6, we can find $F \in R(rN)$ for sufficiently large N such that $F \notin R(r)^N$. Obviously $P = FT^{-rN}$ cannot be expressed as a polynomial in the P_i. This contradiction proves the assertion.

The assumption that $n = s^2$ was crucial in Lemma 4.7. The following conjecture of Nagata is still unsolved.

Conjecture. *Let $p_1, \ldots, p_n$ be $n \geq 9$ general points in projective plane. Let C be a plane curve of degree d which passes through each P_i with multiplicity m_i. Then*

$$d\sqrt{n} \geq \sum_{i=1}^{n} m_i.$$

Here "n general points" means that the sets of points $(P_1, \ldots, P_n)$ for which the assertion in the conjecture may be wrong form a proper closed subset in $(\mathbb{P}^2)^n$.

Bibliographical notes

The relationship between Hilbert's Fourteenth Problem and the Zariski Problem is discussed in [73]. The material about Grosshans subgroups was taken from [41], see also [91]. The original proof of the Weitzenböck Theorem can be found in [122]. The case char$(k) \neq 0$ is discussed in a paper of A. Fauntleroy [29]. The original example of Nagata can be found in [79] (see also [78]). We follow R. Steinberg ([114]) who was able to simplify essentially the geometric part of Nagata's proof. The group law on an irreducible singular plane cubic is discussed in [46], Examples 6.10.2, 6.11.4 and Exercises 6.6, 6.7.

An essentially new example of a linear action with algebra of invariants not finitely generated can be found in [1]. It is based on an example of P. Roberts ([94]). Nagata's conjecture on plane algebraic curves has not yet been proved. It has inspired a lot of research in algebraic geometry (see [45] and references there). It has also an interesting connection with the problem of symplectic sphere packings (see [68]). It implies that the symplectic 4-ball of radius 1 and volume 1 contains n disjoint symplectically embedded 4-balls of total volume arbitrarily close to 1.

Exercises

4.1 Prove that the additive group $\mathbb{G}_a$ is not geometrically reductive.

4.2 Let $D_1, \ldots, D_n$ be divisors on a nonsingular variety X. Consider the algebra

$$R^*(D_1, \ldots, D_n) = \oplus_{k_1,\ldots,k_n \geq 0} \Gamma(X, L_{k_1 D_1 + \cdots + k_n D_n}).$$

(i) Show that the algebra $R^*(D_1, \ldots, D_n)$ is isomorphic to the algebra $R^*(D)$ for some divisor D on some projective bundle over X.

(ii) Let $S = \{(k_1, \ldots, k_n) \in \mathbb{Z}^n : \Gamma(X, L_{k_1 D_1 + \cdots + k_n D_n}) \neq \{0\}\}$. Show that S is a finitely generated semigroup if $R^*(D_1, \ldots, D_n)$ is finitely generated.

(iii) Let X be a nonsingular projective curve of genus 1, let $a, b \in X$ be two points such that the divisor class of $a - b$ is not a torsion element in the group of divisor classes on X. Prove that $R^*(a, b)$ is not finitely generated.

4.3 Show that the algebra constructed in Nagata's counterexample is isomorphic to the algebra $R^*(l, -E)$ where l is the inverse image of a line under the blow-up of n points in the projective plane and E is the exceptional divisor.

4.4 Prove that the algebra $R^*(D)$ is finitely generated if there exists a positive number N such that the complete linear system defined by the line bundle $L_D^{\otimes N}$ has no base points.

4.5 Show that the algebra of regular functions on the coset space G/H is isomorphic to the subalgebra $\mathcal{O}(G)^H$ where H acts on G by left multiplication.

4.6 Let H be a closed reductive subgroup of an affine algebraic group G which acts on G by left translations. Show that the homogeneous space G/H is an affine variety.

4.7 Write explicitly the group law on the set of nonsingular points of a nodal cubic over a field of characteristic different from 2.

4.8 Show that the conjecture of Nagata is not true without the assumption $n \geq 9$.

Chapter 5

Algebra of covariants

5.1 Examples of covariants

Let $G = \mathrm{SL}_n$ act on an affine algebraic variety $X = \mathrm{Spm}(A)$. Let U be its subgroup of upper triangular unipotent matrices. In this chapter we shall give a geometric interpretation of the algebra of invariants A^U. Its elements are called *semiinvariants*.

Suppose $G = \mathrm{SL}(V)$ acts linearly on a vector space W. Fix a nonzero vector v_0 in V and let H be the stabilizer of v_0 in G. Let $R \in \mathrm{Pol}(W)^H$. For any $v \in V \setminus \{0\}$ there exists $g \in \mathrm{SL}(V)$ such that $g \cdot v = v_0$. Define a function F_R on $W \times V \setminus \{0\}$ by

$$F_R(w, v) = R(g \cdot w). \tag{5.1}$$

Since $g^{-1} \cdot v_0 = g'^{-1} \cdot v_0$ implies $g' g^{-1}(v_0) = v_0$ and hence $g' = hg$ for some $h \in H$, we have

$$R(g' \cdot w) = R(hg \cdot w) = R(g \cdot w).$$

This shows that this definition does not depend on the choice of g and that the function F_R is well-defined. Also, for any $g' \in \mathrm{SL}(V)$ we have $(gg'^{-1})g' \cdot v = v_0$ and hence

$$F_R(g' \cdot w, g' \cdot v) = R(gg'^{-1} \cdot (g' \cdot w)) = R(g \cdot w) = F_R(w, v).$$

Therefore F_R is invariant under the natural *diagonal action* of G on $W \times V$:

$$g(w, v) = (g \cdot w, g \cdot v).$$

It is clear that F_R is a polynomial function in the first argument. Moreover, if R is homogeneous of degree m, then F_R is homogeneous of degree m in the first variable. Let us see that F_R is also polynomial in the second argument. Choose coordinates to assume that $v_0 = (1, \ldots, 0)$. Let $v = (x_0, \ldots, x_n) \in V \setminus \{0\}$. Assume $x_0 \neq 0$. Let

$$A = \begin{pmatrix} x_0 & 0 & \ldots & \ldots & 0 \\ x_1 & x_0^{-1} & 0 & \ldots & 0 \\ x_2 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ x_n & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

Clearly, A belongs to SL(V) and $A \cdot v_0 = v$. Thus $A^{-1}v = v_0$ and $F_R(w, v) = R(A^{-1} \cdot w)$ is a regular function on the open set $x_0 \neq 0$. Similarly we see that F_R is regular on the open set $x_i \neq 0$. Thus F_R is a rational function which is regular on $V \setminus \{0\}$. Hence it is regular on the whole of V and so is a polynomial function.

Conversely, if F is a G-invariant polynomial function on $W \times V$, then the function $w \mapsto F(w, v_0)$ is an H-invariant polynomial function on W. It is easy to see that this establishes an isomorphism of vector spaces:

$$\text{Pol}(W)^H \cong (\text{Pol}(W) \otimes \text{Pol}(V))^{\text{SL}(V)}.$$

Note that the space Pol$(W) \otimes$ Pol(V) has a natural bigrading, so that

$$\text{Pol}_m(W)^H \cong \bigoplus_{p=0}^{\infty} (\text{Pol}_m(W) \otimes \text{Pol}_p(V))^{\text{SL}(V)}.$$

Let us specialize this construction by taking $W = \text{Pol}_d(V)$.

Definition. A *covariant* of degree m and order p on the space Pol$_d(V)$ is an element of the space $(\text{Pol}_m(\text{Pol}_d(V)) \otimes \text{Pol}_p(V))^{\text{SL}(V)}$. We shall denote this space by Cov$(V)_{m,p}(d)$.

The geometric meaning of a covariant $F(a, v) \in \text{Cov}(V)_{m,p}(d)$ is very simple. It can be considered as a polynomial map of affine spaces

$$F : \text{Pol}_d(V) \to \text{Pol}_p(V)$$

given by homogeneous polynomials of degree m. This map is SL(V)-equivariant with respect to the natural actions of SL(V) on the domain and the target space.

In coordinates:

$$F\Big(\sum_{|\mathbf{i}|=d} \tbinom{d}{\mathbf{i}} a_{\mathbf{i}} x^{\mathbf{i}}\Big) = \sum_{|\mathbf{j}|=p} \tbinom{p}{\mathbf{j}} A_{\mathbf{j}} x^{\mathbf{j}},$$

where $A_{\mathbf{j}}$ are homogeneous polynomials of degree m in the coefficients $a_{\mathbf{i}}$.

On can easily define the symbolic expression of covariants. By polarizing, an element of $\mathrm{Cov}_{m,p}(d) = (\mathrm{Pol}_m(\mathrm{Pol}_d(V)) \otimes \mathrm{Pol}_p(V))^{\mathrm{SL}(V)}$ becomes an $\mathrm{SL}(V)$-invariant polynomial function on the space of matrices $\mathrm{Mat}_{r,m+1}$ which is homogeneous of degree d in each column different from the last one, and is homogeneous of degree p in the last column. Observe that each of the first m columns corresponds to a basis $(\xi_0^{(j)}, \ldots, \xi_n^{(j)})$ in V. The last one consists of the coordinates $(x_0, \ldots, x_n)$ with respect to this basis. There is an analog of the First Fundamental Theorem which says that one can write this function as a linear combination of products of r-minors taken from the first m columns and dot-products of the last column with one of the first m columns. In each product, each column, except the last one, appears d times, and the last column appears p times. This implies that the number of minors in each product must be equal to

$$w = \max\Big\{\frac{md}{r} - p, 0\Big\}.$$

This number is called the *weight* of a covariant. It has the property that

$$F(g \cdot a, g \cdot v) = (\det g)^w F(a, v), \quad \forall g \in \mathrm{GL}(V).$$

The symbolic expression for the products is

$$\prod_{i=1}^{w} (\tau_{i1} \ldots \tau_{ir}) \prod_{j=1}^{p} \alpha_x^{(s_j)},$$

where $\alpha_x^{(j)} = \sum_{i=0}^n \xi_i^{(j)} x_j$. Here each $\tau_{ij}, s_j \in [m]$, and each number from $[m]$ occur exactly d times among them.

Example 5.1. An invariant of degree m is a covariant of degree m and of order 0.

Example 5.2. The identity map $\mathrm{Pol}_d(V) \to \mathrm{Pol}_d(V)$ is a covariant of degree 1 and order d. Its weight is equal to zero. Its symbolic expression is α_x^d.

Example 5.3. Let $F(x_0, \ldots, x_n) \in \mathrm{Pol}_d(k^r) = k[x_0, \ldots, x_n]_d$. Let

$$F_{ab} = \frac{\partial^2 F}{\partial x_a \partial x_b}, \quad a, b = 0, \ldots, n.$$

The *Hessian* of F is the determinant

$$\mathrm{Hess}(F) = \det \begin{pmatrix} F_{00} & \dots & F_{0n} \\ \vdots & \ddots & \vdots \\ F_{n0} & \dots & F_{nn} \end{pmatrix}. \tag{5.2}$$

The map Hess : $F \mapsto \mathrm{Hess}(F)$ is a covariant of degree r and order $(d-2)r$. Its symbolic expression is

$$\mathrm{Hess}(F) = (d(d-1))^r (12\dots r)^2 (\alpha_x^{(1)} \dots \alpha_x^{(r)})^{d-2}.$$

We leave it to the reader to check this.

More generally, let (x_{ij}) be the square matrix with entries x_{ij} considered as a variables. Take F as above and consider the product $\prod_{i=0}^n F(x_{i1}, \dots, x_{in})$ as a polynomial function on $\mathrm{Mat}_r(k)$. Define the rth *transvectant* as

$$(F)^{(r)} = \Omega^r \Big(\prod_{i=0}^n F(x_{i1}, \dots, x_{in})\Big)\big|_{x_{ij}=x_j}$$

where Ω is the omega-operator. The last subscript means that we have to replace each unknown x_{ij} with x_j. The map $T^r : F \to (F)^{(r)}$ is a covariant of degree r and order $r(m-r)$. For example,

$$T^0(F) = F^r, \quad T^1(F) = 0, \quad T^2(F) = \mathrm{Hess}(F).$$

Example 5.4. One can combine covariants and invariants to get an invariant. For example, consider the Hessian of a binary cubic. It is a binary quadric. Take its discriminant. The result must be an invariant of degree 4; let us compute it. If $F = a_0x_0^3 + 3a_1x_0^2x_1 + 3a_2x_0x_1^2 + a_3x_1^3$ we have

$$\begin{aligned} \mathrm{Hess}(F) &= \det \begin{pmatrix} 6a_0x_0 + 6a_1x_1 & 6a_1x_0 + 6a_2x_1 \\ \\ 6a_1x_0 + 6a_2x_1 & 6a_3x_1 + 6a_2x_0 \end{pmatrix}, \\ \mathrm{Discr}(\mathrm{Hess}(F)) &= 36(a_0x_0 + a_1x_1)(a_3x_1 + a_2x_0) - (a_1x_0 + a_2x_1)^2 \\ &= 36((a_0a_3 - a_1a_2)^2 - 4(a_0a_2 - a_1^2)(a_1a_3 - a_2^2)) \\ &= 36(-6a_0a_1a_2a_3 + a_0^2a_3^2 + 4a_1^3a_2 + 4a_0a_2^3 - 3a_1^2a_2^2). \end{aligned}$$

This is (up to a constant factor) the discriminant of the binary cubic form from Chapter 2, Example 2.1.

Example 5.5. For any two binary forms $F \in \mathrm{Pol}_d(k^2), G \in \mathrm{Pol}_{d'}(k^2)$ define their *Jacobian*

$$J(F,G) = \det \begin{pmatrix} \frac{\partial F}{\partial x_0} & \frac{\partial F}{\partial x_1} \\ \\ \frac{\partial G}{\partial x_0} & \frac{\partial G}{\partial x_1} \end{pmatrix}.$$

Then $F \to J(F, \mathrm{Hess}(F))$ is a covariant of degree 3 and order $3(d-2)$.

5.2 Covariants of an action

The notion of a covariant of a homogeneous form is a special case of the notion of a covariant of an arbitrary rational action of an affine algebraic group G on an affine variety $X = \mathrm{Spm}(A)$. Let $\rho : G \to \mathrm{GL}(W)$ be a linear representation of G in a finite-dimensional vector space W. We call W a G-module. A *covariant of an action* with values in W is an equivariant regular map $X \to W$, where W is considered as an affine space. Equivalently, it is a G-equivariant homomorphism of algebras $\mathrm{Pol}(W) \to A$. Since any such homomorphism is determined by the images of the unknowns, it is defined by a linear map $f : W^* \to A$. Let $\mathrm{Hom}(W^*, A) = A \otimes W$ be the set of such maps. The group G acts by the formula

$$g \cdot (a \otimes w) = g^*(a) \otimes \rho(g)(w).$$

This corresponds to the action on morphisms $X \to W$ given by the formula

$$g \cdot f(x) = \rho(g)(f(g^{-1} \cdot x)).$$

A covariant is an invariant element of this space. In the previous section we considered the case $G = \mathrm{SL}(V)$, $X = \mathrm{Pol}_d(V) = \mathbb{A}^{\binom{n+d}{d}}$ and $W = \mathrm{Pol}_k(V)$ with the natural representation of $\mathrm{SL}(V)$. If we take $W = \mathrm{Pol}_k(V^*)$ with the natural action of G on the space of linear functions, we obtain the notion of a *contravariant* of order k on the space $\mathrm{Pol}_d(V)$. Another special case is when $A = \mathrm{Pol}_{d_1}(V) \otimes \cdots \otimes \mathrm{Pol}_{d_s}(V)$ and $W = \mathrm{Pol}_p(V)$. In this case a covariant is called a *concomitant* of order p. A concomitant of order 0 is called a *combinant*. For example, the *resultant* $R(F_1, \ldots, F_s)$ of s homogeneous polynomials is a combinant.

Let $\mathrm{Hom}(W^*, A)^G = (W \otimes A)^G$ be the set of covariants with values in a G-module W. It has an obvious structure of an A^G-module. It is called the *module of* *covariants* with values in W. If char$(k) = 0$, we can identify the spaces $\mathrm{Pol}_m(W^*)$ and $\mathrm{Pol}(W)^*$ so that the direct sum

$$\begin{aligned}\mathrm{Cov}(G; A, W) &= \bigoplus_{m=0}^{\infty} \mathrm{Hom}(\mathrm{Pol}_m(W), A)^G \\ = \bigoplus_{m=0}^{\infty} (\mathrm{Pol}_m(W^*) \otimes A)^G &= (\mathrm{Pol}(W^*) \otimes A)^G\end{aligned}$$

has a natural structure of a k-algebra. It is called the *algebra of covariants*. Applying Nagata's Theorem we obtain

Theorem 5.1. *Assume G is a geometrically reductive group. Then the algebra of covariants* $\mathrm{Cov}(G; A, W)$ *is a finitely generated k-algebra.*

Corollary 5.1. *Suppose G is a geometrically reductive algebraic group acting rationally on $X = \mathrm{Spm}(A)$. Then the module of covariants* $\mathrm{Hom}(W^*, A)^G$ *is finitely generated.*

Proof. The algebra $\mathrm{Cov}(G; A, W)$ is a graded finitely generated k-algebra. We identify A^G with the subalgebra of covariants $\mathrm{Cov}(G; A, k)$, where k is the trivial G-module. Obviously $\mathrm{Cov}(G; A, W)$ is a finitely generated A^G-algebra. We may assume that it is generated by a finite set of homogeneous elements $F_1, \ldots, F_n$ of positive degrees $q_1, \ldots, q_n$. Thus there is a surjective homomorphism of graded A^G-algebras $A^G[T_1, \ldots, T_n] \to \mathrm{Cov}(G; A, W)$, where $\deg T_i = q_i$. Since each $A^G[T_1, \ldots, T_n]_m$ is a finite free A^G-module, its image $\mathrm{Cov}(G; A, W)_m$ is a finitely generated A^G-module; hence $\mathrm{Hom}(W^*, A)^G = \mathrm{Cov}(G; A, W)_1$ is finitely generated. $\square$

Here is another proof of this result in the case when G is linearly reductive, for example when G is reductive over $\mathbb{C}$. We use that any rational linear linear representation of G in a finite-dimensional linear space E is *completely reducible* in the following sense.

Theorem 5.2. *Any submodule V of E admits a complementary submodule V' (i.e., $E = V \oplus V'$).*

Proof. Without loss of generality, we may assume that $V \neq \{0\}$ and is an irreducible submodule, i.e., V does not contain any nontrivial proper submodules. Consider the natural map of G-modules $f : \mathrm{Hom}_k(E, V) \to \mathrm{Hom}_k(V, V)$. By

Schur's Lemma, the subspace $L = \mathrm{Hom}_k(V,V)^G$ is one-dimensional. Its inverse image $W = f^{-1}(L)$ is a submodule of $\mathrm{Hom}_k(E,V)$, and the restriction f' of f to W is a nonzero linear G-invariant function. By definition of linear reductivity there exists a nonzero G-invariant vector $\phi \in W$ such that $f'(\phi) \neq 0$. The linear map $\phi : E \to V$ is G-invariant and its restriction to V is a nonzero automorphism of irreducible G-module V. The kernel of the G-invariant linear map ϕ is the desired complementary subspace of V. $\square$

Let $M = W \otimes A$ and let M' be the A-submodule of M generated by invariant elements. Since A is noetherian and $W \otimes A$ is a free A-module of finite rank, M^G is a finitely generated A-module. Let $m_1, \ldots, m_n \in M^G$ be its spanning set. For any $m \in M^G$ we can write

$$m = a_1 m_1 + \cdots + a_n m_n \tag{5.3}$$

for some $a_i \in A$. Since G is linearly reductive the G-submodule A^G of A has a complementary invariant submodule, i.e., $A = A^G \oplus N$. This does not follows directly from Theorem 5.2 because A is infinite-dimensional. One uses Lemma 3.4 to show that A is the union of finite-dimensional invariant subspaces, and then applies Theorem 5.2. Let

$$R : A \to A^G$$

be the projection operator (called the *Reynolds operator*). It has the property

$$R(ab) = bR(a), \quad \forall a \in A, \forall b \in A^G.$$

In the case $k = \mathbb{C}$ we take for R the averaging operator over the compact form of G. Let $\widetilde{R} : M^G \otimes_{A^G} A \to M^G$ be the map defined by $m \otimes a \mapsto R(a)m$. By (5.3), M^G is equal to the image under $\widetilde{R}$ of the finitely generated A^G-module $\sum_{i=1}^n A^G m_i$ and hence it is finitely generated.

Let $\rho : G \to \mathrm{GL}(W)$ be a finite-dimensional linear rational representation of a linearly reductive group. By Theorem 5.2, W can be decomposed into a direct sum of irreducible representations W_i. When G is finite, there are only finitely many irreducible representations (up to isomorphism); in general G has infinitely many nonisomorphic irreducible representations. Let $W = \oplus_{i=1}^n W_i$ be a decomposition of W into a direct sum of irreducible representations. We have an isomorphism of G-modules:

$$W \cong \bigoplus_{\rho \in \mathrm{Irr}(G)} \mathrm{Hom}(W_\rho, W)^G \otimes W_\rho, \tag{5.4}$$

where $\mathrm{Irr}(G)$ is the set of isomorphism classes of finite-dimensional irreducible G-modules, W_ρ is a representative of the class ρ, and G acts trivially on the space of linear maps $\mathrm{Hom}(W_\rho, W)^G$. This isomorphism is defined by the map

$$\sum_\rho f_\rho \otimes w_\rho \mapsto \sum_\rho f_\rho(w_\rho), \quad f_\rho \in \mathrm{Hom}(W_\rho, W)^G, w_\rho \in W_\rho.$$

Note that, by Schur's Lemma, when $W = W_\rho$ this gives $\mathrm{Hom}(W_\rho, W_{\rho'})^G = k$ if $\rho = \rho'$ and $\{0\}$ otherwise. The dimension of the space $\mathrm{Hom}(W_\rho, W)^G$ is called the *multiplicity* of W_ρ in W and is denoted by $\mathrm{mult}_\rho(W)$. It is equal to the number of direct irreducible summands (or factors) of W isomorphic to W_ρ.

Recall that any element of A is contained in a finite-dimensional G-invariant subspace of A generated by its G-translates (see Lemma 3.4). This allows us to apply (5.4) to the G-module A. We have

$$A \cong \bigoplus_{\rho \in \mathrm{Irr}(G)} \mathrm{Hom}(W_\rho, A)^G \otimes W_\rho. \tag{5.5}$$

We consider both sides as A^G-modules. By Corollary 5.1 each summand is a finitely generated A^G-module. Thus we see that any module of covariants for A is contained in A as a direct summand.

Example 5.6. Let G be a finite abelian group of order prime to $\mathrm{char}(k)$. Then any irreducible representation of G is one-dimensional, and hence is defined by a character $\chi : G \to \mathbb{G}_m = \mathrm{GL}_k(1)$. For each χ, let

$$A_\chi = \{a \in A : g \cdot a = \chi(g)a, \ \forall g \in G\}.$$

Then (5.5) translates into the equality

$$A = \bigoplus_{\chi : G \to \mathbb{G}_m} A_\chi.$$

The subring of invariants A^G corresponds to the trivial character.

5.3 Linear representations of reductive groups

Let G be a linearly reductive connected affine algebraic group and let $\rho : G \to \mathrm{GL}(W)$ be its rational linear representation. Let U be a maximal unipotent subgroup of a connected linearly reductive group G. The reader unfamiliar with the notion may assume that $G = \mathrm{GL}_r$ or SL_r, in which case U is a subgroup conjugate to the group of unipotent upper triangularupper triangular matrices. We have seen in section 5.1 that in the case $G = \mathrm{SL}_2$ the algebra $\mathrm{Pol}(\mathrm{Pol}_d(k^2))^U$ is isomorphic to the algebra of covariants $\mathrm{Cov}(G; \mathrm{Pol}_d(k^2), k^2)$. In this section we shall give a similar interpretation of the algebra A^U where G acts rationally on a finitely generated k-algebra A.

For this we have to recall some basic facts about finite-dimensional linear rational representations of a reductive group G. We assume that $\mathrm{char}(k) = 0$. Let $\rho : G \to \mathrm{GL}(W)$ be such a representation. Choose a maximal torus T in G (when $G = \mathrm{GL}_r$ it is a subgroup of diagonal matrices or its conjugate subgroup). Restricting ρ to T we get a linear rational representation $\bar{\rho} : T \to \mathrm{GL}(W)$. Since T is commutative we can decompose W into the direct sum of eigenspaces

$$W = \bigoplus_{\chi \in \mathcal{X}(T)} W_\chi,$$

where $\mathcal{X}(T)$ denotes the set of *rational character* of T, i.e., homomorphisms of algebraic groups $T \to \mathbb{G}_m$, and

$$W_\chi = \{w \in W : \bar{\rho}(t)(w) = \chi(t)w, \quad \forall t \in T\}.$$

Any rational character $\chi : T \to \mathbb{G}_m$ is defined by a homomorphism of the algebras of regular functions

$$k[Z, Z^{-1}] \cong \mathcal{O}(\mathbb{G}_m) \to \mathcal{O}(\mathbb{G}_m^r) \cong k[Z_1, Z_1^{-1}, \ldots, Z_r, Z_r^{-1}].$$

It is easy to see that it is given by a *Laurent monomial* $Z^a = Z_1^{a_1} \cdots Z_r^{a_r}$, where $a = (a_1, \ldots, a_r) \in \mathbb{Z}^r$. The monomial is the image of Z. Also it is easy to see that the product of characters corresponds to the vector sum of the exponents a. This gives us an isomorphism of abelian groups

$$\mathcal{X}(T) \cong \mathbb{Z}^r.$$

Let

$$\mathrm{Wt}(\rho) = \{\chi : W_\chi \neq \{0\}\}. \tag{5.6}$$

Since W is finite-dimensional, $\mathrm{Wt}(\rho)$ is a finite set. It is called the set of *weights* of ρ.

A rational character $\alpha : T \to \mathbb{G}_m^*$ is called a *root* if there exists a nontrivial homomorphism of algebraic groups $f_\alpha : \mathbb{G}_a \to G$ such that, for any $t \in T$ and any $x \in k$,

$$t \cdot f_\alpha(x) \cdot t^{-1} = f_\alpha(\alpha(t)x).$$

For example, there are $r(r-1)$ roots for $G = \mathrm{GL}_r$. Each is defined by the homomorphism which sends $x \in k$ to the matrix $I_r + xE_{ij}$, where $1 \le i, j \le r, i \neq j$.

Let R be the set of roots. There is the notion of a *positive root*. We fix a Borel subgroup B containing T (in the case $G = \mathrm{GL}_r$ we may take B to be the group of upper triangular matrices or its conjugate subgroup) and require that the image of f_α is contained in B. Let R_+ be the set of positive roots. Then $R = R_+ \sqcup R_-$, where $R_- = \{-\alpha, \alpha \in R_+\}$ is the set of *negative roots*. There is a finite set of roots $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ such that any root can be written as a linear combination of the α_i with nonnegative integer coefficients. They are called *simple roots*. The number r is called the *rank* of G. In the case SL_r these are the roots with $f_{\alpha_i}(a) = I_r + aE_{i(i+1)}, i = 1, \ldots, r-1$. Under the isomorphism $\mathcal{X}(T) \cong \mathbb{Z}^n$ they correspond to the vectors $e_i - e_{i+1}$, where $(e_1, \ldots, e_r)$ is the standard basis of $\mathbb{Z}^r$.

Let U_α denote the image of the homomorphism $f_\alpha(\mathbb{G}_a)$ corresponding to a root α. One can show that the subgroups

$$U^+ = \prod_{\alpha \in R_+} U_\alpha, \quad U^- = \prod_{\alpha \in R_-} U_\alpha$$

are maximal unipotent subgroups of G. In the case $G = \mathrm{SL}_r$ the group U^+ (resp. U^-) is the subgroup of upper triangular (resp. lower triangular) matrices.

We have the following.

Lemma 5.1. *Let*

$$W = \bigoplus_{\chi \in \mathrm{Wt}(\rho)} W_\chi.$$

For every root $\alpha \in R$, we have

$$\rho(U_\alpha)(W_\chi) \subset \bigoplus_{i \ge 0} W_{\chi + i\alpha}.$$

Proof. Let $W \to k[T] \otimes W$ be the homomorphism defining the action of U_α on W. For any $w \in W_\chi$ its image is equal to $\sum_{i \ge 0} T^i \otimes w_i$. This means that for any $x \in k$,

$$\rho(f_\alpha(x))(w) = \sum_{i \ge 0} x^i w_i. \tag{5.7}$$

By definition of a root, we have

$$\rho(f_\alpha(\alpha(t)x))(w) = \sum_i \alpha(t)^i x^i w_i$$

and

$$\begin{aligned} \rho(t)\rho(f_\alpha(x))(w) &= \sum_i x^i \rho(t)(w_i) \\ = \rho(f_\alpha(\alpha(t)x)\rho(t))(w) &= \rho(f_\alpha(\alpha(t)x))(\chi(t)w) \\ &= \chi(t)\sum_i \alpha(t)^i x^i w_i. \end{aligned}$$

Comparing the coefficients of x^i we get $w_i \in W_{\chi+i\alpha}$. Thus equation (5.7) gives

$$\rho(U_\alpha)(w) \subset \bigoplus_{i\geq 0} W_{\chi+i\alpha}.$$

$\square$

The set R_+ defines an order on the set of characters. We say that $\chi \geq \chi'$ if $\chi - \chi'$ is equal to a linear combination of positive roots with nonnegative coefficients. Let $\lambda \in \mathrm{Wt}(\rho)$ be a maximal element (not necessary unique) with respect to this order. Then, for any $\alpha \in R_+$, we have $W_{\lambda+i\alpha} = \{0\}$ if $i > 0$. It follows from (5.7) that $\rho(U_\alpha)$ acts identically on W_λ. Thus the whole group U^+ acts identically on W_λ. On the other hand, by Lemma 5.1, we get

$$\rho(U^-)(W_\lambda) \subset \bigoplus_{\chi\leq\lambda} W_\chi.$$

Since $\rho(T)(W_\lambda) = W_\lambda$, all elements $g \in G$ of the form $u^+ \cdot t \cdot u^-$, where $u^\pm \in U^\pm, t \in T$, leave the subspace

$$W(\lambda) = \bigoplus_{\chi\leq\lambda} W_\chi$$

invariant. Since the subset $U^+ \cdot T \cdot U^-$ is Zariski dense in G (check this for $G = \mathrm{SL}_r$ or GL_r, where this set consists of matrices with nonzero pivots), all elements of G leave $W(\lambda)$ invariant. Thus $W(\lambda)$ is a G-submodule. Let $v \in W_\lambda \setminus \{0\}$. Consider the G-submodule $W(\lambda)_v$ generated by v. Obviously it is contained in $W(\lambda)$ and

$$W(\lambda)_v \cap W_\lambda = kv.$$

In fact, U^+ does not change v, T multiplies v by a constant, and U^- sends v to the sum $v + \sum_{\chi<\lambda} v_\chi$, where $v_\chi \in W_\chi$. We consider a complementary subspace to $W(\lambda)_v$ in $W(\lambda)$ and choose again a nonzero vector v' in it to get a submodule $W(\lambda)_{v'}$. Continuing in this way we will decompose $W(\lambda)$ into the direct sum of $\dim W_\lambda$ submodules. Each summand V has the following properties:

(i) there exists a weight λ such that $V = \oplus_{\chi \leq \lambda} V_\chi$,

(ii) $\dim V_\lambda = 1$ (a nonzero vector in V_λ is called a *highest weight vector*),

(iii) $\rho(U^+)|V_\lambda$ is the identity representation.

Such a G-module V is called a *highest weight module*. It is determined uniquely (up to isomorphism) by the character λ (*highest weight*) and is denoted by $L(\lambda)$. Thus we infer from the above discussion the following:

Theorem 5.3. *Every finite-dimensional rational representation of a connected linearly reductive group G is isomorphic to the direct sum of highest weight representations $L(\lambda)$.*

Not every weight χ occurs as a highest weight of some $L(\lambda)$. The ones which occur are called *dominant weights*. This set is preserved under taking the dual module, i.e., $L(\lambda)^* = L(\lambda^*)$ for some dominant weight λ^*. We will describe dominant weights in the next section.

Let us return to the situation when a reductive group G acts regularly on an affine algebraic variety $X = \mathrm{Spm}(A)$. For every dominant weight λ a homomorphism of G-modules $L(\lambda) \to A$ is determined by the image of a fixed highest weight vector of $L(\lambda)$. The set of such images forms an A^G-submodule $A^{(\lambda)}$ of A. We have

$$(L(\lambda) \otimes A)^G = \mathrm{Hom}_k(L(\lambda^*), A)^G = A^{(\lambda^*)}.$$

It is easy to see that, if v is a highest weight vector of $L(\lambda)$ and v' is a highest weight vector of $L(\lambda')$, the vector $v \otimes v'$ is a highest weight vector in an irreducible summand of the representation $L(\lambda) \otimes L(\lambda')$ isomorphic to $L(\lambda + \lambda')$. This easily implies that the subalgebra of the A^G-algebra A generated by the images of highest weight vectors is isomorphic to the direct sum of the A^G-modules $A^{(\lambda)}$, where λ runs through the set of dominant weights. Since U^+ acts identically on any highest weight vector we see that

$$\bigoplus_\lambda A^{(\lambda)} \subset A^{U^+}.$$

Conversely, if $a \in A^{U^+}$, by (5.4) a can be written uniquely as a sum $\sum_\rho a_\rho$, where each a_ρ belongs to an irreducible G-submodule of A. This implies that each a_ρ

is U^+-invariant and hence generates a submodule isomorphic to $L(\lambda)$ for some dominant weight λ. This shows that

$$\bigoplus_{\lambda} A^{(\lambda)} \cong A^{U^+}. \tag{5.8}$$

Since every irreducible representation is isomorphic to some highest weight representation $L(\lambda)$, we can apply (5.4) to obtain an isomorphism of A^G-modules

$$A \cong \bigoplus_{\lambda} \mathrm{Hom}_k(L(\lambda^*), A)^G \otimes L(\lambda^*).$$

This gives

$$A^{U^+} \cong \bigoplus_{\lambda} \mathrm{Hom}_k(L(\lambda^*), A)^G \otimes L(\lambda^*)^{U^+}.$$

It follows from the definition of $L(\lambda)$ that $L(\lambda^*)^{U^+} = L(\lambda)_\lambda$ is spanned by a highest weight vector, and hence is one-dimensional. This gives

$$A^{U^+} \cong \bigoplus_{\lambda} A^{(\lambda)} \otimes_k k \cong \bigoplus_{\lambda} A^{(\lambda)}.$$

We will see a little later that A^{U^+} is a finitely generated algebra.

5.4 Dominant weights

Let us now describe dominant weights. For every root α there is the *dual root* $\check{\alpha}$ which is a homomorphism $\check{\alpha} : \mathbb{G}_m \to T$. It is characterized by the property that, for any $t \in \mathbb{G}_m$ and $x \in \mathbb{G}_a$,

(i) $\check{\alpha}(t) f_\alpha(x) \check{\alpha}^{-1}(t) = f_\alpha(x)$,

(ii) $\alpha \circ \check{\alpha}(t) = t^2$.

For example, when $G = \mathrm{GL}(r)$ and U_α is the subgroup of matrices $I_r + aE_{ij}, a \in k$, where E_{ij} denotes the matrix with 1 as the (ij)th entry and 0 elsewhere, the dual root $\check{\alpha}$ is given by $t \mapsto (I_r + (t-1)E_{ii} + (t^{-1}-1)E_{jj})$.

Note that the composition of a homomorphism $f : \mathbb{G}_m \to T$ (called *a one-parameter subgroup*) of T and a rational character $\chi : T \to \mathbb{G}_m$ can be identified with an integer. We denote it by (f, χ).

Let $\mathcal{X}(T)^*$ be the set of one-parameter subgroups. An element of $\mathcal{X}(T)^*$ is given by a homomorphism of algebras of functions

$$k[Z_1^{\pm 1}, \ldots, Z_r^{\pm 1}] \cong \mathcal{O}(T) \to \mathcal{O}(\mathbb{G}_m) \cong k[Z, Z^{-1}].$$

It is defined by the images of Z_i. Since it defines a homomorphism of groups it is easy to see that the image of each Z_i is a monomial Z^m for some $m \in \mathbb{Z}$. Thus a one-parameter subgroup is given by a vector $m = (m_1, \dots, m_r) \in \mathbb{Z}^r$. Since each one-parameter subgroup takes values in a commutative group, we can multiply them; this of course corresponds to the sum of vectors in $\mathbb{Z}^r$. The composition of a character and a one-parameter subgroup corresponds to the dot-product in $\mathbb{Z}^r$. So it is natural to distinguish the group of characters $\mathcal{X}(T)$ and the group $\mathcal{X}(T)^*$ of one-parameter subgroups by identifying one of them, say $\mathcal{X}(T)$, with $\mathbb{Z}^r$ and the other one with the dual group $\mathrm{Hom}(\mathbb{Z}^r, \mathbb{Z}) = (\mathbb{Z}^r)^*$. Then the pairing (f, χ) from above is equal to $(f, \chi) = f(\chi)$.

A character $\lambda : T \to \mathbb{G}_m$ is called a *dominant weight* if for any positive root α one has $(\breve{\alpha}, \lambda) \geq 0$.

Finally, one defines a *fundamental weight* as an element ω_j of $\mathcal{X}(T) \otimes \mathbb{Q}$ with the property $(\breve{\alpha}_i, \omega_j) = \delta_{ij}$ (the Kronecker symbol). In the case when R spans the group of characters of T (e.g. $G = \mathrm{SL}_r$ but not GL_r), a fundamental weight is uniquely determined by this property. Let $\mathcal{X}(T)_0$ be the subgroup of $\mathcal{X}(T)$ which consists of characters χ such that $(\breve{\alpha}, \chi) = 0$ for all roots $\alpha \in R$. Choose a basis $(\omega_0^{(1)}, \dots, \omega_0^{(k)})$ of $\mathcal{X}(T)_0$ and let $\omega_1, \dots, \omega_r$ be the set of fundamental roots no two of which are congruent modulo the subgroup $\mathcal{X}(T)_0$. Then any dominant weight can be written uniquely in the form

$$\lambda = n_1\omega_1 + \dots + n_r\omega_r + a_1\omega_0^{(1)} + \dots + a_s\omega_0^{(s)}, \tag{5.9}$$

where $n_i \in \mathbb{Z}_{\geq 0}, i = 1, \dots, r, a_i \in \mathbb{Z}, i = 1, \dots, s$.

Any dominant weight λ_0 from $\mathcal{X}(T)_0$ defines a one-dimensional representation $G \to \mathbb{G}_m$. We have $r = \mathrm{rank}(G)$ fundamental representations $L(\omega_i)$ corresponding to the fundamental weights ω_i. If λ is as in (5.9), then $L(\lambda)$ is isomorphic to an irreducible quotient of the tensor product $\otimes_{i=1}^r V(\omega_i)^{\otimes n_i}$ tensored with the one-dimensional representation defined by the vector $\sum_i a_i\omega_0^{(i)}$.

It follows from (5.9) that the semigroup of dominant weights is finitely generated. We will use this to prove the result which we promised earlier:

Theorem 5.4. *Let U be a maximal unipotent group of a reductive group G. Assume that G acts rationally on a finitely generated k-algebra A. Then the subalgebra A^U of U-invariant elements is finitely generated over k.*

Proof. Since all maximal unipotent subgroups are conjugate, we may assume that $U = U^+$. Applying Nagata's Theorem and Lemma 4.1, it suffices to show that $A = \mathcal{O}(G)^{U^+}$ is finitely generated. We know from (5.8) that $A = \oplus_\lambda A^{(\lambda)}$. Any homomorphism of G-modules $\phi : L(\lambda) \to \mathcal{O}(G)^U$ is determined by the linear U-invariant function on $L(\lambda)$ defined by the formula $v \mapsto \phi(v)(e)$. Since $\dim L(\lambda^*)^U = 1$, we obtain that each $L(\lambda)$ enters in $A^{(\lambda)}$ with multiplicity 1, i.e., $\dim A^{(\lambda)} = \dim L(\lambda)$. Thus we may assume that there is an isomorphism of graded algebras $A = \oplus_\lambda L(\lambda)$. Hence A is generated by the subspaces $L(\lambda)$ corresponding to the generators of the semi-group of dominant weights. □

5.5 The Cayley–Sylvester formula

In this section we give an explicit description of irreducible representations for the group GL_r. We choose the maximal torus T which consists of diagonal matrices $\mathrm{diag}(t_1, \ldots, t_r)$. The corresponding Borel subgroup is the group of upper triangular matrices. We have, for any $1 \le i, j \le r, i \ne j$,

$$\mathrm{diag}(t_1, \ldots, t_r)(I_r + xE_{ij})\mathrm{diag}(t_1, \ldots, t_r)^{-1} = I_r + (t_i/t_j)xE_{ij}.$$

This shows that the characters $\alpha_{ij} : \mathrm{diag}[t_1, \ldots, t_r] \mapsto t_i t_j^{-1}$ are roots. Under the isomorphism $\mathcal{X}(T) \cong \mathbb{Z}^r$ each α_{ij} corresponds to the vector $e_i - e_j$. So we have $r(r-1)$ roots. Since $I_r + xE_{ij} \in B$ if and only if $i < j$, we see that R_+ consists of roots α_{ij} with $i < j$. Simple roots are

$$\alpha_i = \alpha_{i(i+1)}, \quad i = 1, \ldots, r.$$

The dual roots are the homomorphisms $\alpha_{ij} : \mathbb{G}_m \to T$ defined by $t \mapsto I_r + (t-1)E_{ii} + (t-1)E_{jj}$. Thus all dual roots can be identified with linear functions $\mathbb{Z}^r \to \mathbb{Z}$ defined by $e_i^* - e_j^*$ where $(e_1^*, \ldots, e_r^*)$ is the dual basis to the standard basis $(e_1, \ldots, e_r)$. A dominant weight $\lambda = (m_1, \ldots, m_r)$ must satisfy

$$\lambda \cdot (e_i - e_{i+1}) \ge 0$$

which translates into the inequalities $m_i \ge m_{i+1}$. There are $r - 1$ fundamental weights

$$\omega_i = e_1 + \cdots + e_i, \quad i = 1, \ldots, r-1,$$

and $\mathcal{X}(T)_0$ is generated by the weight

$$\omega_0 = e_1 + \cdots + e_r.$$

The irreducible representation corresponding to ω_0 is of course the natural representation

$$\det : \mathrm{GL}_r \to \mathbb{G}_m.$$

We have

$$L(d\omega_1) = \mathrm{Pol}_d((k^r)^*).$$

Here the highest weight is the monomial ξ_1^d, where $(\xi_1, \ldots, \xi_r)$ is the standard basis of k^r. All other weights are $\mathbf{i} = (i_1, \ldots, i_r)$ with $i_1 + \cdots + i_r = d$. The corresponding subspace $L(d\omega_1)_{\mathbf{i}}$ is spanned by the monomial $\xi^{\mathbf{i}}$. We can write

$$\begin{aligned}
\mathbf{i} &= de_1 - \sum_{s=1}^{r-1}(d - i_1 - \cdots - i_s)(e_s - e_{s+1}) \\
&= d\omega_1 - (d - i_1)\alpha_1 - \cdots - (d - i_1 - \cdots - i_{r-1})\alpha_n, \\
L(d\omega_i) &= \mathrm{Pol}_d((\Lambda^i k^r)^*), \quad i = 2, \ldots, n.
\end{aligned}$$

Here the highest weight is $(\xi_1 \wedge \cdots \wedge \xi_i)^d$. When $i = r-1$ we get $(\Lambda^{r-1}(k^r))^* \cong (k^r) \otimes \det$ and hence

$$\mathrm{Pol}_d(k^r) = L(d\omega_{r-1}) \otimes \det{}^{-d}.$$

The highest weight here is the monomial x_0^d.

Consider the case $n = 1$. Let V be a two-dimensional vector space. Since $\Lambda^2 V$ is isomorphic to the representation $\det : \mathrm{GL}(V) \to \mathbb{G}_m$, we have an isomorphism of representations:

$$V \cong V^* \otimes \det.$$

In particular, $V \cong V^*$ as representations of $\mathrm{SL}(V)$. We have one fundamental weight ω_1 so that any irreducible representation with dominant weight $\lambda = (m_1, m_2), m_1 \geq m_2$, is isomorphic to

$$S^{m_1 - m_2}(V) \otimes \det{}^{m_2} \cong S^{m_1 - m_2}(V^*) \otimes \det{}^{m_1} \cong \mathrm{Pol}_{m_1 - m_2}(V) \otimes \det{}^{m_1}.$$

Let us consider the representation $\mathrm{Pol}_m(\mathrm{Pol}_d(V))$. The space has a basis formed by monomials in coefficients of a general binary d-form

$$A_0 T_0^d + dA_1 t_0^{d-1} t_1 + \cdots + A_d t_1^d = (\xi_0 t_0 + \xi_1 t_1)^d.$$

So we can write any monomial of degree m in the A_i as a monomial of degree md in the basis (ξ_1, ξ_2) of V:

$$A_{i_1} \cdots A_{i_m} = (\xi_1^{d-i_1} \xi_2^{i_1}) \cdots (\xi_1^{d-i_m} \xi_2^{i_m}) = \xi_1^{md-w} \xi_2^{w},$$

where

$$w = i_1 + \cdots + i_m$$

is the *weight* of the monomial $A_{i_1}\cdots A_{i_m}$. This shows that $A_{i_1}\cdots A_{i_m}$ belongs to the weight space with character $(md-w,w)$. Let

$$\mathcal{P}(m,d,w)=\{(i_1,\dots,i_m):0\le i_1\le\cdots\le i_m\le d, i_1+\cdots+i_m=w\}.$$

The cardinality $p_w(m,d)$ of this set is equal to the number of monomials with weight w. Let $\lambda=(m_1,m_2)$ be a dominant weight. Suppose $V(\lambda)$ is a direct summand of $\mathrm{Pol}_m(\mathrm{Pol}_m(V))$. Then $(m_1,m_2)=(md-w,w)$ for some w with $md-2w\ge 0$. The weights of $V(\lambda)$ are the vectors $(md-w-i,w+i), i=0,\dots,md-w$. This shows that $\mathrm{Pol}_m(\mathrm{Pol}_m(V))$ contains

$$p_0(m,d)=1 \quad \text{summand } V(md,0)\cong \mathrm{Pol}_{md}(V)\otimes \det{}^{md},$$

$$p_1(m,d)-p(m,d) \quad \text{summands } V(md-1,1)\cong \mathrm{Pol}_{md-2}(V)\otimes \det{}^{md-1},$$

$$p_2(m,d)-p_1(m,d) \quad \text{summands } V(md-2,2)\cong \mathrm{Pol}_{md-4}(V)\otimes \det{}^{md-2},$$

and so on. It is known that the generating function for the numbers $p_i(m,d)$ is equal to the Gaussian polynomial

$$\sum_{i=0}^{\infty} p_i(m,d)t^i=\begin{bmatrix}m+d\\ d\end{bmatrix},$$

where

$$\begin{bmatrix}a\\ b\end{bmatrix}=\frac{(1-x^a)(1-x^{a-1})\cdots(1-x^{a-b+1})}{(1-x)(1-x^2)\cdots(1-x^b)}.$$

(see [113]). This gives us

Theorem 5.5. *(Plethysm decomposition) Let* $\dim V=2$*. There is an isomorphism of representations of* $\mathrm{GL}(V)$*:*

$$\mathrm{Pol}_m(\mathrm{Pol}_d(V))\cong\bigoplus_{w=0}^{[md/2]}(\mathrm{Pol}_{md-2w}(V)\otimes\det{}^{md-w})^{\oplus N(m,d,w)},$$

where

$$N(m,d,w)=\textit{coefficient of } x^w \textit{ in the polynomial } (1-x)\begin{bmatrix}m+d\\ d\end{bmatrix}.$$

Restricting the representation to the subgroup $\mathrm{SL}(V)$ we have an isomorphism of $\mathrm{SL}(V)$-representations

$$\mathrm{Pol}_m(\mathrm{Pol}_d(V)) \cong \bigoplus_{w=0}^{[md/2]} \mathrm{Pol}_{md-2w}(V)^{\oplus N(m,d,w)}.$$

As a corollary we obtain the *Cayley-Sylvester formula* for the dimension of the space of covariants:

Corollary 5.2.

$$\dim \mathrm{Cov}_{m,p}(d) = N(m,d,(md-p)/2)$$

and it is zero if $md-p$ *is odd.*

We also get *Hermite's Reciprocity* :

Theorem 5.6. *There is an isomorphism of* $\mathrm{SL}(V)$*-modules*

$$\mathrm{Pol}_m(\mathrm{Pol}_d(V)) \cong \mathrm{Pol}_d(\mathrm{Pol}_m(V)).$$

Proof. This follows from the following symmetry property:

$$p_w(m,d) = p_w(d,m).$$

This can be checked by defining the bijection $\mathcal{P}(m,d,w) \to \mathcal{P}(d,m,w)$ by sending a vector $(i_1,\ldots,i_m)$ from $\mathcal{P}(m,d,w)$ to the vector $(j_1,\ldots,j_d)$, where

$$j_s = \#\{t : i_s \geq s\}, s = 1,\ldots,d.$$

It follows also from the following property of the Gaussian polynomials:

$$\begin{bmatrix} m+d \\ d \end{bmatrix} = \begin{bmatrix} m+d \\ m \end{bmatrix}.$$

$\square$

Corollary 5.3.

$$\dim \mathrm{Pol}_m(\mathrm{Pol}_d(V))^{\mathrm{SL}(V)} = \dim \mathrm{Pol}_d(\mathrm{Pol}_m(V))^{\mathrm{SL}(V)}.$$

Remark 5.1. The covariant

$$\mathrm{Pol}_m(\mathrm{Pol}_d(V)) \to \mathrm{Pol}_{md}(V) \tag{5.10}$$

admits a simple interpretation in terms of the Veronese map. Let V be a linear space of dimension $n+1$. Recall that the *Veronese map* of degree d in dimension n is a regular map

$$\nu_d : \mathbb{P}(V^*) \to \mathbb{P}(\mathrm{Pol}_d(V))$$

given by $l \mapsto l^d$, where $l \in V^*$ is a linear function on V. It is easy to see that this map is $\mathrm{SL}(V)$-equivariant, where $\mathrm{SL}(V)$ acts naturally on $\mathbb{P}(V^*)$ and on $\mathbb{P}(\mathrm{Pol}_d(V))$. The inverse image under v_d defines an equivariant linear map

$$v_d(m)^* : \mathrm{Pol}_m(\mathrm{Pol}_d(V)) \to \mathrm{Pol}_{md}(V^*).$$

When $n = 1$, there is an isomorphism $V^* \cong V$ of $\mathrm{SL}(V)$-modules and the map $v_d(m)^*$ is the covariant (5.10). Note that the image of the Veronese map (called the *Veronese variety*) is always defined by equations of degree 2 (see [104]). The number of linearly independent equations is equal to

$$\dim \mathrm{Pol}_2(\mathrm{Pol}_d(V)^*) - \dim \mathrm{Pol}_{2d}(V) = \tfrac{1}{2}\tbinom{d+n}{n}(1 + \tbinom{d+n}{n}) - \tbinom{2d+n}{n}.$$

Thus, if $m = 2$ the kernel of the map (5.10) is a $\mathrm{SL}(V)$-submodule of the dimension given by the above formula.

Remark 5.2. One can strengthen Theorem 5.6 as follows (see [49]). Let V be a vector space of dimension r, and let

$$\mathcal{S}_{d,r} = \mathrm{Pol}(\mathrm{Pol}_m(V)) = \bigoplus_{m=0}^{\infty} \mathrm{Pol}_m(\mathrm{Pol}_d(V))$$

be the algebra of polynomials on the space $\mathrm{Pol}_m(V))$. Let

$$\mathcal{A}_{d,r} = \bigoplus_{m=0}^{\infty} \mathrm{Pol}_d(\mathrm{Pol}_m(V)). \tag{5.11}$$

We use the symbolic expression to identify elements of $\mathrm{Pol}_d(\mathrm{Pol}_m(V))$ with multihomogeneous functions on V^d of multi-degree (m^d) (see Lemma 1.1). The product of functions defines bilinear maps

$$\mathrm{Pol}_d(\mathrm{Pol}_m(V)) \times \mathrm{Pol}_d(\mathrm{Pol}_n(V)) \to \mathrm{Pol}_d(\mathrm{Pol}_{n+m}(V))$$

which endow $\mathcal{A}_{d,r}$ with a structure of a graded algebra. The natural action of $\mathrm{GL}(V)$ on V defines an action of $\mathrm{GL}(V)$ on both algebras $\mathcal{S}_{d,r}$ and $\mathcal{A}_{d,r}$ by automorphisms of graded algebras. Notice that $\mathcal{A}_{d,r}^{\mathrm{SL}(V)}$ is isomorphic to the algebra $k[G_{r,d}]^{\Sigma_r \times T}$ (Corollary 2.4). Identifying the linear spaces $\mathrm{Pol}_1(\mathrm{Pol}_d(V)) = \mathrm{Pol}(V)^*$ and $\mathrm{Pol}_d(\mathrm{Pol}_1(V)) = \mathrm{Pol}_d(V^*)$ (see (1.7)), we get a $\mathrm{GL}(V)$-equivariant algebra homomorphism:

$$h_{d,r} : \mathcal{S}_{d,r} \to \mathcal{A}_{d,r}. \tag{5.12}$$

When $r = 2$, the homomorphism $h_{d,2}$ is a $\mathrm{GL}(V)$-equivariant isomorphism of graded algebras. Hermite's Reciprocity only states that all graded pieces are isomorphic as $\mathrm{GL}(V)$-modules.

Example 5.7. Take $m = d = 2$. We get $p_0(2,2) = p_1(2,2) = 1, p_2(2,2) = 2$. Thus we have the following isomorphism of $\mathrm{SL}(V)$-representations:

$$\mathrm{Pol}_2(\mathrm{Pol}_2(V)) \cong \mathrm{Pol}_4(V) \oplus k.$$

Using the previous remark this has a simple geometric interpretation. In this case the Veronese variety is a conic, and the kernel of $v_2(2)^*$ is one-dimensional. It is spanned by a quadratic polynomial vanishing on the conic.

Example 5.8. Take $m = 2, d = 3$. Then we have an isomorphism of $\mathrm{SL}(V)$-modules

$$\mathrm{Pol}_2(\mathrm{Pol}_3(V)) \cong \mathrm{Pol}_3(\mathrm{Pol}_2(V)).$$

Thus quadrics in $\mathbb{P}^3 \cong \mathrm{Pol}_3(V)$ can be canonically identified with cubics in $\mathbb{P}^2 \cong \mathrm{Pol}_2(V)$. The Veronese curve $C = v_3(\mathbb{P}^1)$ is a rational space curve of degree 3. It is defined by three linearly independent quadric equations. Thus the kernel of the projection $\mathrm{Pol}_2(\mathrm{Pol}_3(V)) \to \mathrm{Pol}_6(V)$ is equal to the space $\mathcal{N}$ of quadrics vanishing on C. Using the plethysm decomposition

$$\mathrm{Pol}_2(\mathrm{Pol}_3(V)) \cong \mathrm{Pol}_6(V) \oplus \mathrm{Pol}_2(V)$$

we can identify $\mathcal{N}$, $\mathrm{SL}(V)$-equivariantly, with the space of binary quadratic forms.

5.6 Standard tableaux again

Finally let us explain the tableau functions from the point of view of representation theory. Note that any $L(\omega_i)$ can be embedded (as a representation) into some tensor power of some copies of $V = k^r$. So when we take their symmetric products and their tensor products we can embed each again into some $V^{\otimes N}$. So each irreducible representation is realized as an irreducible submodule of $V^{\otimes N}$ for some N. Let us find them by decomposing $V^{\otimes N}$ into a direct sum of irreducible representations.

Fix a basis $(\xi_1, \dots, \xi_r)$ of V. For any ordered subset $I = (i_1, \dots, i_N)$ of $[r]$ let ξ_I denote the tensor $\xi_{i_1} \otimes \cdots \otimes \xi_{i_N}$. A diagonal matrix diag$[t_1, \dots, t_r] \in T$ acts on ξ_I by multiplying it by the monomial $t_I = t_{i_1} \cdots t_{i_N}$. Writing any element of $V^{\otimes N}$ as a sum of tensors ξ_I we easily see that the weights of our representation are the vectors $e_I = e_{i_1} + \cdots + e_{i_N}$. The weight subspace W_{e_I} is spanned by the tensors ξ_J, where J is obtained from I by a permutation of $[N]$. A vector e_I is a dominant weight if

$$e_I \cdot (e_i - e_{i+1}) \geq 0, \quad i = 1, \dots, r-1.$$

This means that

$$e_I = (m_1, \dots, m_r), \quad m_1 \geq m_2 \geq \cdots \geq m_r \geq 0, m_1 + \cdots + m_r = N.$$

Assume for the moment that $N = 1$. Then the highest weight vector is ξ_1. Assume that $N = 2$. Then $\xi_1 \otimes \xi_2$ is sent by $f_{\alpha_1}(1) = I_r + E_{12}$ to $\xi_1 \otimes (\xi_2 + \xi_1) = \xi_1 \otimes \xi_2 + \xi_1 \otimes \xi_1$. Similarly, $\xi_2 \otimes \xi_1$ is sent to $\xi_2 \otimes \xi_1 + \xi_1 \otimes \xi_1$. So in order that $t = \lambda \xi_1 \otimes \xi_2 + \mu \xi_2 \otimes \xi_1$ be invariant under U^+ we must have $\lambda + \mu = 0$, i.e., t must be proportional to $\xi_1 \otimes \xi_2 - \xi_2 \otimes \xi_1 = \xi_1 \wedge \xi_2$. If $N = 3$ we must have

$$t = \xi_1 \otimes (\xi_1 \otimes \xi_2 - \xi_2 \otimes \xi_1) = \xi_1 \otimes \xi_1 \otimes \xi_2 - \xi_1 \otimes \xi_2 \otimes \xi_1$$

or

$$t = \xi_1 \otimes \xi_1 \otimes \xi_2 - \xi_2 \otimes \xi_1 \otimes \xi_1,$$

or

$$t = \xi_1 \otimes \xi_2 \otimes \xi_1 - \xi_2 \otimes \xi_1 \otimes \xi_1.$$

Now in the case of arbitrary N we do the following: consider a matrix

$$\mathbf{E} = \begin{pmatrix} \xi_1^{(1)} & \xi_1^{(2)} & \dots & \xi_1^{(N)} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_r^{(1)} & \xi_r^{(2)} & \dots & \xi_r^{(N)} \end{pmatrix}.$$

Each column represents a basis $(\xi_1, \dots, \xi_r)$. We will be taking

$$p_1 = m_1 - m_2 \quad \text{minors of order 1 from the first row,}$$

$$p_2 = m_2 - m_3 \quad \text{minors of order 2 from the first 2 rows},$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

$$p_r = m_r \quad \text{minors of order } r$$

in such a way that the minors do not have common columns. Of course we compute the minors using the tensor product operation. We first take the product of the minors in an arbitrary order, but then we reorganize the sum by permuting the vectors in each decomposable tensor in such a way that each summand has its upper indices in increasing order. These indices will be our highest weight vectors.

It is convenient to describe such a vector by a *Young diagram*. We view a dominant vector $\lambda = (m_1, \ldots, m_r)$ as a partition of N. It is described by putting m_i boxes in the ith row. It has $p_j = m_j - m_{j+1}$ columns of length $j = 1, \ldots, r$ ($m_{r+1} = 0$). We fill the boxes with different numbers $\tau_{ij} \in [N]$. Each τ_{ij} indicates which column enters into the minor of the matrix $\mathbf{E}$ of the corresponding size. A filled Young diagram is called *standard* if each row and each column are in increasing order. Here is an example of a Young diagram for the partition $(5, 3, 1)$ of $N = 9$:

1	3	4	8	9
2	5	6		
7				

It turns out that the multiplicity of each $L(\lambda)$ in $V^{\otimes N}$ is equal to the number of standard filled Young diagrams of the shape given by the vector λ. It is given by the *hook formula*

$$\mathrm{mult}_\lambda(V^{\otimes N}) = \frac{N!}{\prod_{1 \le i \le r, 1 \le j \le m_i} (m_i + r + 1 - i - j)}$$

(see [67]).

Example 5.9. We described invariants in $\mathrm{Pol}_m(\mathrm{Pol}_d(V))$ by embedding this space into $V^{\otimes md}$ via the polarization map. Since the space of invariants is contained in the representation $\det^w$ of GL_r where $w = md/r$, the corresponding dominant vector is $\lambda = (w, \ldots, w) \in \mathbb{Z}^r$. The representation $L(\lambda)$ is of course one-dimensional. The Young diagram is of rectangular shape with r rows and $w = md/r$ columns. The number of such diagrams is equal to the dimension of the space $(V^{\otimes N})^{\mathrm{SL}_r}$. It is not difficult to see that the hook formula gives the formula

$$\dim(V^{\otimes N})^{\mathrm{SL}_r} = (md)! \prod_{i=1}^{r} \frac{(r-1)!(r-2)!\cdots 1!}{(w+r-1)!\cdots w!}.$$

The standard tableaux on the set $[m]$ of size $w \times r$ defined in Chapter 1 correspond to standard Young diagrams which are filled in such a way that if we write the set $[N] = [md]$ as the disjoint union of m subsets $\{1, \ldots, d\}, \{d+1, \ldots, 2d\}, \ldots, \{(m-1)d, \ldots, md\}$, then each column consists of r numbers taken from different subsets of $[md]$. Moreover, for a homogeneous standard tableau we have to take exactly d numbers from each subset. The general formula for the dimension of the space $\mathrm{Pol}_m(\mathrm{Pol}_d(V))^{\mathrm{SL}_r}$ is not known for $n > 1$.

Bibliographical notes

The notion of a covariant of a quantic (i.e., a homogeneous form) goes back to A. Cayley. It is discussed in all classical books in invariant theory. The fact that a covariant of a binary form corresponds to a semiinvariant was first discovered by M. Roberts in 1861 ([93]). It can already be found in Salmon's book [99]. The result that the algebra of covariants of a binary form is finitely generated was first proved by P. Gordan [38] (see also classical proofs in [28], [39]). A modern proof can be found in [115]. Theorem 5.4 applied to the action of $G = \mathrm{SL}(V)$ on the algebra $\mathrm{Pol}(\mathrm{Pol}_d(V))$ is a generalization of Gordan's Theorem. The first proof of this theorem was given by M. Khadzhiev [62]. Our exposition of the modern theory of covariants follows [91]. The algebra of covariants of binary forms of degree d was computed by P. Gordan for $d \leq 6$ ([38]) and by F. von Gall for degree $d = 7, 8$ ([36], [35]) (the proof of completeness of the generating set for $d = 7$ may not be correct). For ternary forms the computations are known only for forms of degree 3 ([37], [42]) and incomplete for degree 4 ([100], [19]) (a thesis of Emmy Noether was devoted to such computations). Combinants of two binary forms of degrees (d_1, d_2) are known in the cases $d_1, d_2 \leq 4$ ([98]; see a modern account of the case $d_1 = d_2 = 3$ in [83]). Also known are combinants of two ternary forms of degrees $(d_1, d_2) = (2, 2), (2, 3)$ ([28]).

The theory of linear representations of reductive groups is a subject of numerous textbooks (see, for example, [34], [53]). For the historical account we refer the reader to [7]. The Cayley-Sylvester formula was first proven by Sylvester in 1878 (see historical notes in [111]). Other proofs of the Cayley-Sylvester formula can be found in [110], [111], [115]. Hermite's Reciprocity goes back to 1854. One can find more about plethysms for representations of GL_n in [34]. The relationship between Young diagrams and standard tableaux is discussed in numerous books (see [65], [115], [123]).

Exercises

5.1 Let $\Phi : \mathrm{Pol}_d(F) \to \mathrm{Pol}_p(V)$ be a covariant of degree m and order p and $I \in \mathrm{Pol}_{m'}(\mathrm{Pol}_p(V))^{\mathrm{SL}(V)}$ be an invariant. Consider the composition and compute its degree and weight.

5.2 Let Hess : $\mathrm{Pol}_3(k^3) \to \mathrm{Pol}_3(k^3)$ be the Hessian covariant. Show that it defines a rational map of degree 3 from the projective space of plane cubic curves to itself. [Hint: By a projective transformation reduce a plane cubic to a *Hesse form* $x_0^3 + x_1^3 + x_2^3 + ax_0x_1x_2 = 0$ and evaluate the covariant.]

5.3 Using the symbolic expression of covariants describe all covariants of degree r on the space $\mathrm{Pol}(k^r)$.

5.4 Find a covariant of degree 2 and order 2 on the space $\mathrm{Pol}_4(k^2)$. Describe the locus of indeterminacy for the corresponding rational map $\mathbb{P}^4 - \to \mathbb{P}^2$.

5.5 Find the symbolic expression for the transvectant T^r.

5.6 Find all covariants of degree 3 for binary forms.

5.7 Define the rth transvectant $(f_1, \ldots, f_r)^{(r)}$ of r homogeneous forms in r variables by generalizing the definition of the covariant T^r. Prove that it is a concomitant and find its multi-degree and order.

5.8 Consider the operation of taking the dual hypersurface in projective space. Show that it defines a contravariant on the space $\mathrm{Pol}_d(V)$. Find its order and degree for $n \leq 2$.

5.9 Let $F = 0$ be a plane curve of degree 4. Consider the set of lines which intersect it in four points which make an anharmonic (or a harmonic) cross-ratio. Show the set of such lines forms a plane curve in the dual plane. Find its degree and show that this construction defines a contravariant on the space $\mathrm{Pol}_4(k^3)$. Find its degree.

5.10 Let G be a finite group which acts on a finitely generated domain A. Assume that the action is *faithful* (i.e., only $g = 1$ acts identically). Show that for any irreducible representation W_ρ of G the rank of the module of covariants $\mathrm{Hom}(W_\rho^*, A)^G$ is equal to $\dim W_\rho$. [Hint: Use the fact that each irreducible representation is contained in the regular representation (realized in the group algebra $k[G]$ of G) with multiplicity equal to its dimension.]

5.11 Let M be a finitely generated abelian group and let $k[M]$ be its group algebra over a field k. Show

(i) $D(M) = \mathrm{Spm}(k[M])$ is an affine algebraic group.

(ii) $D(M) \cong \mathbb{G}_m^r$ if and only if M is free.

(iii) The group of rational homomorphisms $D(M) \to \mathbb{G}_m$ is naturally isomorphic to M, and the group of rational homomorphisms $\mathbb{G}_m \to D(M)$ is isomorphic to $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$.

(iv) Each closed subgroup of $D(M)$ is isomorphic to $D(M')$ where M' is a factor group of M.

(v) There is a bijective correspondence between closed subgroups H of $D(M)$ and subgroups of M.

5.12 Find the roots, dual roots, dominant weights, and fundamental weights for the group $G = \mathrm{SL}_r$.

5.13 Let $L(\lambda)$ be a representation of G with highest weight vector v.

(i) Let $l = kv$ be the line spanned by v. Show that the stabilizer $G_l = \{g \in G : g \cdot l = l\}$ is a parabolic subgroup P (i.e., a closed subgroup containing a Borel subgroup).

(ii) Show that the map $g \to g \cdot v$ defines a projective embedding of the homogeneous space $G/P \to \mathbb{P}(L(\lambda))$.

(iii) Consider the case $G = \mathrm{GL}_r$ and $\lambda = \omega_i$ is one of the fundamental weights. Show that G/P is isomorphic to the Grassmann variety $\mathrm{Gr}(i, r)$ and the map defined in (ii) is the Plücker embedding.

5.14 In the notation of section 5.1 show that $V = L(\omega_1)$ for the group $G = \mathrm{SL}(V)$. Show that there is an isomorphism of $\mathrm{Pol}(W)^G$-modules

$$\mathrm{Pol}(W)^H \cong \bigoplus_{p=0}^{\infty} \mathrm{Pol}(W)^{(p\omega_n)}.$$

5.15 Let H be a subgroup of $G = \mathrm{SL}_r$ which contains the subgroup U of upper triangular matrices.

(i) Show that for any highest weight module $L(\lambda)$ one has $\dim_k L(\lambda)^H \leq 1$ and the equality takes place if and only if H is contained in the stabilizer of a highest weight vector.

(ii) Let $\Lambda(H)$ be the set of λ for which the equality holds. Show that for any action of G on $X = \mathrm{Spm}(A)$ there is an isomorphism of A^G-modules $A^H \cong \oplus_{\lambda\in\Lambda(H)} A^{(\lambda)}$.

(iii) Consider the example of H from the previous problem and find $\Lambda(H)$.

5.16 Let $\dim V = 2$ and $\mathrm{char}(k) = 0$. Show that there is an isomorphism of $\mathrm{SL}(V)$-modules

$$\mathrm{Pol}_2(\mathrm{Pol}_d(V)) \cong \bigoplus_{i\geq 0} \mathrm{Pol}_{2n-4i}(V).$$

5.17 Let V be as in the previous exercise. Find the decomposition of the $\mathrm{GL}(V)$-module $\mathrm{Pol}_n(V) \otimes \mathrm{Pol}_m(V)$ into irreducible summands (the *Clebsch–Gordan decomposition*).

5.18 Find an irreducible representation of GL_3 with highest weight equal to $\omega_1 + \omega_2$.

Chapter 6

Quotients

6.1 Categorical and geometric quotients

Let G be an affine algebraic group acting (rationally, as always) on an algebraic variety X over an algebraically closed field k. We would like to define the quotient variety X/G whose points are orbits. As we explained in Chapter 1 this is a hopeless task due to the existence of nonclosed orbits. So we need to modify the definition of X/G; for this we look first at the categorical notion of a quotient object with respect to an equivalence relation.

Let (X, R) be a set together with an equivalence relation $R \subset X \times X$. The canonical map $p : X \to X/R$ has the universal property with respect to all maps $f : X \to Y$ such that $R \subset X \times_Y X = (f \times f)^{-1}(\Delta_Y)$. Also we have $R = X \times_{X/R} X = (p \times p)^{-1}(\Delta_{X/R})$. This equality expresses the property that the fibres of the map p are the equivalence classes. Let us express this in categorical language. Let $\mathcal{C}$ be any category with fibred products. We define an *equivalence relation* on an object X as a subobject $R \subset X \times X$ (or more generally just a morphism $R \to X \times X$) satisfying the obvious axioms (expressed by means of commutative diagrams). Then we define a quotient X/R as an object in $\mathcal{C}$ for which there is a morphism $p : X \to X/R$ having the universal property with respect to morphisms $X \to Y$ such that $R \to X \times X$ factors through a morphism $R \to X \times_Y X$. By definition there is a canonical morphism

$$R \to X \times_{X/R} X. \tag{6.1}$$

Note that, in general, there is no reason to expect that the morphism (6.1) will be an isomorphism or an epimorphism.

Let $\sigma : G \times X \to X$ be an algebraic action. We say that the pair (X, σ) is a *G-variety* and often drop σ from the notation. Let $\Psi : G \times X \to X \times X$ be the morphism (σ, pr_2). This morphism should be thought of as an equivalence relation on X defined by the action. A G-equivariant morphism of G-varieties corresponds to a morphism of sets with an equivalence relation. The definition of a G-equivariant morphism $f : X \to Y$ can be rephrased by saying that the map Ψ factors through the natural morphism $X \times_Y X \to X \times X$; this corresponds to the property $(f, f)(R) \subset \Delta$. This suggests the following definition.

Definition. A *categorical quotient* of a G-variety X is a G-invariant morphism $p : X \to Y$ such that for any G-invariant morphism $g : X \to Z$ there exists a unique morphism $\bar{g} : Y \to Z$ satisfying $\bar{g} \circ p = g$. A categorical quotient is called a *geometric quotient* if the image of the morphism Ψ equals $X \times_Y X$. We shall denote the categorical quotient (resp. geometric quotient) by $p : X \to X/\!\!/G$ (resp. $p : X \to X/G$). It is defined uniquely up to isomorphism.

A different approach to defining a geometric quotient is as follows. We know how to define a geometric quotient as a set; we next discuss topological spaces. We put the structure of a topological space on X/G so that the canonical projection $p : X \to X/G$ is continuous. The weakest topology on X/G for which this will be true is the topology in which a subset $U \subset X/R$ is open if and only if $p^{-1}(U)$ is open. Then we examine ringed spaces, whose definition is given in terms of choosing a class of functions on X (e.g. regular functions, smooth functions, analytic functions). If $\phi \in \mathcal{O}(U)$ is a function on $U \subset X/R$, then the composition $p^*(\phi) = \phi \circ p$ must be a function on $p^{-1}(U)$. It is obviously a G-invariant function. Using this remark we can define the structure of a ringed space on X/R by setting $\mathcal{O}(U) = \mathcal{O}(p^{-1}(U))^G$. This makes $p : X \to X/R$ a categorical quotient in the category of ringed spaces. Finally, we want the fibres of p to be orbits; this is the condition that the morphism (6.1) is an isomorphism.

Definition. A *good geometric quotient* of a G-variety X is a G-invariant morphism $p : X \to Y$ satisfying the following properties:

(i) p is surjective;

(ii) for any open subset U of Y, the inverse image $p^{-1}(U)$ is open if and only if U is open;

(iii) for any open subset U of Y, the natural homomorphism $p^* : \mathcal{O}(U) \to \mathcal{O}(p^{-1}(U))$ is an isomorphism onto the subring $\mathcal{O}(p^{-1}(U))^G$ of G-invariant functions;

(iv) the image of $\Psi : G \times X \to X \times X$ is equal to $X \times_Y X$.

Proposition 6.1. *A good geometric quotient is a categorical quotient.*

Proof. Let $q : X \to Z$ be a G-invariant morphism. Pick any affine open cover $\{V_i\}_{i \in I}$ of Z. For any V_i the inverse image $q^{-1}(V_i)$ will be an open G-invariant subset of X. Then we have the obvious inclusion $q^{-1}(V_i) \subset p^{-1}(U_i)$, where $U_i = p(q^{-1}(V_i))$. Comparing the fibres over points $y \in Y$ and using property (iv) (which says that the fibres of p are orbits), we conclude that in fact $q^{-1}(V_i) = p^{-1}(U_i)$. By property (ii), U_i is open in Y. Since p is surjective we get an open cover $\{U_i\}_{i \in I}$ of Y. The map $q^{-1}(V_i) \to V_i$ is defined by a homomorphism

$$\alpha_i : \mathcal{O}(V_i) \to \mathcal{O}(q^{-1}(V_i)) = \mathcal{O}(p^{-1}(U_i)).$$

Since q is a G-invariant morphism, the image of α_i is contained in the subring $\mathcal{O}(p^{-1}(U_i))^G$ of $\mathcal{O}(U_i)$. This defines a unique homomorphism $\mathcal{O}(V_i) \to \mathcal{O}(U_i)$, and hence a unique morphism $\bar{q}_i : U_i \to V_i$ (because V_i is affine). It is immediately checked that the maps $\bar{p}_i$ agree on the intersections $U_i \cap U_j$, and hence define a unique morphism $\bar{q} : Y \to Z$ satisfying $q = \bar{q} \circ p$. □

Proposition 6.2. *Let $p : X \to Y$ be a G-equivariant morphism satisfying the following properties:*

(i) for any open subset U of Y, the homomorphism of rings $p^* : \mathcal{O}(U) \to \mathcal{O}(p^{-1}(U))$ is an isomorphism onto the subring $\mathcal{O}(p^{-1}(U))^G$ of G-invariant functions;

(ii) if W is a closed G-invariant subset of X then $p(W)$ is a closed subset of Y;

(iii) if W_1, W_2 are closed invariant subsets of X with $W_1 \cap W_2 = \emptyset$, then $p(W_1) \cap p(W_2) = \emptyset$.

Under these conditions p is a categorical quotient. It is a good geometric quotient if additionally

(iv) the image of $\Psi : G \times X \to X \times X$ is equal to $X \times_Y X$.

Conversely, a good geometric quotient satisfies properties (i)–(iv).

Proof. This is similar to the previous proof. With the same notation, let

$$W_i = X \setminus q^{-1}(V_i).$$

This is a closed G-invariant subset of X, hence, by (ii), $U_i = Y \setminus p(W_i)$ is an open subset of Y. Clearly, $p^{-1}(U_i) \subset q^{-1}(V_i)$. Since $\bigcap_i W_i = \emptyset$, by (iii) we have $\bigcap_i p(W_i) = \emptyset$, hence $Y = \bigcup_i U_i$. Now composing the homomorphisms $\alpha_i : \mathcal{O}(V_i) \to \mathcal{O}(q^{-1}(V_i))^G$ with the restriction homomorphism $\mathcal{O}(q^{-1}(V_i))^G \to \mathcal{O}(p^{-1}(U_i))^G = \mathcal{O}(U_i)$ we get a homomorphism $\mathcal{O}(V_i) \to \mathcal{O}(U_i)$. Since V_i is affine this defines a morphism $U_i \to V_i$ whose composition with $p : p^{-1}(U_i) \to U_i$ is the map $q : p^{-1}(U_i) \to V_i$. Gluing together these morphisms we construct $Y \to Z$ as in the proof of Proposition 6.1. This shows that Y is a categorical quotient.

Let us check that under condition (iv) $p : X \to Y$ is a good geometric quotient. First we see that p is surjective. Indeed, (i) implies that p is dominant and (iii) implies that $p(X)$ is closed. Also property (ii) implies property (ii) of the definition of a good geometric quotient. In fact, if $p^{-1}(U)$ is open, then $X \setminus p^{-1}(U)$ is closed and G-invariant. Since p is surjective, its image is equal to $Y \setminus U$ and is closed. Therefore U is open. This checks the definition.

Conversely, assume $p : X \to Y$ is a good geometric quotient. Properties (i) and (iv) follow from the definition. Let us check properties (ii) and (iii). The set $U = X \setminus W$ is open and invariant. Since the fibres of p are orbits, $U = p^{-1}(p(U))$ and hence $p(U)$ is open. For the same reason, $W = p^{-1}(p(W))$ and hence $p(W) = Y \setminus p(W)$ is closed. Furthermore, $W_1 \cap W_2 = p^{-1}(p(W_1)) \cap p^{-1}(p(W_2)) = p^{-1}(p(W_1) \cap p(W_2))$. This checks property (iii). $\square$

Corollary 6.1. *Under the assumptions from the preceding Proposition, the map $p : X \to Y$ satisfies the following properties:*

(i) two points $x, x' \in X$ have the same image in Y if and only if $\overline{G \cdot x} \cap \overline{G \cdot x'} \neq \emptyset$;

(ii) for each $y \in Y$ the fibre $p^{-1}(y)$ contains a unique closed orbit.

Proof. In fact, the closures of orbits are closed G-invariant subsets in X. So if $\overline{G \cdot x} \cap \overline{G \cdot x'} = \emptyset$, $p(\overline{G \cdot x}) \cap p(\overline{G \cdot x'}) = \emptyset$. But both sets contain the point $p(x) = p(x')$. Conversely, if $p(x) \neq p(x')$, we get that $G \cdot x$ and $G \cdot x'$ lie in different fibres. Since the fibres are closed subsets, $\overline{G \cdot x}$ and $\overline{G \cdot x'}$ lie in different fibres, and hence they are disjoint. This proves (i). To prove (ii) we notice that by (i) two closed orbits in the same fibre must have nonempty intersection, but this is absurd. Since each fibre contains at least one closed orbit, we are done. $\square$

Definition. A categorical quotient satisfying properties (i), (ii) and (iii) from Proposition 6.2 is called a *good categorical quotient*.

Remarks 6.1. 1. Note that condition (ii) in the definition of a good geometric quotient is satisfied if we require

(ii)' for any closed G-invariant subset Z of X, the image $p(Z)$ is closed. Also, together with condition (iii) this implies the surjectivity of the factor map p. In fact, condition (iii) ensures that the map p is dominant, i.e., its image is dense in Y. But by (ii)', the image of p must be closed.

2. Suppose X is an irreducible normal G-variety over an algebraically closed field of characteristic 0, and $p : X \to Y$ is a surjective G-invariant morphism such that Y is normal and each fibre over any point $y \in Y$ is an orbit. Then $p : X \to Y$ is a geometric quotient. The proof is rather technical and we omit it (see [75], Proposition 0.2).

3. The definitions of categorical and geometric quotients are obviously "local" in the following sense: If $p : X \to Y$ is a G-equivariant morphism, and $\{U_i\}$ is an open cover of Y with the property that each $p_i : p^{-1}(U_i) \to U_i$ is a categorical (resp. geometric) quotient, then p is a categorical (resp. geometric) quotient.

6.2 Examples

Let us give some examples.

Example 6.1. Let G be a finite group considered as an algebraic group over a field k. Assume that X is quasi-projective. Then the geometric quotient X/G always exists. In fact, assume first that X is affine. By Theorem 3.1, the algebra $\mathcal{O}(X)^G$ is finitely generated over k. Let Y be an affine algebraic variety with $\mathcal{O}(Y) = \mathcal{O}(X)^G$. By the theorems on lifting of ideals in integral extensions, the map $p : X \to Y = X/G$ satisfies properties (ii) and (iii) from Proposition 6.2. Also, the group G acts transitively on the set of prime ideals in $\mathcal{O}(X)$ which lie over a fixed prime ideal of $\mathcal{O}(Y)$ (see, for example, [9], Chapter V, §2, Theorem 2). This shows that $\Psi : G \times X \to X \times_Y X$ is an isomorphism.

Now let $X \subset \mathbb{P}^n$ be quasi-projective but not necessarily affine. Let $\bar{X}$ be the closure of X. Let $O \subset X$ be an orbit and let F be a homogeneous polynomial vanishing on $\bar{X} \setminus X$ but not vanishing at any point of O. Thus O is contained in an affine subset $U = \bar{X} \setminus V(F)$. Recall that the complement of a hypersurface in a projective space is an open affine subset. This implies that U, being closed in an affine set, is affine. Let $U(O) = \bigcap_{g \in G}(g \cdot U)$. This is an open G-invariant affine subset of X containing O. By letting O vary, we get an open affine G-invariant covering $\{U_i\}$ of X. We already know that each quotient $p_i : U_i \to U_i/G = V_i$

exists. We will glue the V_i together to obtain the geometric quotient $p : X \to X/G$ (we refer to the gluing construction of algebric varieties in section 8.2). To do this we observe first that $U_i \cap U_j$ is affine and $U_i \cap U_j/G$ is open in V_i and V_j; this follows from considering the affine case. Thus we can glue V_i and V_j together along the open subset $V_{ij} = U_i \cap U_j/G$; we do this for all i and j. The resulting algebraic variety Y is separated. In fact we use that in the affine situation

$$(X_1 \times X_2)/(G_1 \times G_2) \cong X_1/G_1 \times X_2/G_2,$$

where $G_1 \times G_2$ acts on $X_1 \times X_2$ by the Cartesian product of the actions. Thus the image of $\Delta_X \cap (U_i \times U_j)$ in $(U_i \times U_j)/(G \times G) \cong U_i/G \times U_j/G$ is closed, and, as is easy to see, coincides with $\Delta_Y \cap (V_i \times V_j)$. This shows that Δ_Y is closed. It remains to prove that X/G is quasi-projective; we shall do this later. Note that, if X is not a quasi-projective algebraic variety, X/G may not exist in the category of algebraic varieties even in the simplest case when G is of order 2. The first example of such an action was constructed by M. Nagata ([77]) in 1956 and later a simpler construction was given by H. Hironaka (unpublished). However, if we assume that each orbit is contained in a G-invariant open affine subset, the previous construction works and X/G exists.

Example 6.2. Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a finitely generated k-algebra with a geometric grading (see Example 3.1). Consider the corresponding action of $\mathbb{G}_m$ on $X = \mathrm{Spm}(A)$. Let x_0 be the vertex of X defined by the maximal ideal $\bigoplus_{i=1}^{\infty} A_i$. Then the open subset $X' = X \setminus x_0$ is invariant and the geometric quotient $X /\!\!/ \mathbb{G}_m$ exists and is isomorphic to the projective variety $\mathrm{Projm}(A)$. We leave the details to the reader.

Example 6.3. Let H be a closed subgroup of an affine algebraic group G and G/H be the coset space (see Example 3.3). The canonical projection $G \to G/H$ is a good geometric quotient. We omit the proof, referring the reader to [53], IV, 12, where all conditions of the definition are verified.

Let us show now that the categorical quotient of an affine variety always exists. We will need the following lemma.

Lemma 6.1. *Let X be an affine G-variety, and let Z_1 and Z_2 be two closed G-invariant subsets with $Z_1 \cap Z_2 = \emptyset$. Assume G is geometrically reductive. Then there exists a G-invariant function $\phi \in \mathcal{O}(X)^G$ such that $\phi(Z_1) = 0, \phi(Z_2) = 1$.*

Proof. First choose a function $\varphi \in \mathcal{O}(X)$, not necessarily G-invariant, such that $\varphi(Z_1) = 0, \varphi(Z_2) = 1$. This is easy: since the sum of the ideals defining Z_1 and

Z_2 is the unit ideal, we can find a function $\alpha \in I(Z_1)$ and a function $\beta \in I(Z_2)$ such that $1 = \alpha + \beta$. Then we take $\varphi = \alpha$. Let W be the linear subspace of $\mathcal{O}(X)$ spanned by the translates $g^*(\varphi), g \in G$. We know that it is finite-dimensional (Lemma 3.4); let $\varphi_1, \ldots, \varphi_n$ be a basis. Consider the map $f : X \to \mathbb{A}^n$ defined by these functions. Clearly, $f(Z_1) = (0, \ldots, 0), f(Z_2) = (1, \ldots, 1)$. The group G acts linearly on the affine space, defining a linear representation. By definition of geometrically reductive groups, we can find a nonconstant G-invariant homogeneous polynomial $F \in k[Z_1, \ldots, Z_n]$ such that $F(1, \ldots, 1) = 1$. Then $\phi = f^*(F) = F(\varphi_1, \ldots, \varphi_n)$ satisfies the assertion of the lemma. □

Now we are ready to prove the following main result of this chapter:

Theorem 6.1. *Let G be a geometrically reductive group acting on an affine variety X. Then the subalgebra $\mathcal{O}(X)^G$ is finitely generated over k, and the canonical morphism $p : X \to Y = \mathrm{Spm}(\mathcal{O}(X)^G)$ is a good categorical quotient.*

Proof. The first statement is Nagata's Theorem proven in Chapter 3. To show that p is a good categorical quotient, we apply Proposition 6.2. First of all, property (i) easily follows from the fact that taking invariants commutes with localizations. More precisely, if $f \in \mathcal{O}(X)^G$, then $(\mathcal{O}(X)_f)^G = (\mathcal{O}(X)^G)_f$; this is easy and we skip the proof. Next let Z be a closed G-invariant subset of X. Suppose $p(Z)$ is not closed. Let $y \in \overline{p(Z)} \setminus p(Z)$. Then $W_1 = Z$ and $W_2 = p^{-1}(y)$ are two closed G-invariant subsets of X with empty intersection. By the preceding Lemma, there exists a function $\phi \in \mathcal{O}(X)^G$ such that $\phi(Z) = 0, \phi(p^{-1}(y)) = 1$. Since $\phi = p^*(\varphi)$ for some $\varphi \in \mathcal{O}(Y)$, we obtain $\varphi(p(Z)) = 0, \varphi(y) = 1$. But this is absurd since y belongs to the closure of $p(Z)$. This verifies condition (ii). Now let Z_1 and Z_2 be two disjoint G-invariant closed subsets of X. As above we find a function $\varphi \in \mathcal{O}(Y)$ with $\varphi(p(W_1)) = 0, \varphi(p(W_2)) = 1$. This obviously implies that $p(Z_1) \cap p(Z_2) = \emptyset$. This verifies (iii). □

Example 6.4. We have already discussed this example in Chapter 1. Let $G = \mathrm{GL}_n$ act on itself by the adjoint action, i.e. $g \cdot x = gxg^{-1}$. For each matrix $g \in \mathrm{GL}_n$ we consider the characteristic polynomial

$$\det(g - tI_n) = (-t)^n + c_1(g)(-t)^{n-1} + \cdots + c_n(g).$$

Define a regular G-equivariant map $c : \mathrm{GL}_n \to \mathbb{A}^n$ by the formula $c(g) = (c_1(g), \ldots, c_n(g))$. We claim that this is a categorical quotient. To check this it is enough to verify that $\mathcal{O}(G)^G = k[c_1, \ldots, c_n)] \cong k[Z_1, \ldots, Z_n]$; this is what we did in Chapter 1. It is clear that the fibre of c does not consist of one orbit, so the quotient is not a geometric quotient.

6.3 Rational quotients

We know that neither X/G nor $X/\!\!/G$ exists in general. So a natural problem is to find all possible open subsets of X for which the categorical or geometric quotient exists. Geometric invariant theory gives a solution to this problem when we additionally assume that the quotient is a quasi-projective algebraic variety.

Let us first show that any open subset U for which a geometric quotient U/G exists must be contained in a certain open subset X_{reg}. We will assume in the sequel that G is connected. Otherwise, we consider its connected component G^o containing the identity element. It is a normal closed subgroup of G and the quotient G/G^o is a finite group. It is easy to see (see Exercise 6.11) that we can divide by G in two steps: first divide by G^o, and then divide the quotient by the finite group G/G^o.

For any point $x \in X$ we have a regular map

$$\sigma_x : G \to X, \quad g \mapsto \sigma(g,x) := g \cdot x.$$

Clearly the image of this map is the G-orbit $\mathrm{O}(x)$ of the point x.

The set theoretical fibre of this map at a point x is denoted by G_x and is called the *isotropy subgroup* of x in the action σ. It is a closed subgroup of G, hence an affine algebraic group. If $\mathrm{char}(k) = 0$, the set theoretical fibre of σ_x coincides with the scheme theoretical fibre (or, in other words, the latter is a reduced closed subscheme of G). We are not going to prove this; to do so we would have to go into the theory of *group schemes* and prove the fundamental result of the theory that every group scheme over a field of characteristic zero is reduced.

Since all fibres of σ_x over points in $\mathrm{O}(x)$ are isomorphic (they are conjugate subgroups of G_x), the theorem on the dimension of fibres (see [104]) gives

$$\dim \mathrm{O}(x) = \dim G - \dim G_x. \tag{6.2}$$

If $\overline{\mathrm{O}(x)} \neq \mathrm{O}(x)$, the complement $\overline{\mathrm{O}(x)} \setminus \mathrm{O}(x)$ is a proper closed subset of $\overline{\mathrm{O}(x)}$, hence its dimension is strictly less than $\dim \overline{\mathrm{O}(x)}$. Take any $y \in \overline{\mathrm{O}(x)} \setminus \mathrm{O}(x)$ and consider its orbit $\mathrm{O}(y)$. Since $\dim \overline{\mathrm{O}(y)} < \dim \overline{\mathrm{O}(x)}$, applying (6.2) to y we see that

$$\dim G_x < \dim G_y. \tag{6.3}$$

Let

$$I = \Phi^{-1}(\Delta_X) = \{(g,x) \in G \times X : \sigma(g,x) = x\}.$$

This is a closed subset of $G \times X$. Consider the second projection $\mathrm{pr}_2 : I \to X$. Its fibre over a point $x \in X$ is isomorphic to the isotropy subgroup G_x. By the theorem on the dimension of fibres applied to pr_2, there exists an open subset X_{reg} of X such that $\dim G_x = d$ for all $x \in X_{\mathrm{reg}}$ and $\dim G_x > d$ for all $x \notin X_{\mathrm{reg}}$.

Applying (6.2) we obtain that for any $x \in X_{\mathrm{reg}}$ the orbit $\mathrm{O}(x)$ is closed in X_{reg} and has dimension equal to $\dim G - d$. Also, any other orbit in X has dimension strictly less than $\dim G - d$. Let U be any G-invariant open subset of X for which a geometric quotient $U \to U/G$ exists. We assume that X is irreducible. So $p_U : U \cap X_{\mathrm{reg}} \neq \emptyset$ and hence some of the orbits in U must be of dimension $\dim G - d$. By the theorem on dimension of fibres all fibres of p_U have dimension greater than or equal to $\dim G - d$ and hence all fibres of p_U have dimension equal to $\dim G - d$. Therefore they are contained in X_{reg} and hence $U \subset X_{\mathrm{reg}}$.

Thus we get a necessary condition for the existence of U/G: U must be an open subset of X_{reg}.

Theorem 6.2. *(M. Rosenlicht) Assume X is irreducible. Then X_{reg} contains an open subset U such that a good geometric quotient $U \to U/G$ exists with quasi-projective U/G. The field of rational functions on U/G is isomorphic to the subfield $k(X)^G$ of G-invariant rational functions on X.*

Proof. The proof is easy if we assume additionally that G is geometrically reductive and X is affine. Let Y be an algebraic variety with field of rational functions isomorphic to $k(X)^G$; such a Y always exists since $k(X)^G$ is of finite transcendence degree over k. Consider the rational dominant map $X_{\mathrm{reg}} \to Y$ defined by the inclusion of the fields $k(X)^G \subset k(X)$. By deleting some subset from X_{reg} we find a G-invariant open subset $U \subset X_{\mathrm{reg}}$ and a regular map from $f : U \to Y$. Replacing Y by an open subset we may assume that f is surjective. This is condition (i) from the definition of a good geometric quotient. For any open subset $V \subset U$ we have an inclusion $\mathcal{O}(V) \subset k(Y) = k(X)^G$. Since $f^*(\mathcal{O}(V)) \subset \mathcal{O}(f^{-1}(V))$ we see that $f^*(\mathcal{O}(V)) \subset \mathcal{O}(f^{-1}(V))^G$. Conversely $\mathcal{O}(f^{-1}(V))^G \subset k(X)^G = k(Y)$ and hence $\mathcal{O}(f^{-1}(V))^G \subset \mathcal{O}(V)$. Thus we have checked condition (i) of Proposition 6.2. Since U is G-invariant, the fibres of f are unions of orbits. Since any orbit in X_{reg} is closed in X_{reg}, it is closed in U. By Lemma 6.1 we can separate closed invariant subsets by functions from $\mathcal{O}(V)$. This shows that the fibres of f are orbits. This checks condition (iv). The conditions (ii) and (iii) of Proposition 6.2 are checked by using the argument from the proof of Theorem 6.1.

Let us give an idea for the proof in the general case. For the details we refer to the original paper of Rosenlicht ([95]; see also [91], 2.3). Since we do not assume that X is affine, even if G is geometrically reductive we cannot separate the closed orbits contained in the fibres of the map $f : U \to Y$. Consider the generic fibre of f as an algebraic variety U_η over the field $K = k(Y) = k(X)^G$.

Let $\bar{K}$ be the algebraic closure of K. The group $G(\bar{K})$ acts on $U_\eta(\bar{K})$ and the field of invariant rational functions is isomorphic to $\bar{K}$. All orbits of $G(\bar{K})$ have the same dimension. Suppose that a group G acts on an irreducible quasi-projective variety $X \subset \mathbb{P}^N$ such that all orbits are of the same dimension and closed. We define a map from X to the Chow variety parametrizing closed subsets of $\mathbb{P}^n$ of the same dimension d (see [75], Chapter 4, $\S 6$) by assigning to a point $x \in X$ the closure of the orbit $G \cdot x$. If the image is of positive dimension, we can construct a nonconstant invariant function on X by taking the inverse image of a rational function on the image. Otherwise the image is one point, and we obtain that X consists of one orbit. Applying this argument to $U_\eta(\bar{K})$ we see that it consists of one orbit. This implies that there is an open subset of Y such that each fibre consists of one orbit. Again deleting a closed subset from Y we may assume that Y is nonsingular. Since the dimension of all orbits is the same, the morphism f is open; this is called *Chevalley's criterion* (see [6], p. 44). This verifies condition (ii) of the definition of a good geometric quotient. The remaining conditions have been checked already. $\square$

Corollary 6.2. *The transcendence degree of $k(X)^G$ is equal to $\dim X - \dim G + d$, where $d = \min_{x\in X}\{\dim G_x\}$.*

Any model of $k(X)^G$ is called a *rational quotient* of X by G. We see that X contains an open subset such that a good geometric quotient U/G exists and coincides with a rational quotient.

Bibliographical notes

The notions of a categorical and geometric quotients are originally due to Mumford ([75]). Many books discuss different versions of these notions (see [64], [82]). Many interesting results about the structure of fibres of the quotient maps have been omitted; we refer to [91] for a survey of these results.

Exercises

6.1 Let $\mathbb{G}_a$ act on $\mathbb{A}^2$ by the formula $t \cdot (z_1, z_2) = (z_1, z_2 + tz_1)$. Consider the map $\mathbb{A}^2 \to \mathbb{A}^1,\ (z_1, z_2) \mapsto z_1$. Is it a categorical quotient? If so, is it a geometric quotient?

6.2 Let $\mathbb{G}_m$ act on $\mathbb{A}^n$ by the formula $t \cdot (z_1, \dots, z_n) = (t^{q_1} z_1, \dots, t^{q_n} z_n)$ for some positive integers $q_1, \dots, q_n$ coprime to char(k). Let $A = k[T_1, \dots, T_n]$ with the corresponding geometric grading defined by the action. Show that the geometric quotient $\Big(\mathbb{A}^n \setminus \{0\}\Big)/\mathbb{G}_m$ (see Example 6.2) is isomorphic to a quotient of $\mathbb{P}^{n-1}$ by a finite group.

6.3 Let $A = \bigoplus_{i\in\mathbb{Z}} A_i$ be a graded finitely generated k-algebra, and $A^{(e)} = \bigoplus_{i\in\mathbb{Z}} A_{ei}$. Show that, if e is coprime to char(k), $A^{(e)} = A^G$, where G is a cyclic group of order e.

6.4 Construct a counterexample to Lemma 6.1 when $G = \mathbb{G}_a$ is the additive group.

6.5 In the notation of Nagata's Theorem show that for any open subset U of Y, the restriction map $p^{-1}(U) \to U$ is a categorical quotient with respect to the induced action of G.

6.6 Describe the orbits and the fibres of the categorical quotient from Example 6.4 when $n = 2$.

6.7 Show that the categorical quotient of $\mathrm{Pol}_3(k^2)$ by SL_2 is isomorphic to $\mathbb{A}^1$. Describe the orbits and the fibres of the categorical quotient.

6.8 Let G act on an irreducible affine variety X and let $f : X \to Y$ be a G-invariant morphism to a normal affine variety. Assume that codim$(Y \setminus f(X), Y) \geq 2$ and that there exists an open subset U of Y such that for all $y \in U$ the fibre $f^{-1}(y)$ contains a dense orbit. Show that $Y \cong X/\!\!/G$.

6.9 Let G be a finite group of automorphisms of an irreducible algebraic variety. Prove that $k(X/G) = k(X)^G$.

6.10 Show by example that in general the field of fractions $Q(A^G)$ of the ring of invariants A^G is not equal to $Q(A)^G$. Prove that $Q(A^G) = Q(A)^G$ if A is a UFD and any rational homomorphism $G \to \mathbb{G}_m$ is trivial.

6.11 Let G be an algebraic group acting regularly on an algebraic variety X and let H be a closed invariant subgroup of finite index. Suppose that a geometric quotient $Y = X/G$ exists. Show that geometric quotients X/H and $(X/H)/(G/H)$ exist and $X/G \cong (X/H)/(G/H)$. Is the same true without assuming that H is of finite index?

Chapter 7

Linearization of actions

7.1 Linearized line bundles

We have seen already in the proof of Lemma 3.5 that a rational action of an affine algebraic group G on an affine variety X can be "linearized". This means that we can G-equivariantly embed X in affine space $\mathbb{A}^n$ on which G acts via a linear representation. We proved this by considering the linear space spanned by the G-translates of generators of the algebra $\mathcal{O}(X)$. In this chapter we will do a similar construction for a normal projective algebraic variety. This will be our main tool for constructing quotients.

Recall that a regular map of a projective variety X to the projective space $\mathbb{P}^n$ is defined by choosing a line bundle L (or equivalently an invertible sheaf $\mathcal{L}$ of $\mathcal{O}_X$-modules, or a Cartier divisor D) and a set of its sections $s_0, \ldots, s_n$. The map is defined by sending $x \in X$ to the point $(s_0(x), \ldots, s_n(x)) \in \mathbb{P}^n$. This point is well-defined if for any $x \in X$ there is a section s_i such that $s_i(x) \neq 0$. Often we will be taking for $(s_0, \ldots, s_n)$ a basis of the space of sections $\Gamma(X, L)$ of L. The condition above says in this case that for any $x \in X$ there exists a section $s \in \Gamma(X, L)$ such that $s(x) \neq 0$. We say in this case that L is *base-point-free*. Let $\phi_L : X \to \mathbb{P}^n$ be a map defined by a base-point-free L. Of course, it depends on the choice of a basis; different choices define maps which are the same up to composition with a projective transformation of $\mathbb{P}^n$. If ϕ_L is a closed embedding, L is called *very ample*. If $L^N := L^{\otimes N}$ is very ample for some $N > 0$, then L is called *ample*.

We will often identify L with its total space $\mathbb{V}(L)$, which comes with a projection $\pi : \mathbb{V}(L) \to X$; locally $\mathbb{V}(L)$ is the product of X and the affine line

$\mathbb{A}^1$.

Definition. A G-linearization of L is an action $\bar{\sigma} : G \times L \to L$ such that

(i) the diagram

$$\begin{array}{ccc} G \times L & \xrightarrow{\bar{\sigma}} & L \\ {\scriptstyle \mathrm{id}\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times X & \xrightarrow{\sigma} & X \end{array}$$

is commutative,

(ii) the zero section of L is G-invariant.

A G-linearized line bundle (or a *line G-bundle*) over a G-variety X is a pair $(L, \bar{\sigma})$ consisting of a line bundle L over X and its linearization. A morphism of G-linearized line bundles is a G-equivariant morphism of line bundles.

It follows from the definition that for any $g \in G$ and any $x \in X$ the induced map of the fibres

$$\bar{\sigma}_x(g) : L_x \to L_{g \cdot x}$$

is a linear isomorphism.

We can view the set of such isomorphisms as an isomorphism of line bundles

$$\bar{\sigma}(g) : L \to g^*(L),$$

where we consider $g \in G$ as an automorphism $x \to g \cdot x$ of X. The axioms of the actions translate into the following 1-*cocycle condition*:

$$\bar{\sigma}(gg') = \bar{\sigma}(g') \circ g'^*(\bar{\sigma}(g)) : L \to g'^*(L) \to g'^*(g^*(L)) = (gg')^*(L). \qquad (7.1)$$

The collection of the isomorphisms $\bar{\sigma}(g)$ can also be viewed as an isomorphism of vector bundles

$$\Phi : \mathrm{pr}_2^*(L) \to \sigma^*(L).$$

The cocycle condition (7.1) is translated into a condition on Φ which can be expressed by some commutative diagrams; this is left to the reader.

Using the definition of linearization by means of an isomorphism Φ it is easy to define an abelian group structure on the set of line G-bundles. If $\Phi : \mathrm{pr}_2^*(L) \to \sigma^*(L)$ and $\Phi' : \mathrm{pr}_2^*(L') \to \sigma^*(L')$ are two line G-bundles, we define their tensor product as the line bundle $L \otimes L'$ with the G-linearization given by the isomorphism:

$$\Phi \otimes \Phi' : \mathrm{pr}_2^*(L \otimes L') = \mathrm{pr}_2^*(L) \otimes \mathrm{pr}_2^*(L') \to \sigma^*(L \otimes L) = \sigma^*(L) \otimes \sigma^*(L').$$

Here we use the obvious property of the inverse image

$$f^*(L \otimes L') = f^*(L) \otimes f^*(L').$$

The zero element in this group is the trivial line bundle $X \times \mathbb{A}^1$ whose linearization is given by the product $\sigma \times \mathrm{id} : G \times X \times \mathbb{A}^1 \to X \times \mathbb{A}^1$. This is called the *trivial linearization*. The inverse (L, Φ) is equal to (L^{-1}, Φ') with Φ' defined as the inverse of the transpose of Φ. One checks that this again satisfies the cocycle condition. The structure of an abelian group which we have just defined induces an abelian group structure on the set of isomorphism classes of line G-bundles. We denote this group by $\mathrm{Pic}^G(X)$. It comes with the natural homomorphism

$$\alpha : \mathrm{Pic}^G(X) \to \mathrm{Pic}(X)$$

which is defined by forgetting the linearization.

Let us now describe the kernel of the homomorphism α. Observe first that if $f : L \to L'$ is an isomorphism of line bundles and $\Phi : \mathrm{pr}_2^*(L) \to \sigma^*(L)$ is a G-linearization of L, then we can define a G-linearization of L' by setting $\Phi' = \sigma^*(f)^{-1} \circ \Phi \circ \mathrm{pr}_2^*(f)$. Thus, if $\alpha((L, \bar{\sigma}))$ is isomorphic to the trivial bundle, we can replace it by an isomorphic line G-bundle to assume that L is trivial. This shows that $\mathrm{Ker}(\alpha)$ consists of isomorphism classes of linearizations on the trivial line bundle $L = X \times \mathbb{A}^1$.

We denote a point of $X \times \mathbb{A}^1$ by (x, t). For any $g \in G$,

$$\bar{\sigma}(g)(x, t) = (g \cdot x, \Psi(g, x)t),$$

where $\Psi(g, x) \in k^*$. The function $\Psi : (g, x) \mapsto \Psi(g, x)$ must be a regular function on $G \times X$ which is nowhere vanishing. In other words, $\Psi \in \mathcal{O}(G \times X)^*$. The axioms of the action give us that

$$\Psi(gg', x) = \Psi(g, g' \cdot x)\Psi(g', x). \tag{7.2}$$

Let us see when two functions Ψ, Ψ' define isomorphic linearizations. Let $a : X \times \mathbb{A}^1 \to X \times \mathbb{A}^1$ be an automorphism of the trivial bundle. It is defined by a formula $(x,t) \mapsto (x, \phi(x)t)$, where $\phi \in \mathcal{O}(X)^*$. It commutes with the actions defined by Ψ and Ψ' if and only if

$$\phi(g \cdot x)\Psi(g,x) = \phi(x)\Psi'(g,x).$$

Or, equivalently, for any $g \in G$,

$$\Psi'(g,x) = \Psi(g,x)\phi(g \cdot x)/\phi(x).$$

Let $Z^1_{\text{alg}}(G, \mathcal{O}(X)^*)$ denote the group of functions Ψ satisfying (7.2) considered as a subgroup of the group $\mathcal{O}(G \times X)^*$ and let $B^1_{\text{alg}}(G, \mathcal{O}(X)^*)$ be its subgroup consisting of functions of the form $g^*(\phi)/\phi$ for some $\phi \in \mathcal{O}(X)^*$. It follows from the definition of the group structure on $\text{Pic}^G(X)$ that the product in $Z^1_{\text{alg}}(G, \mathcal{O}(X)^*)$ corresponds to the tensor product of linearized line G-bundles. So the above discussion proves the following.

Theorem 7.1. *The kernel of the forgetful homomorphism $\alpha : \text{Pic}^G(X) \to \text{Pic}(X)$ is isomorphic to the group*

$$H^1_{\text{alg}}(G, \mathcal{O}(X)^*) := Z^1_{\text{alg}}(G, \mathcal{O}(X)^*)/B^1_{\text{alg}}(G, \mathcal{O}(X)^*).$$

Note the special case when for any integral k-algebra K

$$(\mathcal{O}(X) \otimes_k K)^* = K^* \otimes 1.$$

This happens, for example, when X is affine space, or when X is connected and proper over k. Then

$$\mathcal{O}(G \times X)^* = \text{pr}_1^*(\mathcal{O}(G)^*)$$

and (7.2) gives that

$$Z^1_{\text{alg}}(G, \mathcal{O}(X)^*) \cong \text{Hom}_{\text{alg}}(G, \mathbb{G}_m) := \mathcal{X}(G),$$

the subscript indicating that we are considering rational homomorphisms of algebraic group. The latter group is called the *group of rational characters* of G. We studied this group when G was a torus. Also we have $g^*(\phi) = \phi$ and hence $B^1_{\text{alg}}(G, \mathcal{O}(X)^*) = 0$. Thus we obtain

Corollary 7.1. *Assume $\mathcal{O}(G \times X)^* = \text{pr}_1^*(\mathcal{O}(G)^*)$. Then*

$$\text{Ker}(\alpha) \cong \mathcal{X}(G).$$

Remark 7.1. According to a theorem of Rosenlicht ([95]), for any two irreducible algebraic varieties X and Y over an algebraically closed field k, the natural homomorphism

$$\mathcal{O}(X)^* \otimes \mathcal{O}(Y)^* \to \mathcal{O}(X \times Y)^*$$

is surjective. Let us give a sketch of the proof. First we use that for any irreducible algebraic variety the group $\mathcal{O}(X)^*/k^*$ is finitely generated. (This is not difficult to prove by reducing to the case of a normal variety and then finding a complete normal variety $\bar{X}$ containing X such that $D = \bar{X} \setminus X$ is a divisor. Then for any $f \in \mathcal{O}(X)^*$ the divisor of f is contained in the support of D and hence is equal to a linear combination of irreducible components of D. This defines an injective homomorphism from the group $\mathcal{O}(X)^*/k^*$ to a finitely generated abelian group.) Now assume we have an invertible function $\phi(x,y)$ on $X \times Y$. For a fixed $x \in X$ we have a function $\phi_x(y) = \phi(x,y) \in \mathcal{O}(Y)^*$. Since $\mathcal{O}(Y)^*/k^*$ is a finitely generated group, the map $X \to \mathcal{O}(Y)^*/k^*, x \mapsto \phi_x(y)$ modulo k^* must be constant. Of course to justify this we have to show that this map is given by an algebraic function; this can be done. So assuming this, we obtain that $\phi(x,y)$ is equal to a function $\psi(y)$ up to a multiplicative factor $c(x)$ depending on x. So $\phi(x,y) = c(x)\psi(y)$ as asserted.

7.2 The existence of linearization

To find conditions for the existence of a G-linearization of a line bundle we have to study the image of the forgetful homomorphism α. This consists of isomorphism classes of line bundles on X which admit some G-linearization. We start with the following lemma.

Lemma 7.1. *Let G be a connected affine algebraic group, and let X be an algebraic G-variety. A line bundle L over X admits a G-linearization if and only if there exists an isomorphism of line bundles $\Phi : \mathrm{pr}_2^*(L) \to \sigma^*(L)$.*

Proof. We already know that this condition is necessary, so we show that it is sufficient. Assume that such an isomorphism exists. The problem is that it may not satisfy the cocycle condition (7.1). Let us interpret Φ as a collection of isomorphisms $\Phi_g : L \to g^*(L)$. When $g = e$, the unity element, we get an automorphism $\Phi_e : L \to L$. It is given by a function $\phi \in \mathcal{O}(X)^*$. Composing all Φ_g with Φ_e^{-1}, we may assume that $\Phi_e = \mathrm{id}_L$. Now the isomorphisms $\Phi_{gg'}$ and $g'^*(\Phi_g) \circ \Phi_{g'}$ differ by an automorphism of L. Denote it by $F(g,g')$ so that we have

$$\Phi_{gg'} \circ F(g,g') = g'^*(\Phi_g) \circ \Phi_{g'}.$$

The cocycle condition means that $F(g,g') \equiv \mathrm{id}_L$. So far we have only that $F(e,g) = F(g,e) = \mathrm{id}_L$ for any $g \in G$. Let us identify the automorphism $F(g,g')$ with an invertible function on $G\times G\times X$. By Rosenlicht's Theorem which we cited in Remark 7.1, we can write $F(g,g')(x) = F_1(g)F_2(g')F_3(x)$. Since $F(e,g',x) \equiv 1$ and $F(g,e,x) \equiv 1$, the functions $F_2(g)F_3(x)$ and $F_1(g)F_3(x)$ are constants. Thus $F_3(x)$ is constant and hence F_1 and F_2 are constants. This implies that $F \equiv 1$. This proves the assertion. □

Remark 7.2. The existence of an isomorphism $\Phi : \mathrm{pr}_2^*(L) \to \sigma^*(L)$ means that L is a G-invariant line bundle. So the preceding lemma asserts that any G-invariant line bundle admits a G-linearization provided that G is a connected algebraic group. The assertion is not true if G is not connected. For example, assume that G is a finite group. The functions $F(g,g')$ which we considered in the preceding proof form a 2-cocycle of G with values in k^* (with trivial action of G in k^*). The obstruction for the existence of a G-linearization lies in the cohomology group $H^2(G,k^*)$. The latter group is called the group of *Schur multipliers* of G. It has been computed for many groups G and, of course, it is not trivial in general. If we denote the subgroup of G-invariant line bundles by $\mathrm{Pic}(X)^G$, then one has an exact sequence of abelian groups

$$0 \to \mathrm{Hom}(G,k^*) \to \mathrm{Pic}^G(X) \to \mathrm{Pic}(X)^G \to H^2(G,k^*). \tag{7.3}$$

Lemma 7.2. *Assume that X is normal (for example, nonsingular) and G is a connected affine algebraic group. Let $x_0 \in X$. For any line bundle L on $G \times X$ we have*

$$L \cong \mathrm{pr}_1^*(L|G \times x_0) \otimes \mathrm{pr}_2^*(L|e \times X).$$

Proof. It is enough to show that $L \cong \mathrm{pr}_1^*(L_1) \otimes \mathrm{pr}_2^*(L_2)$ for some $L_1 \in \mathrm{Pic}(G)$ and $L_2 \in \mathrm{Pic}(X)$; then it is immediately checked that $L_1 \cong L|G \times x_0$ and $L_2 \cong L|e \times X$. To do this we use the following fact about algebraic groups: G contains an open Zariski subset U isomorphic to $(\mathbb{A}^1 \setminus \{0\})^N$. For GL_n this follows from the fact that any matrix with nonzero pivots can be reduced to triangular form by elementary row transformations. We also use the fact that the homomorphism $\mathrm{pr}_2^* : \mathrm{Pic}(X) \to \mathrm{Pic}(\mathbb{A}^1 \setminus \{0\} \times X)$ is an isomorphism (see [46], Chapter 2, Proposition 6.6). These two facts imply that $L|U \times X \cong \mathrm{pr}_2^*(L_2)$ for some line bundle L_2 on X. Let D be a Cartier divisor on $G \times X$ representing L (i.e., $L \cong \mathcal{O}_{G\times X}(D)$). Then the preceding isomorphism implies that there exists a Cartier divisor D_2 on X such that $D' = D - \mathrm{pr}_2^*(D_2)|U \times X = 0$. For every irreducible component D_i' of D' its image in G is contained in the closed subset $Z = G \setminus U$. By the theorem on the dimension of fibres, the fibres of $\mathrm{pr}_1 : D_i' \to Z$ must be of dimension equal to $\dim X$. This easily implies that $D_i' = \mathrm{pr}_1^*(D_i)$, where $D_i \subset Z$. Thus $D' = \mathrm{pr}_1^*(D_1)$ for some Weil (and hence Cartier because G is nonsingular) divisor on G. So we have the equality of Cartier divisors $D = \mathrm{pr}_1^*(D_1) + \mathrm{pr}_2^*(D_2)$. This translates into an isomorphism of line bundles $L \cong \mathrm{pr}_1^*(\mathcal{O}_G(D_1)) \otimes \mathrm{pr}_2^*(\mathcal{O}_X(D_2))$. □

Define now a homomorphism $\delta : \mathrm{Pic}(X) \to \mathrm{Pic}(G)$ by

$$\delta(L) = (\mathrm{pr}_2^*(L) \otimes \sigma^*(L^{-1}))|G \times x_0,$$

where x_0 is a chosen point in X. Suppose $\delta(L)$ is trivial. By the preceding lemma applied to $M = \mathrm{pr}_2^*(L) \otimes \sigma^*(L^{-1})$ we obtain that $M = \mathrm{pr}_2^*(M|e \times X)$. But the restriction of σ and pr_2 to $e \times X$ are equal. This implies that M is trivial, hence there exists an isomorphism $\Phi : \mathrm{pr}_2^*(L) \to \sigma^*(L)$. By Lemma 7.1, L admits a G-linearization. This proves

Theorem 7.2. *Let G be a connected affine algebraic group acting on a normal variety X. Then the following sequence of groups is exact*

$$0 \to \mathrm{Ker}(\alpha) \to \mathrm{Pic}^G(X) \xrightarrow{\alpha} \mathrm{Pic}(X) \xrightarrow{\delta} \mathrm{Pic}(G).$$

Corollary 7.2. *Under the assumption of the theorem, the image of* $\mathrm{Pic}^G(X)$ *in* $\mathrm{Pic}(X)$ *is of finite index. In particular, for any line bundle L on X there exists a number n such that $L^{\otimes n}$ admits a G-linearization.*

Proof. Use the fact that for any affine algebraic k-group G the Picard group $\mathrm{Pic}(G)$ is finite (see [66], p.74). □

Remark 7.3. The assertion that $\mathrm{Pic}(G)$ is finite can be checked directly for many groups. For example, the group is trivial for $G = \mathrm{GL}_n, \mathbb{G}_m^n, \mathbb{G}_a$ since these groups are open subsets of affine space. To compute $\mathrm{Pic}(G)$ for $G = \mathrm{PGL}_n, \mathrm{SL}_n$, we use the following facts. Let V be an irreducible hypersurface of degree d in $\mathbb{P}^N$. Then

$$\mathrm{Pic}(\mathbb{P}^N \setminus V) \cong \mathbb{Z}/d\mathbb{Z}. \tag{7.4}$$

This isomorphism is defined by restricting a sheaf to an open subset. Another fact, which is not trivial, is that

$$\mathrm{Pic}(V) = \mathbb{Z}h, \tag{7.5}$$

where h is the class of a hyperplane section of V, provided $N \geq 4$. This is called the *Lefschetz Theorem* on hyperplane sections (see [40], p. 169).

Now notice that $G = \mathrm{PGL}_n$ is isomorphic to $\mathbb{P}^{n^2-1} \setminus V$, where V is given by the determinant equation $\det(x_{ij}) = 0$. This gives

$$\mathrm{Pic}(\mathrm{PGL}_n) \cong \mathbb{Z}/n\mathbb{Z}.$$

On the other hand, SL_n is isomorphic to the complement of a hyperplane section $x_{00} = 0$ of the hypersurface

$$\det((x_{ij})_{1\leq i,j\leq n}) - x_{00}^n = 0$$

in $\mathbb{P}^{n^2}$. So when $n \geq 2$ we can apply (7.4) to obtain

$$\mathrm{Pic}(\mathrm{SL}_n) \cong 0.$$

There is a notion of a simply connected semi-simple algebraic group (which makes sense over an arbitrary algebraically closed field). For all such groups $\mathrm{Pic}(G)$ is trivial. Any G is isomorphic to a quotient $\tilde{G}/A$, where $\tilde{G}$ is simply connected and A is a finite abelian group whose dual abelian group is isomorphic to $\mathrm{Pic}(G)$. For example, $\tilde{G} = \mathrm{SL}_n$ for $G = \mathrm{PGL}_n$. For simple algebraic groups $\mathrm{Pic}(G)$ is a subgroup of the abelian group $A(\mathcal{D})$ defined by the Cartan matrix of the root system of the Lie algebra of G. Here are the values of $A(\mathcal{D})$ for different types of simple Lie algebras:

A_n	B_n	C_n	D_{2k}	D_{2k+1}	F_4	G_2	E_6	E_7	E_8
$\mathbb{Z}/(n+1)\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/4\mathbb{Z}$	1	1	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1

We refer to [88] for a description of the Picard group of any homogeneous space G/H.

7.3 Linearization of an action

Now we are ready to prove that any algebraic action on a normal quasi-projective variety can be linearized. Let L be a G-linearized line bundle, let $V = \Gamma(X, L)$ be its space of sections, and let G be an affine algebraic group. The group G acts naturally and linearly on V by the formula

$$\rho(g)(s)(x) = \bar{\sigma}(g, s(\sigma(g^{-1}, x))),$$

or, in simplified notation,

$$(g \cdot s)(x) = g \cdot s(g^{-1} \cdot x). \tag{7.6}$$

We know that any finite-dimensional subspace W' of V is contained in a G-invariant finite-dimensional subspace W generated by the translates of a basis of W'. Thus we obtain a linear representation

$$\rho : G \to \mathrm{GL}(W).$$

Assume that the linear system W is *base-point-free* (i.e., for any $x \in X$ there exists $s \in W$ such that $s(x) \neq 0$). Then W defines a regular map $\phi_W : X \to \mathbb{P}(W^*)$ by the formula

$$\phi_W(x) = \{s \in W : s(x) = 0\}.$$

Here we identify a point in $\mathbb{P}(W^*)$ with a hyperplane in W. Note that although "$s(x)$" does not make sense (since it depends on a local trivialization of L), the equality $s(x) = 0$ is well-defined. The representation (7.6) in W defines a representation in W^* and the induced projective representation in $\mathbb{P}(W^*)$. It is given by the formula

$$g \cdot H = g^{-1}(H),$$

where H is a hyperplane in W. Now

$$\begin{array}{rcl} \phi_W(g \cdot x) & = & \{s \in W : s(g \cdot x) = 0\} \\ = \{s \in W : g^{-1}s(g \cdot x) = 0\} & = & \{s \in W : (g^{-1} \cdot s)(x) = 0\} \\ = g^{-1}(\phi_W(x)) & = & g \cdot \phi_W(x). \end{array}$$

This shows that the map ϕ_W is G-equivariant.

Choosing a basis $(s_0, \ldots, s_n)$ in W we obtain a G-equivariant rational map

$$f : X \to \mathbb{P}^n, \quad x \mapsto (s_0(x), \ldots, s_n(x)).$$

If the rational map defined by a basis of W' is an embedding, then this map is an embedding too. Now let $i : X \hookrightarrow \mathbb{P}^N$ be an embedding of X as a locally closed subvariety of projective space. We take $L = i^*(\mathcal{O}_{\mathbb{P}^N}(1))$. When n is large enough, $L^{\otimes n} = i^*(\mathcal{O}_{\mathbb{P}^N}(n))$ admits a G-linearization. Let $W' \subset \Gamma(X, L^{\otimes n})$ be the image of $\Gamma(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n))$ under the canonical restriction map $\Gamma(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n)) \to \Gamma(X, L^{\otimes n})$. Obviously, W' is a finite-dimensional base-point-free linear system. It defines an embedding of X into projective space which is the composition of i and a Veronese map $v_n : \mathbb{P}^N \to \mathbb{P}^{\binom{N+n}{n}-1}$ (obtained from the Veronese map $v_d : \mathbb{P}(k^{N+1}) \to \mathbb{P}(\mathrm{Pol}_n(k^{N+1}))$ by choosing bases). Replacing W' with a G-invariant linear system W as above, we obtain a linearization of the action of G on X.

Theorem 7.3. *Let X be a quasi-projective normal algebraic variety, acted on by an irreducible algebraic group G. Then there exists a G-equivariant embedding $X \hookrightarrow \mathbb{P}^n$, where G acts on $\mathbb{P}^n$ via its linear representation $G \to \mathrm{GL}_{n+1}$.*

Example 7.1. Let $G = \mathrm{PGL}_{n+1}$ act on $X = \mathbb{P}^n$ in the natural way. Let us see that the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ is not G-linearizable but $\mathcal{O}_{\mathbb{P}^n}(n+1) = \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes(n+1)}$ is. We view G as an open subset of the projective space $\mathbb{P}^N(N = n^2 + 2n = \dim G)$ whose complement is the determinant hypersurface Δ given by the equation $\det((T_{ij})) = 0$. The action $\sigma\colon G \times X \to X$ is the restriction to $G \times X$ of the rational map $\sigma'\colon \mathbb{P}^N \times \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ given by the formula

$$\sigma'((a_{ij}), (x_0, \ldots, x_n)) = \Big(\sum_{j=0}^{n} a_{1j}x_j, \ldots, \sum_{j=0}^{n} a_{nj}x_j\Big).$$

Note that this map is not defined at any point (A, x) such that $\det(A) = 0, A \cdot x = 0$. The restriction of the projection $\mathbb{P}^N \times X \to \mathbb{P}^N$ to the set Z of such points is a birational map onto the determinant hypersurface (it is an isomorphism over the subset of matrices of corank equal to 1). Since Z is of codimension ≥ 2 in $\mathbb{P}^N \times \mathbb{P}^n$ the line bundle $\sigma^*(\mathcal{O}_{\mathbb{P}^n}(1))$ is the restriction of a line bundle on $\mathbb{P}^N \times \mathbb{P}^n$. The formula for the action shows that this bundle must be $\mathrm{pr}_1^*(\mathcal{O}_{\mathbb{P}^N}(1)) \otimes \mathrm{pr}_2^*(\mathcal{O}_{\mathbb{P}^n}(1))$. Thus $\sigma^*(\mathcal{O}_{\mathbb{P}^n}(1))$ restricted to $(\mathbb{P}^N \setminus \Delta) \times \{x_0\}$ is isomorphic to the restriction of $\mathcal{O}_{\mathbb{P}^N}(1)$ to $\mathbb{P}^N \setminus Z$. If $\mathcal{O}_{\mathbb{P}^n}(1)$ admits a linearization, we have $\sigma^*(\mathcal{O}_{\mathbb{P}^n}(1)) \cong \mathrm{pr}_2^*(\mathcal{O}_{\mathbb{P}^n}(1))$, and hence the latter line bundle must be trivial. However, by (7.4), it is a generator of the group $\mathrm{Pic}(\mathbb{P}^N \setminus \Delta) \cong \mathbb{Z}/(n+1)\mathbb{Z}$.

Bibliographical notes

The existence of a linearization of some power of a line bundle on a normal complete algebraic variety was first proven in [75] by using the theory of Picard varieties for complete normal varieties. Our proof, which is borrowed from [66], does not use the theory of Picard varieties and applies to any normal quasi-projective varieties. One can also consider vector G-bundles of arbitrary rank (see for example [101]); however, no generalization of Corollary 7.2 to this case is known to me.

Exercises

7.1 Let L be a line bundle over a connected affine algebraic group. Show that the complement L^* of the zero section of L has the structure of an algebraic group such that the projection map $\pi : L^* \to G$ is a homomorphism of groups with kernel isomorphic to $\mathbb{G}_m$.

7.2 Let G be a connected affine algebraic group. Show that $H^1_{\mathrm{alg}}(G, \mathcal{O}(X)^*)$ is a homomorphic image of the group $\mathcal{X}(G)$. In particular it is trivial if X is connected and complete.

7.3 Use Rosenlicht's Theorem from Remark 7.1 to show that any invertible regular function $f \in \mathcal{O}(G)^*$ on a connected affine algebraic group G with value 1 at the unity $e \in G$ defines a rational character of G.

7.4 Let X be a nonsingular algebraic variety and let G be its finite group of automorphisms. Show that the group $\mathrm{Pic}^G(X)$ is isomorphic to the group of G-invariant Weil divisors modulo linear equivalence defined by G-invariant rational functions. [Hint: Use Hilbert's Theorem 90 which asserts that $H^1(G, k(X)^*) = 0$.]

7.5 Let $\mathbb{G}_m$ act on an affine algebraic variety X defining the corresponding grading of $\mathcal{O}(X)$. Let M be a projective module of rank 1 over $\mathcal{O}(X)$ and let L be the associated line bundle on X. Show that there is a natural bijective correspondence between G-linearizations of L and structures of $\mathcal{O}(X)$-graded modules on M.

7.6 Show that any line bundle on a normal irreducible variety X on which SL_n acts admits a unique SL_n-linearization.

7.7 Let $f : X \to \mathbb{P}(V)$ be a G-equivariant map, where G acts on $\mathbb{P}(V)$ via its linear representation. Show that $L = f^*(\mathcal{O}_{\mathbb{P}(V)}(1))$ admits a G-linearization and the map f is the map given by the line bundle L.

7.8 Show that the total space of the line bundle $L = \mathcal{O}_{\mathbb{P}^n}(1)$ is isomorphic to the complement of a point in $\mathbb{P}^{n+1}$. Describe the unique SL_{n+1}-linearization of L in terms of an action of the group SL_{n+1} on the total space.

Chapter 8

Stability

8.1 Stable points

From now on we will assume that G is a reductive algebraic group acting on an irreducible algebraic variety X. In this chapter we will explain a general construction of quotients due to D. Mumford. The idea is to cover X by open affine G-invariant sets U_i and then to construct the categorical quotient $X/\!\!/G$ by gluing together the quotients $U_i/\!\!/G$. The latter quotients are defined by Nagata's Theorem. Unfortunately, such a cover does not exist in general. Instead we find such a cover of some open subset of X. So we can define only a "partial" quotient $U/\!\!/G$. The construction of U will depend on a parameter, a choice of a G-linearized line bundle L.

Definition. Let L be a G-linearized line bundle on X and $x \in X$;

(i) x is called *semi-stable* (with respect to L) if there exists $m > 0$ and $s \in \Gamma(X, L^m)^G$ such that $X_s = \{y \in X : s(y) \neq 0\}$ is affine and contains x.

(ii) x is called *stable* (with respect to L) if there exists s as in (i) and additionally G_x is finite and all orbits of G in X_s are closed.

(iii) x is called *unstable* (with respect to L) if it is not semi-stable.

We shall denote the set of semi-stable (respectively stable, unstable) points by

$$X^{\mathrm{ss}}(L), \quad X^{\mathrm{s}}(L), \quad X^{\mathrm{us}}(L).$$

Remark 8.1. 1. Obviously $X^{\mathrm{ss}}(L)$ and $X^{\mathrm{s}}(L)$ are open G-invariant subsets (but could be empty).

2. If L is ample and X is projective, the sets X_s are always affine, so this condition in the definition of semi-stable points can be dropped. In fact, for any $n > 0$, $X_{s^n} = X_s$ so we may assume that L is very ample. Let $f : X \to \mathbb{P}_k^N$ be a closed embedding defined by some complete linear system associated to L. Then X_s is equal to the inverse image of an affine open subset in $\mathbb{P}_k^N$ which is the complement to some hyperplane. Since a closed subset of an affine set is affine we obtain the assertion.

3. The restriction of L to $X^{\mathrm{ss}}(L)$ is ample. This is a consequence of the following criterion of ampleness: L is ample on a variety X if and only if there exists an affine open cover of X formed by the sets X_s, where s is a global section of some tensor power of L. For the proof we refer to [46], p. 155.

4. The definitions of the sets $X^{\mathrm{ss}}(L), X^{\mathrm{s}}(L), X^{\mathrm{us}}(L)$ do not change if we replace L by a positive tensor power (as a G-linearized line bundle).

5. Assume L is ample. Let $x \in X^{\mathrm{ss}}(L)$ be a point whose orbit $G \cdot x$ is closed and whose isotropy subgroup G_x is finite. I claim that $x \in X^{\mathrm{s}}(L)$. In fact let $x \in X_s$ be as in the definition of semi-stable points. Then the set $Z = \{y \in X_s : \dim G_y > 0\}$ is closed in X_s and does not intersect $G \cdot x$. Since G is reductive, there exists a function $\phi \in \mathcal{O}(X_s)^G$ such that $\phi(G \cdot x) \neq 0, \phi(Z) = 0$. One can show that there exists some number $r > 0$ such that $\phi s^{\otimes r}$ extends to a section s' of some tensor power of L (see [46], Chapter 2, 5.14). Since X is irreducible, this section must be G-invariant. Thus $x \in X_{s'} \subset X_s$ and all points in $X_{s'}$ have zero-dimensional stabilizer. This implies that the orbits of all points in $X_{s'}$ are closed in $X_{s'}$. This checks that x is stable.

6. In [75] a stable point is called *properly stable* and in the definition of stability the finiteness of G_x is omitted.

Let us explain the definition of stability in more down-to-earth terms. Assume that L is very ample, and embed X equivariantly in $\mathbb{P}(V)$. We have a G-equivariant isomorphism of vector spaces

$$\Gamma(X, L^m) \cong \mathrm{Pol}_m(V)/I_m,$$

where I_m is the subspace of $\mathrm{Pol}_m(V)$ which consists of polynomials vanishing on X. Passing to invariants, we obtain

$$\Gamma(X, L^m)^G \cong (\mathrm{Pol}_m(V)/I_m)^G.$$

Let x^* denote a point in V such that $kx^* = x \in \mathbb{P}(V)$. Every $s \in \Gamma(X, L^m)^G$ can be represented by a polynomial $F_s \in \mathrm{Pol}_m(V)$ which is G-invariant modulo

I_m. In particular, F_s is constant on the orbit of x^* for any point $x \in X$. Clearly $s(x) \neq 0$ if and only if F_s does not vanish on x^*. So the set of unstable points is equal to the image in $\mathbb{P}(V)$ of the set

$$\mathcal{N}(G;V) = \{v \in V : F(v) = 0, \forall F \in \bigoplus_{m>0} \mathrm{Pol}_m(V)^G\}.$$

This set is called the *null-cone* of the linear action of G in V. It is an affine variety given by a system of homogeneous equations (an *affine cone*). Let $v \in V$ and $\mathrm{O}(v)$ be its orbit in V. Suppose $0 \in \overline{\mathrm{O}(v)}$. Then for any G-invariant polynomial F we have $F(v) = F(\overline{\mathrm{O}(v)}) = F(0) = 0$. Thus the corresponding point $x = kv$ in X is unstable. Conversely, if x is unstable, $0 \in \overline{\mathrm{O}(v)}$. In fact, otherwise we can apply Lemma 6.1 and find an invariant polynomial P such that $P(v) \neq 0$ but $P(0) = 0$. If we write P as a sum of homogeneous polynomials P_m of positive degree, we find some P_m which does not vanish at v. Then x is semi-stable. This interpretation of stability goes back to the original work of D. Hilbert ([47]).

8.2 The existence of a quotient

Let us show that the open subset of semi-stable (respectively stable) points admits a categorical (respectively geometric) quotient.

First we have to recall the definition of the gluing construction of algebraic varieties. Let $\{X_i\}_{i\in I}$ be a finite set of affine algebraic varieties. The *gluing data* is a choice of an open affine subset $U_{ij} \subset X_i$ for each $j \in I$, and an isomorphism $\phi_{ji} : U_{ij} \to U_{ji}$ for each pair $(i,j) \in I \times I$. It is required that

(i) $U_{ii} = X_i$, and ϕ_{ii} is the identity for each $i \in I$,

(ii) for any $i, j, k \in I$, $\phi_{ji}(U_{ij} \cap U_{ik}) \subset U_{jk}$ and

$$(\phi_{kj} \circ \phi_{ji})|_{U_{ij}\cap U_{ik}} = \phi_{ki}|_{U_{ij}\cap U_{ik}}.$$

Let R be an equivalence relation on the set $\bigsqcup_{i\in I} X_i$ defined by $x \sim y$ if and only if there exists a pair $(i,j) \in I \times I$ such that $x \in U_{ij}, y \in U_{ij}$ and $y = \phi_{ij}(x)$. The assumptions (i) and (ii) show that it is indeed an equivalence relation. Let X be the corresponding factor set and let $p : \bigsqcup_{i\in I} X_i \to X$ be the canonical projection. We equip X with the topology for which a subset V is open if and only if $p^{-1}(V)$ is open in the Zariski topology. The restriction p_i of p to X_i defines a homeomorphism of X_i with an open subset V_i of X so that $X = \bigcup_{i\in I} V_i$ and $p_i(U_{ij}) = V_i \cap V_j$. We also introduce the notion of a regular function on an open subset $V \subset X$. By definition, this is a collection of regular functions f_i on $p_i^{-1}(V) \subset X_i$ such that $(f_j|_{U_{ji}}) \circ \phi_{ji} = f_i|_{U_{ij}}$ for any $i, j \in I$. Let $\mathcal{O}_X(V)$ be the k-algebra of regular functions on V. The assignment $V \to \mathcal{O}_X(V)$ is a sheaf of k-algebras, called the *structure sheaf* of X. The pair $(X, \mathcal{O}_X)$ is an example of a *ringed space*, i.e., a topological space equipped with a sheaf of rings. Ringed spaces form a category. A morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ such that for any open subset $V \subset Y$ and $\phi \in \mathcal{O}_Y(V)$, the composition $\phi \circ f \in \mathcal{O}_X(f^{-1}(V))$. Each open subset V of X is equipped with the structure of a ringed space whose structure sheaf $\mathcal{O}_V$ is equal to the restriction of $\mathcal{O}_X$ to V. Each quasi-projective algebraic variety can be considered as a ringed space, the structure sheaf is the sheaf of regular functions. It follows from the definition that the ringed space $(X, \mathcal{O}_X)$ obtained by gluing of affine varieties is locally isomorphic to an affine variety, i.e., it admits an open cover by subsets which are isomorphic to affine varieties as ringed spaces; in the notation from above each open set V_i is isomorphic to X_i. Thus we are led to the notion of an *abstract algebraic variety* which is a ringed space locally isomorphic to an affine algebraic variety. One usually adds a separatedness property which ensures that the intersection of two open affine subsets is an affine variety. An abstract algebraic variety X is isomorphic to a quasi-projective algebraic variety if and only if there exists an ample line bundle L over X which is used to embed X into projective space. We leave it to the reader to define the notion of a line bundle over an abstract algebraic variety. A useful criterion of ampleness of a line bundle was given in Remark 8.1.3.

Theorem 8.1. *There exists a good categorical quotient*

$$\pi : X^{\mathrm{ss}}(L) \to X^{\mathrm{ss}}(L)/\!\!/G.$$

There is an open subset U in $X^{\mathrm{ss}}(L)/\!\!/G$ such that $X^{\mathrm{s}}(L) = \pi^{-1}(U)$ and the restriction of π to $X^{\mathrm{s}}(L)$ is a geometric quotient of $X^{\mathrm{s}}(L)$ by G. Moreover there exists an ample line bundle M on $X^{\mathrm{ss}}(L)//G$ such that $\pi^*(M) = L^{\otimes n}$, restricted to $X^{\mathrm{ss}}(L)$, for some $n \geq 0$. In particular, $X^{\mathrm{ss}}(L)/\!\!/G$ is a quasi-projective variety.

Proof. Since any open subset of X is quasi-compact in the Zariski topology we can find a finite set $\{s_1, \ldots, s_r\}$ of invariant sections of some tensor power of L such that $X^{\mathrm{ss}}(L)$ is covered by the sets X_{s_i}. Obviously we may assume that all the s_i belong to $\Gamma(X, L^{\otimes N})^G$ for some sufficiently large N. Let $U_i = X_{s_i}, i = 1, \ldots, r$. For every U_i, we consider the ring $\mathcal{O}(U_i)^G$ of G–invariant regular functions and let $\pi_i : U_i \to Y_i := U_i/\!\!/G$ with $\mathcal{O}(Y_i) = \mathcal{O}(U_i)^G$ as constructed in

Nagata's Theorem. For each i, j we can consider s_i/s_j as a regular G-invariant function on U_j. Let $\phi_{ij} \in \mathcal{O}(Y_j)$ be the corresponding regular function on the quotient. Consider the open subset $D(\phi_{ij}) \subset Y_j$. Obviously

$$\pi_j^{-1}(D(\phi_{ij})) = \pi_i^{-1}(D(\phi_{ji})) = U_i \cap U_j.$$

This easily implies that both sets $D(\phi_{ij})$ and $D(\phi_{ji})$ are categorical quotients of $U_i \cap U_j$. By the uniqueness of categorical quotient there is an isomorphism $\alpha_{ij} : D(\phi_{ij}) \to D(\phi_{ji})$. It is easy to see that the set of isomorphisms $\{\alpha_{ij}\}$ satisfies the conditions of gluing. So we can glue together the quotients Y_i and the maps π_i to obtain a morphism $\pi : X^{\text{ss}}(L) \to Y$, where $Y = X^{\text{ss}}(L)/\!\!/G$. To show that Y is separated it is enough to observe that it admits an affine open cover by the sets Y_i which satisfies the following properties: $Y_i \cap Y_j \cong U_i \cap U_j/\!\!/G$ are affine and $\mathcal{O}(Y_i \cap Y_j)$ is generated by restrictions of functions from $\mathcal{O}(Y_i)$ and $\mathcal{O}(Y_j)$. The latter property follows from the fact that $\mathcal{O}(U_i \cap U_j)$ is generated by restrictions of functions from $\mathcal{O}(U_i)$ and $\mathcal{O}(U_j)$. In fact, the separatedness also follows from the assertion that Y is quasi-projective. So let us concentrate on proving the latter.

Note that the cover $\{U_i\}_{i=1,\dots,r}$ of $X^{\text{ss}}(L)$ is a trivializing cover for the line bundle L' obtained by restriction of L to $X^{\text{ss}}(L)$. In fact, by Remark 8.1.3, L' is ample; hence we may assume that some tensor power $L'^{\otimes t}$ is very ample. This implies that $L'^{\otimes t}$ is equal to the line bundle $f^*(\mathcal{O}_{\mathbb{P}^n}(1))$ for some embedding $f : X^{\text{ss}}(L) \to \mathbb{P}^n$. The section $s_i^{\otimes t}$ of $L'^{\otimes t}$ is equal to the section $f^*(h)$ where h is a section of $\mathcal{O}_{\mathbb{P}^n}(1)$. Thus the open subset U_i is equal to $f^{-1}(V_i)$ where V_i is an open subset of $\mathbb{P}^n$ isomorphic to affine space. This shows that L' restricted to U_i is equal to $(f|_{U_i})^*(\mathcal{O}_{\mathbb{P}^n}(1)|V_i)$. However, $\mathcal{O}_{\mathbb{P}^n}(1)|V$ is isomorphic to the trivial line bundle since any line bundle over affine space is isomorphic to the trivial bundle. By fixing some trivializing isomorphisms we can identify the functions $(s_i/s_j)|_{U_i \cap U_j}$ with the transition functions g_{ij} of L'. As we have shown before, $s_i/s_j = \pi^*(\phi_{ij})$ for some functions $\phi_{ij} \in \mathcal{O}(Y_j)$. We use the transition functions $h_{ij} = \phi_{ij}|Y_i \cap Y_j$ to define a line bundle M on Y. Obviously $\pi^*(M) \cong L'$. Let us show that M is ample. First we define some sections t_j by setting $t_j|_{Y_i} = \phi_{ij}$ for a fixed j and variable i. Since for any i_1, i_2

$$\phi_{i_2 j} = \phi_{i_1 j}\phi_{i_2 i_1}$$

the $t_j|_{Y_{i_1} \cap Y_{i_2}}$ differ by the transition function of M, hence t_j is in fact a section of M. Clearly $\pi^*(t_j) = s_j$ and $Y_{t_j} = Y_j$. As above, since all Y_j are affine, we obtain that M is ample. Since $\pi : X^{\text{ss}}(L) \to Y$ is obtained by gluing together good categorical quotients, the morphism π is a good categorical quotient.

It remains to show that the restriction of π to $X^{\mathrm{s}}(L)$ is a geometric quotient. By definition $X^{\mathrm{s}}(L)$ is covered by affine open G-invariant sets where G acts with closed orbits. Since π is a good categorical quotient, for any $x \in X^{\mathrm{s}}(L)$ the fibre $\pi^{-1}(\pi(x))$ consists of one orbit. Thus $\pi|_{X^{\mathrm{s}}(L)}$ is a good geometric quotient. $\square$

In the case when L is ample and X is projective, the following construction of the categorical quotient $X^{\mathrm{ss}}(L)/\!\!/G$ is equivalent to the previous one.

Proposition 8.1. *Assume that X is projective and L is ample. Let*

$$R = \bigoplus_{n\geq 0} \Gamma(X, L^{\otimes n}).$$

Then

$$X^{\mathrm{ss}}(L)/\!\!/G \cong \mathrm{Projm}(R^G).$$

In particular, the quotient $X^{\mathrm{ss}}(L)/\!\!/G$ is a projective variety.

Proof. First of all, we observe that by Nagata's Theorem the algebra R^G is finitely generated. It also has a natural grading, induced by the grading of R. Replacing L by $L^{\otimes d}$ we may assume that R^G is generated by elements $s_0, \ldots, s_n$ of degree 1. Let $Y = \mathrm{Projm}(R^G)$ be the projective subvariety of $\mathbb{P}^n$ corresponding to the homogeneous ideal I equal to the kernel of the homomorphism $k[T_0, \ldots, T_n] \to R^G, T_i \mapsto s_i$. (The reader should go back to Chapter 3 to recall the definition of $\mathrm{Projm}(A)$ for any finitely generated graded k-algebra A.) The elements s_i generate the ideal $\mathfrak{m} = R^G_+$ generated by homogeneous elements of positive degree. Thus the affine open sets $U_i = X_{s_i}$ cover $X^{\mathrm{ss}}(L)$. On the other hand the open sets $Y_i = Y \cap \{T_i \neq 0\}$ form an open cover of Y with the property that $\mathcal{O}(Y_i) = \mathcal{O}(U_i)^G$. The maps $U_i \to Y_i$ define a morphism $X^{\mathrm{ss}}(L) \to Y$ which coincides with the categorical quotient defined in the proof of the preceding theorem. $\square$

Remark 8.2. If we assume that L is very ample, and embeds X in the projective space $\mathbb{P}(\Gamma(X, L)^*) = \mathbb{P}(V)$, then we can interpret the null-cone as follows. The sections s_i from the proof of the preceding proposition define a G-equivariant rational map $X \to \mathbb{P}^n, x \mapsto (s_0(x), \ldots, s_n(x))$. The closed subset of X where this map is not defined is exactly the closed subvariety of X equal to $X \cap \overline{\mathcal{N}(G;V)}$, where the bar denotes the image of the null-cone $\mathcal{N}(G;V)$ in $\mathbb{P}(V)$. So deleting this closed subset from X we obtain the set $X^{\mathrm{ss}}(L)$ and the quotient map $X^{\mathrm{ss}}(L) \to X^{\mathrm{ss}}(L)/\!\!/G$.

Remark 8.3. Note that the morphism $X^{\mathrm{ss}}(L) \to X^{\mathrm{ss}}(L)/\!\!/G$ is affine, i.e., inverse image of an affine open set is affine. There is also the following converse of the preceding proposition. Let U be a G-invariant open subset of X such that the geometric quotient $\pi : U \to U/G$ exists and is an affine map. Assume U/G is quasi-projective. Then there exists a G-linearized line bundle L such that $U \subseteq X^s(L)$. For the proof we refer to [75], p. 41.

8.3 Examples

Example 8.1. Let $X = \mathbb{P}^n$ and $G = \mathrm{SL}_{n+1}$ acting on $\mathbb{P}^n$ naturally via its linear representation. We know that $L = \mathcal{O}_{\mathbb{P}^n}(1)$ admits a unique SL_{n+1} linearization (Exercise 7.7). We also know from Chapter 5 that $\mathrm{Pol}_m(k^{n+1})$ is an irreducible representation for G. Therefore, for any $m > 0$,

$$\Gamma(X, \mathcal{O}_{\mathbb{P}^n}(m))^G = \mathrm{Pol}_m(k^{n+1})^G = \{0\}.$$

This shows that $X^{\mathrm{ss}}(L) = \emptyset$.

Example 8.2. Let $X = \mathbb{P}^n$, $G = \mathbb{G}_m$ with action defined by the formula

$$t \cdot (x_0, \dots, x_n) = (t^{q_0} x_0, \dots, t^{q_n} x_n).$$

Here $q_0, \dots, q_n$ are some integers. We assume that $q_0 \le q_1 \le \dots \le q_n$. Since $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}\mathcal{O}_{\mathbb{P}^n}(1)$ and $\mathcal{X}(\mathbb{G}_m) \cong \mathbb{Z}$ we have $\mathrm{Pic}^G(\mathbb{P}^n) \cong \mathbb{Z}^2$. A G-linearized bundle must be of the form $\mathcal{O}_{\mathbb{P}^n}(m)$; it defines a G-equivariant *Veronese embedding* $\mathbb{P}^n \to \mathbb{P}^{N(m)}$, where $N(m) = \dim k[T_0, \dots, T_n]_m - 1 = \binom{n+m}{m} - 1$. The group $\mathbb{G}_m$ acts on $\mathbb{P}^{N(m)}$ by the formula $t : x_{i_1 \dots i_m} \mapsto t^{q_{i_1} + \dots + q_{i_m}} x_{i_1 \dots i_m}$, where $x_{i_1 \dots i_m}$ is the coordinate in the Veronese space corresponding to the monomial $x_{i_1} \dots x_{i_m}$, $i_1 \le \dots \le i_m$. Now the linearization is given by a linear representation of $\mathbb{G}_m$ in the space $(k[T_0, \dots, T_n]_m)^*$ which lifts the action in the corresponding projective space. Obviously it is defined by the formula

$$t : x_{i_1 \dots i_m} \mapsto t^{-a} t^{q_{i_1} + \dots + q_{i_m}} x_{i_1 \dots i_m}, \tag{8.1}$$

for some integer a. Thus the G-linearized bundles can be indexed by the pairs $(m, a) \in \mathbb{Z}^2$. Denote the corresponding line bundle by $L_{m,a}$. Raising $L_{m,a}$ to the rth power as a G-linearized bundle corresponds to replacing (m, a) with (rm, ra).

We know that $X^{\mathrm{ss}}(L)$ does not change if we replace L by $L^{\otimes r}$. So we may assume that $L = L_{1,p/q}$, where by definition $H^0(\mathbb{P}^n, L_{1,p/q}^{\otimes N})^G$ is defined only for N

divisible by q and $H^0(\mathbb{P}^n, L_{1,p/q}^{\otimes N}) = H^0(\mathbb{P}^n, L_{N,Np/q})^G$. In other words we permit a to be a rational number in formula (8.1) and consider invariant polynomials of degree a multiple of the denominator of a. Here the invariance means that for any $t \in k^*$,

$$F(t^{-a+q_0}x_0, \ldots, t^{-a+q_n}x_n) = F(x_0, \ldots, x_n).$$

Assume now that $q_0 \leq 0$. It is obvious that $\Gamma(\mathbb{P}^n, L_{1,a}^{\otimes N})^G = 0$ for all $N > 0$ if $a \leq q_0$ or $a \geq q_n$. This implies that $X^{\text{ss}}(L_{1,a}) = \emptyset$ if $a \notin [q_0, q_n]$.

When $a = q_0$, we have

$$\bigoplus_{N=0}^{\infty} H^0(\mathbb{P}^n, L_{1,a}^{\otimes N})^G = k[T_0, \ldots, T_m],$$

if $q_0 = \cdots = q_m < q_{m+1}$. Hence

$$X^{\text{ss}}(L_{1,a}) = \mathbb{P}^n \setminus \{x_0 = \cdots = x_m = 0\}$$

and

$$X^{\text{ss}}(L_{1,a})/\!\!/G = \text{Projm}(k[T_0, \ldots, T_m]) = \mathbb{P}^m.$$

In particular, if $q_1 > q_0$, the quotient is the point.

Next, we increase the parameter a. If $q_m < a \leq q_{m+1}$, we have further invariant polynomials. For example, if $a = s/d$, the monomial $T_0^{dq_{m+1}-s}T_{m+1}^{dq_0-s}$ belongs to $\bigoplus_{N=0}^{\infty} H^0(\mathbb{P}^n, L_{1,a}^{\otimes N})^G$. So the set $X^{\text{ss}}(L_{1,a})$ becomes larger and the categorical quotient changes. In fact one can show that the quotients do not change when a stays strictly between two different weights q_i and do change otherwise.

Example 8.3. Consider the special case of the previous example where $q_0 = 0$ and $q_1 = \cdots = q_n = 1$. The restriction of the action to $\mathbb{A}^n$ is given by the formula

$$t \cdot (z_1, \ldots, z_n) = (t \cdot z_1, \ldots, t \cdot z_n).$$

If we take $L = L_{1,a}$ for $a = 1/2$ we get

$$\bigoplus_{m=0}^{\infty} \Gamma(\mathbb{P}^n, L_{1,a}^{\otimes 2m})^G = k[T_0T_1, \ldots, T_0T_n].$$

This shows that $X^{\text{us}}(L) = V(T_0) \cup V(T_1, \ldots, T_n)$. In other words, the set of semi-stable points is equal to the complement of the hyperplane at infinity $T_0 = 0$ and the point $(1, 0, \ldots, 0)$. So it can be identified with $\mathbb{A}^n \setminus \{0\}$. The quotient is of course $\mathbb{P}^{n-1}$. Since the group G acts on this set with trivial stabilizers, we obtain that all orbits are closed and the quotient is a good geometric quotient.

Similar conclusions can be made for any rational $a \in (0,1)$. If $a = 1$, we have

$$\bigoplus_{m=0}^{\infty} \Gamma(\mathbb{P}^n, L_{1,1}^{\otimes m})^G = k[T_1, \ldots, T_n].$$

Thus

$$X^{\text{us}}(L) = \mathbb{P}^n \setminus V(T_1, \ldots, T_n) = \mathbb{P}^n \setminus \{(1,0,\ldots,0)\}.$$

The categorical quotient is the same $\mathbb{P}^n$ but the set of semi-stable points is different.

Example 8.4. Let $X = \mathbb{A}^n$ and $G = \mathbb{G}_m$. Every line bundle is isomorphic to the trivial bundle $L = X \times \mathbb{A}^1$. As we saw in Chapter 7, its G-linearization is defined by the formula

$$t \cdot (z, v) = (t \cdot z, \chi(t)v),$$

where $\chi : \mathbb{G}_m \to \mathbb{G}_m$ is a homomorphism of algebraic groups. It is easy to see that any such homomorphism is given by the formula $t \mapsto t^\alpha$ for some integer α. In fact $\chi^* : k[T, T^{-1}] \to k[T, T^{-1}]$ is defined by the image of T, and the condition that this map is a homomorphism implies that the image is a power of T. So let L_α denote the G-linearized line bundle which is trivial as a bundle and whose linearization is given by the formula

$$t \cdot (z, v) = (t \cdot z, t^\alpha v).$$

A section $s : X \to L_\alpha$ of L_α is given by the formula

$$s(z) = (z, F(z))$$

for some polynomial $F(Z) \in k[Z] = \mathcal{O}(\mathbb{A}^n)$. The group G acts on the space of sections by the formula $s \mapsto {}^t s$, where

$${}^t s(z) = (z, t^\alpha \cdot F(t^{-1} \cdot z)).$$

Thus $s \in \Gamma(X, L_\alpha^{\otimes m})^G$ if and only if

$$F(t \cdot z) = t^{m\alpha} \cdot F(z) \quad \text{for all } z \in k^n, t \in k^*.$$

When $\alpha = 0$, the constant polynomial 1 defines an invariant section of $L^{\otimes m}$ for any m. Thus $X^{\text{ss}}(L_0) = X$ and

$$X/\!\!/G = \text{Spm}(\mathcal{O}(X)^G) = \text{Spm}(k[Z_1, \ldots, Z_n]^{\mathbb{G}_m}).$$

Recall that a $\mathbb{G}_m$-action on an affine variety is equivalent to a $\mathbb{Z}$-grading of its ring of regular functions; the ring of invariants is the subring of elements of degree 0 (see Example 3.1). In our case $\mathcal{O}(X) \cong k[Z_1, \ldots, Z_n]$ but the variables Z_i are not necessarily homogeneous. If we can make a linear change of variables such that they are homogeneous, then the action is given by a formula

$$t \cdot (z_1, \ldots, z_n) = (t^{\alpha_1} z_1, \ldots, t^{\alpha_n} z_n).$$

In this case we say that the action of $\mathbb{G}_m$ on $\mathbb{A}^n$ is *linearizable* . It is an open problem (a very difficult one) whether any action of $\mathbb{G}_m$ on affine space is linearizable. It is known to be true for $n \leq 3$.

Assume now that $\alpha > 0$. Since we know that the set of semi-stable points and the quotient do not change when we replace L by its tensor power, we may assume that $\alpha = 1$. Then

$$\bigoplus_{m=0}^{\infty} \Gamma(X, L_\alpha^{\otimes m})^{\mathbb{G}_m} = \bigoplus_{m=0}^{\infty} k[Z_1, \ldots, Z_n]_m := k[Z_1, \ldots, Z_n]_{\geq 0}.$$

The subring $k[Z_1, \ldots, Z_n]_{\geq 0}$ is a finitely generated algebra over $k[Z_1, \ldots, Z_n]_0$. Thus

$$\bigoplus_{m>0}^{\infty} \Gamma(X, L_\alpha^{\otimes m})^{\mathbb{G}_m} = k[Z_1, \ldots, Z_n]_{>0}$$

is a finitely generated ideal in $k[Z_1, \ldots, Z_n]_{\geq 0}$. Let $f_1, \ldots, f_m$ be its homogeneous generators. Then

$$\begin{aligned} X^{\text{ss}}(L_\alpha) &= D(f_1) \cup \cdots \cup D(f_m), \\ X^{\text{ss}}(L_\alpha)/\!\!/\mathbb{G}_m &= D_+(f_1) \cup \cdots \cup D_+(f_m), \end{aligned}$$

where $D_+(f_i) = \text{Spm}(k[Z_1, \ldots, Z_n]_{(f_i)})$ (see Example 3.1).

Similar conclusion can be reached in the case $\alpha < 0$.

Example 8.5. A special case of the previous example is when $\mathbb{G}_m$ acts on $\mathbb{A}^n$ by the formula

$$t \cdot (z_1, \ldots, z_n) = (t^{q_1} z_1, \ldots, t^{q_n} z_n),$$

where $q_i > 0$. If $\alpha = 0$, we get $k[Z_1, \ldots, Z_n]_0 = k$ so the quotient is one point. If $\alpha < 0$, we get $k[Z_1, \ldots, Z_n]_{<0} = \{0\}$, so the set of semi-stable points is empty. Finally, if $\alpha > 0$, we get

$$X^{\text{ss}} = D(Z_1) \cup \cdots \cup D(Z_n) = \mathbb{A}^n \setminus \{0\},$$

and the construction of the categorical quotient coincides with the construction of the weighted projective space $\mathbb{P}(q_1, \ldots, q_n)$ (see Example 3.1). So we see two different ways to define $\mathbb{P}^n$: as a quotient of $\mathbb{P}^{n+1}$ and as a quotient of $\mathbb{A}^{n+1}$.

Example 8.6. Let G be again $\mathbb{G}_m$ and $X = \mathbb{A}^4$ with the action given by the formula

$$t \cdot (z_1, z_2, z_3, z_4) = (tz_1, tz_2, t^{-1}z_3, t^{-1}z_4).$$

As in the previous example, each G-linearized line bundle is isomorphic to the trivial line bundle with the G-linearization defined by an integer α. We have

$$\Gamma(X, L_\alpha^{\otimes r})^G = k[Z]_{r\alpha}.$$

However, this time the grading is weighted; the weights are $(1, 1, -1, -1)$.

Assume $\alpha = 0$. Then for any $r > 0, 1 \in \Gamma(X, L_0^{\otimes r})^G = \Gamma(X, L_0)^G$. Hence $X = X^{\mathrm{ss}}(L)$, and

$$\mathcal{O}(X)^G = k[Z]_0 = k[Z_1Z_3, Z_1Z_4, Z_2Z_3, Z_2Z_4] \subset k[Z].$$

We have a surjection $k[T_1, T_2, T_3, T_4] \to \mathcal{O}(X)^G$, defined by $T_1 \mapsto Z_1Z_3, T_2 \mapsto Z_1Z_4,\ T_3 \mapsto Z_2Z_3,\ T_4 \mapsto Z_2Z_4$. This shows that

$$\mathcal{O}(X)^G \cong k[T_1, T_2, T_3, T_4]/(T_1T_4 - T_2T_3).$$

Thus $X^{\mathrm{ss}}(L)/\!\!/\mathbb{G}_m$ is isomorphic to the closed subvariety Y_0 of $\mathbb{A}^4$ given by the equation

$$T_1T_4 - T_2T_3 = 0.$$

This is a quadric cone. It has one singular point at the origin.

Assume $\alpha > 0$. Again, without loss of generality we may take $\alpha = 1$. It is easy to see that

$$\bigoplus_{r>0} k[Z]_r = k[Z]_{>0} = Z_1k[Z]_{\geq 0} + Z_2k[Z]_{\geq 0}.$$

Thus

$$X^{\mathrm{ss}}(L_1) = \mathbb{A}^4 \setminus V(Z_1, Z_2).$$

This set is covered by $U_1 = D(Z_1)$ and $U_2 = D(Z_2)$. We have

$$\begin{aligned} \mathcal{O}(U_1)^G &= k[Z]_{(Z_1)} = k[Z]_0[Z_2/Z_1], \\ \mathcal{O}(U_2)^G &= k[Z]_{(Z_2)} = k[Z]_0[Z_1/Z_2]. \end{aligned}$$

We claim that $X^{\mathrm{ss}}(L_1)/G$ is isomorphic to a closed subvariety Y' of $\mathbb{A}^4 \times \mathbb{P}^1_k$ given by the equations

$$T_1Z_2 - T_3Z_1 = 0, \quad T_2Z_2 - T_4Z_1 = 0, \quad T_1T_4 - T_2T_3 = 0.$$

Here we use (Z_1, Z_2) for homogeneous coordinates in $\mathbb{P}^1$. In fact, this variety is covered by the two affine open sets Y_i' given by $Z_i \neq 0, i = 1, 2$. It is easy to see that $\mathcal{O}(Y_i') \cong \mathcal{O}(U_i)^G$. We also verify that these two sets are glued together as they should be according to our construction of the categorical quotient. Thus we obtain an isomorphism $Y' \cong Y_+ := X^{\mathrm{ss}}(L_1)/\!\!/\mathbb{G}_m$. In fact, we have $X^{\mathrm{ss}}(L_1) = X^s(L_1)$ so that Y_+ is a geometric quotient. Note that we have a canonical morphism

$$f_+ : Y_+ \to Y_0$$

which is given by the inclusion of the rings $k[Z]_0 \subset \mathcal{O}(U_i)^G$. Geometrically it is induced by the projection $\mathbb{A}^4 \times \mathbb{P}^1 \to \mathbb{A}^4$. Over the open subset $Y_0 \setminus \{0\}$ this morphism is an isomorphism. In fact, $Y_0 \setminus \{0\}$ is covered by the open subsets $U_i = Y_0 \cap D(T_i), i = 1, \ldots, 4$. The inverse image $\overline{U}_1 = f_+^{-1}(U_1)$ is contained in the open subset where $Z_1 \neq 0$. Since $Z_2/Z_1 = T_3/T_1$ we see that f_+ induces an isomorphism $\mathcal{O}(U_1) \to \mathcal{O}(\overline{U}_1)$. Similarly we treat the other pieces U_i. Over the origin, the fibre of f_+ is isomorphic to $\mathbb{P}^1$. Also, we immediately check that Y_+ is a nonsingular variety. Thus $f_+ : Y_+ \to Y_0$ is a resolution of singularities of Y_0. It is called a *small resolution* because the exceptional set is of codimension > 1. The reader familiar with the notion of the blowing up will recognize Y_+ as the variety obtained by blowing up the closed subvariety of Y_0 defined by the equations $T_1 = T_3 = 0$.

Assume $\alpha < 0$. Similar arguments show that $Y_- = X^s(L_{-1})/\mathbb{G}_m$ is isomorphic to the closed subvariety of $\mathbb{A}^4 \times \mathbb{P}^1_k$ given by the equations

$$T_1Z_4 - T_2Z_3 = 0, T_3Z_4 - T_4Z_3 = 0, T_1T_4 - T_2T_3 = 0.$$

We have a morphism

$$f_- : Y_- \to Y_0$$

which is an isomorphism over $Y_0 \setminus \{0\}$ and whose fibre over $\{0\}$ is isomorphic to $\mathbb{P}^1$. The diagram

$$\begin{array}{ccc} Y_+ & & Y_- \\ f_+ \searrow & & \swarrow f_- \\ & Y & \end{array}$$

represents a type of birational transformations between algebraic varieties which nowadays is called a "flip". Note that Y_+ is not isomorphic to Y_-, but they are isomorphic outside the fibres $f_\pm^{-1}(0) \cong \mathbb{P}^1$.

Bibliographical notes

The theory of stable points with respect to an algebraic action was developed in [75]. There is nothing original in our exposition. The examples given in the chapter show the dependence of the sets of stable points on the choice of linearization of the action. Although this fact was implicitly acknowledged in [75], the serious study of this dependence began only recently; see [23], [117] and the references there. One of the main results of the theory developed in these papers is the finiteness of the set of open subsets which can be realized as the set of semi-stable points for some linearization.

Exercises

8.1 Let X be a homogeneous space with respect to an action of an affine algebraic group G. Assume X is not affine. Show that for any $L \in \mathrm{Pic}^G(X)$ the set $X^{\mathrm{ss}}(L)$ is empty.

8.2 A G-linearized line bundle is called G-*effective* if $X^{\mathrm{ss}}(L) \neq \emptyset$. Show that $L \otimes L'$ is G-effective if both L and L' are G-effective.

8.3 Let $\mathbb{G}_m$ act on an affine algebraic variety X and let $\mathcal{O}(X) = \bigoplus_{i\in\mathbb{Z}} \mathcal{O}(X)_i$ be the corresponding grading. Define $A_0 = \mathcal{O}(X)_0, A_{\geq 0} = \bigoplus_{i\geq 0} \mathcal{O}(X)_i$ and similarly $A_{\leq 0}, A_{>0}, A_{<0}$. Let $L \in \mathrm{Pic}^G(X)$ be trivial as a line bundle. Show that there are only three possibilities (up to isomorphism): $X^{\mathrm{ss}}(L) = X, X \setminus V(I_+), X \setminus V(I_-)$, where I_+ (resp. I_-) is the ideal in $\mathcal{O}(X)$ generated by A_+ (resp. A_-). Show that in the first case $X^{\mathrm{ss}}(L)/\!\!/\mathbb{G}_m$ is isomorphic to $\mathrm{Spm}(A_0)$, in the second (resp. the third) case $X^{\mathrm{ss}}(L)/\!\!/\mathbb{G}_m$ is isomorphic to $\mathrm{Projm}(A_{\geq 0})$ (resp. $\mathrm{Projm}(A_{\leq 0})$).

8.4 In Example 8.6 show that the fibred product $\widetilde{Y} = Y_+ \times_{Y_0} Y_-$ is a nonsingular variety. Its projection to Y_0 is an isomorphism outside the origin, and the inverse image E of the origin is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Show that the restrictions of the projections from $\widetilde{Y}$ to $Y_\pm$ to E coincide with the two projection maps $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$.

8.5 Let G be a finite group acting regularly on X. Show that for any $L \in \mathrm{Pic}^G(X)$, $X^{\mathrm{ss}}(L) = X^{\mathrm{s}}(L)$. Also $X^{\mathrm{s}}(L) = X$ if L is ample. Show that the assumption of ampleness is essential (even for the trivial group!).

8.6 Let $G = \mathrm{SL}_n$ act by conjugation on the affine space M_n of $n \times n$ matrices. Consider the corresponding action of G on the projective space $X = \mathbb{P}(M_n)$. Find the sets $X^{\mathrm{ss}}(L), X^{\mathrm{s}}(L)$ where $L \in \mathrm{Pic}^G(X)$.

8.7 Let $i : Y \hookrightarrow X$ be a closed G-invariant embedding, and let $L_Y = i^*(L)$ where L is an ample G-linearized line bundle on X. Assume that X is projective and G is linearly reductive, e.g. char(k) = 0. Prove that, for any $y \in Y$,

$$\begin{aligned} y \in Y^{\mathrm{s}}(i^*(L)) \quad &\Leftrightarrow \quad i(y) \in X^{\mathrm{s}}(L), \\ y \in Y^{\mathrm{s}}(i^*(L))_{(0)} \quad &\Leftrightarrow \quad i(y) \in X^{s}(L)_{(0)}. \end{aligned}$$

8.8 Consider Example 8.1 with $n = 3$ and $q_0 = 0, q_1 = 2, q_2 = 2, q_3 = 3$. Find all possible categorical quotients.

Chapter 9

Numerical criterion of stability

9.1 The function $\mu(x, \lambda)$

In this chapter we shall prove a numerical criterion of stability due to David Hilbert and David Mumford. It is stated in terms of the restriction of the action to one-parameter subgroups. The idea of the criterion is as follows. Suppose an affine algebraic group G acts on a projective variety $X \subset \mathbb{P}^n$ via a linear representation $\rho : G \to \mathrm{GL}_{n+1}$. This can be achieved by taking a very ample G-linearized line bundle L on X. As in Chapter 8, we denote by x^* a representative in k^{n+1} of a point $x \in X$. We know that $x \in X^{\mathrm{us}}(L)$ if and only if $0 \in \overline{G \cdot x^*}$. If H is a subgroup of G, then $\overline{H \cdot x^*} \subset \overline{G \cdot x^*}$, so one may detect an unstable point by checking that $0 \in \overline{H \cdot x^*}$ for some subgroup H of G. Let us take for H the image of a regular homomorphism $\lambda : \mathbb{G}_m \to G$ (a *one-parameter subgroup* of G). In appropriate coordinates it acts by the formula

$$\lambda(t) \cdot x^* = (t^{m_0} x_0, \dots, t^{m_n} x_n).$$

Suppose all m_i for which $x_i \neq 0$ are strictly positive. Then the map

$$\lambda_{x^*} : \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^{n+1}, \quad t \mapsto \lambda(t) \cdot x^*$$

can be extended to a regular map $\bar{\lambda}_{x^*} : \mathbb{A}^1 \to \mathbb{A}^{n+1}$ by sending the origin of $\mathbb{A}^1$ to the origin of $\mathbb{A}^{n+1}$. It is clear that the latter belongs to the closure of the orbit of x^*, hence our point x is unstable. Similarly, if all m_i are negative, we change λ to λ^{-1} defined by the formula $\lambda^{-1}(t) = \lambda(t^{-1})$ to reach the same conclusion. Let us set

$$\mu(x, \lambda) := \min_i \{m_i : x_i \neq 0\}.$$

So we can restate the preceding remark by saying that if there exists λ in the set $\mathcal{X}(G)^*$ of one-parameter subgroups of G such that $\mu(x,\lambda) > 0$ or $\mu(x,\lambda^{-1}) > 0$, then x is unstable. In other words, we have a necessary condition for semi-stability:

$$x \in X^{\mathrm{ss}}(L) \Longrightarrow \mu(x,\lambda) \leq 0, \quad \forall \lambda \in \mathcal{X}(G)^*. \tag{9.1}$$

Assume the preceding condition is satisfied and $\mu(x,\lambda) = 0$ for some λ. Let us show that x is not stable. In the preceding notation, let $I = \{i : x_i \neq 0, m_i > 0\}$, and let $y = (y_0,\ldots,y_n)$, where $y_i = x_i$ if $i \notin I$, and $y_i = 0$ if $i \in I$. Obviously, y belongs to the closure of the orbit of x under the action of the subgroup $\lambda(\mathbb{G}_m)$. If x were stable, then by definition of stability, y must be in this orbit. However, obviously $\lambda(\mathbb{G}_m)$ fixes y, so that y cannot be stable. Thus we obtain a necessary condition for stable points:

$$x \in X^{\mathrm{s}}(L) \Longrightarrow \mu(x,\lambda) < 0, \quad \forall \lambda \in \mathcal{X}(G)^*. \tag{9.2}$$

We have to show first that the numbers $\mu(x,\lambda)$ are independent of a choice of coordinates in $\mathbb{A}^{n+1}$, and also that the previous condition is sufficient for semi-stability. Let us start with the former task and do the latter one in the next section.

Let $x \in X$ and $x^* \in \mathbb{A}^{n+1}$ be as above. Take a one-parameter subgroup $\lambda : \mathbb{G}_m \to G$; for any $t \in k^*$ the corresponding point $\lambda(t)\cdot x$ is equal to the point with projective coordinates $(t^{m_0'}x_0,\ldots,t^{m_n'}x_n)$, where $m_i' = m_i - \mu(x,\lambda)$ if $x_i \neq 0$ and anything otherwise. Thus when we let t go to 0, we obtain a point in X with coordinates $y = (y_0,\ldots,y_n)$, where $y_i \neq 0$ if and only if $x_i \neq 0$ and $m_i = \mu(x,\lambda)$. The precise meaning of "let t go to 0" is the following. We have a map

$$\lambda_x : \mathbb{A}^1 \setminus \{0\} \to X, \quad t \mapsto \lambda(t)\cdot x.$$

Since X is projective this map can be extended to a unique regular map

$$\bar{\lambda}_x : \mathbb{P}^1 \to X.$$

We set

$$\lim_{t\to 0} \lambda(t)\cdot x := \bar{\lambda}_x(0), \quad \lim_{t\to\infty} \lambda(t)\cdot x := \bar{\lambda}_x(\infty).$$

Obviously

$$\lim_{t\to\infty} \lambda(t)\cdot x = \lim_{t\to 0} \lambda(t)^{-1}\cdot x.$$

So our point y is equal to $\lim_{t\to 0}\lambda(t)\cdot x$. Now it is clear that for any $t \in k$

$$\lambda(t)\cdot y = y,$$

that is, y is a fixed point for the subgroup $\lambda(\mathbb{G}_m)$ of G. Also the definition of y is coordinate-free. Furthermore, for any vector y^* over y,

$$\lambda(t)\cdot y^* = t^{\mu(x,\lambda)}y^*. \tag{9.3}$$

This can be interpreted as follows. Restrict the action of G on X to the action of $\mathbb{G}_m$ defined by λ. Then L has a natural $\mathbb{G}_m$-linearization and, since y is a fixed point, $\mathbb{G}_m$ acts on the fibre L_y; this defines a linear representation $\rho_y : \mathbb{G}_m \to \mathrm{GL}_1 = \mathbb{G}_m$. We know the geometric interpretation of the total space $\mathbb{V}(\mathcal{O}_{\mathbb{P}^n}(-1))$ of the line bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$. It follows from this that the fibre of the canonical projection $\mathbb{A}^{n+1}\setminus\{0\}\to\mathbb{P}^n$ over a point $x\in X$ can be identified with $\mathbb{V}(\mathcal{O}_{\mathbb{P}^n}(-1))_x\setminus\{0\}$. Thus from (9.3) we get that $\mathbb{G}_m$ acts on the fibre L_y^{-1} by the character $t\mapsto t^{\mu(x,\lambda)}$. Hence it acts on the fibre L_y by the rational character $t\mapsto t^{-\mu(x,\lambda)}$. This gives us a coordinate-free definition of $\mu(x,\lambda)$. In fact, this allows one to define the number $\mu^L(x,\lambda)$ for any G-linearized line bundle L as follows. Let $y=\lim_{t\to 0}\lambda(t)\cdot x$. Then $\lambda(\mathbb{G}_m)\subset G_y$ and, as above, there is a representation of $\mathbb{G}_m$ on the fibre L_y. It is given by an integer which is taken to be $-\mu^L(x,\lambda)$.

In the case when $\mu(x,\lambda)\geq 0$, we can give another coordinate-free geometric interpretation of $\mu(x,\lambda)$. Let $I_X\subset k[T_0,\ldots,T_n]$ be the homogeneous ideal defining X in $\mathbb{P}^n$ and $A=k[T_0,\ldots,T_n]/I_X$ be the homogeneous coordinate ring of X. We have $X\cong \mathrm{Projm}(A)$. Let $C_X=\mathrm{Spm}(A)\subset\mathbb{A}^{n+1}$ be the affine cone over X. Let x and x^* be as above. A one-parameter subgroup λ as above defines a morphism

$$\bar{\lambda}_{x^*}:\mathbb{A}^1\to C_X.$$

Let $\phi : A\to k[t]$ be the corresponding homomorphism of the rings of regular functions. The image of the maximal ideal $\mathfrak{m}$ defining the vertex of C_X generates a principal ideal $(t^{m(\lambda)})\subset k[t]$. I claim that

$$m(\lambda)=\mu(x,\lambda). \tag{9.4}$$

In fact, the composition of $\phi : A\to k[t]$ with the canonical homomorphism $k[T_0,\ldots,T_n]\to A$ is given by the formula $T_i\mapsto t^{m_i}$, where

$$\bar{\lambda}_{x^*}(t)=(t^{m_0}a_0,\ldots,t^{m_n}a_n),\quad x^*=(a_0,\ldots,a_n).$$

Since $\mathfrak{m}$ is generated by the cosets of the T_i, we see that $\phi(\mathfrak{m})$ is generated by the monomials t^{m_j} such that $a_j\neq 0$. Now the assertion follows from the definition of $\mu(x,\lambda)$.

9.2 The numerical criterion

Now we are ready to prove the sufficiency of conditions (9.1) and (9.2). The following is the main result of this chapter.

Theorem 9.1. *Let G be a reductive group acting on a projective algebraic variety X. Let L be an ample G-linearized line bundle on X and let $x \in X$. Then*

$$x \in X^{\mathrm{ss}}(L) \Leftrightarrow \mu^L(x,\lambda) \leq 0 \quad \textit{for all} \quad \lambda \in \mathcal{X}(G)^*,$$

$$x \in X^{\mathrm{s}}(L) \Leftrightarrow \mu^L(x,\lambda) < 0 \quad \textit{for all} \quad \lambda \in \mathcal{X}(G)^*.$$

Before starting the proof of the theorem, let us recall the notion of properness of a map between algebraic varieties. We refer to [46] for the details.

Definition. A regular map $f : X \to Y$ of algebraic varieties over an algebraically closed field k is called *proper* if for any variety Z over k the map $f \times \mathrm{id} : X \times Z \to Y \times Z$ is closed (i.e., the image of a closed subset is closed). A variety X is *proper* (or *complete*) over k if the constant map $X \to \mathrm{Spm}(k)$ is proper.

We shall use the *valuative criterion of properness* (see [46]). For any algebraic variety X over k, and any k-algebra K, the set of morphisms of algebraic varieties $\mathrm{Spm}(K) \to X$ can be viewed as the set $X(K)$ of points with values in K. If X is affine, $X(K) = \mathrm{Hom}_k(\mathcal{O}(X), K)$, as was defined in section 3.3. If X is glued together from affine varieties X_i, and K is a field, then $X(K)$ is glued together from the $X_i(K)$.

Let R be a discrete valuation algebra over k with residue k-algebra isomorphic to k (e.g., $R = k[[t]]$ is the algebra of formal power series over k) and let Q be its field of fractions. If X is glued together from affine varieties X_i, then it is separated if and only if the natural map $X(R) \to X(Q)$ is injective (the *valuative criterion of separatedness*). In particular, it is always injective for quasi-projective algebraic varieties, with which we are dealing. A regular map $f : X \to Y$ of varieties over k defines a map $f_K : X(K) \to Y(K)$ of K-points. In particular, the residue homomorphism $R \to k$ induces a map $X(R) \to X(k)$, which is called the residue map. Then the valuative criterion of properness asserts that a regular map $f : X \to Y$ is proper if for any $y \in Y(R) \subset Y(K)$, the natural map $(f_R)^{-1}(y) \to (f_Q)^{-1}(y)$ is bijective.

Example 9.1. Any closed subvariety X of $\mathbb{P}^n$ is proper over k. First of all $\mathbb{P}^n$ is proper over k. Any Q-point of $\mathbb{P}^n$ comes from a unique R-point after multiplying its projective coordinates $(x_0, \ldots, x_n)$ by some power of a generator t of the maximal ideal of R. Now, it follows immediately from the definition of properness that a closed subvariety of a proper variety is proper. On the other hand, an affine variety is obviously not proper. Let us show that $X = \mathbb{P}^n \setminus \{(1, 0, \ldots, 0)\}$ is not complete. First notice that the point $(t, \ldots, t) \in \mathbb{A}^n(Q) = Q^n$ is a Q-point of $\mathbb{A}^n \setminus \{0\} = D(x_1) \cup \cdots \cup D(x_n)$. In fact, it belongs to any open subset $D(x_i)$ since it corresponds to a homomorphism $\phi_i : \mathcal{O}(D(x_i)) = k[x_1, \ldots, x_n]_{x_i} \to Q$ defined by $x_j \mapsto t$. However, this point does not come from any R-point of $\mathbb{A}^n \setminus \{0\}$. In fact $\phi_i(x_i^{-1}) = t^{-1} \notin R$ for any $i = 1, \ldots, n$. Now $\mathbb{A}^n \setminus \{0\} \subset \mathbb{P}^n \setminus \{(1, 0, \ldots, 0)\}$ and $(1, t, \ldots, t) \in X(Q)$ but $(1, t, \ldots, t) \notin X(R)$.

We will need the following fact.

Lemma 9.1. *(Cartan-Iwahori-Matsumoto) Let $R = k[[T]]$ be the ring of formal power series with coefficients in k and let $Q = k((T))$ be its field of fractions. For any reductive algebraic group G, any element of the set of double cosets $G(R) \backslash G(Q)/G(R)$ can be represented by a one-parameter subgroup $\lambda : \mathbb{G}_m \to G$ in the following sense. One considers λ as a $k(T)$-point of G and identifies $k(T)$ with a subfield of $k((T))$ by considering the Laurent expansion of rational functions at the origin of $\mathbb{A}^1$.*

Proof. We prove this only for the case $G = \mathrm{GL}_n$; we refer to the original paper of Iwahori and Matsumoto for the case $\mathrm{char}(k) = 0$ (see [56]). In the case of positive characteristic one has to modify the lemma (see Appendix to Chapter 1 of [75] by J. Fogarty).

A Q-point of G is a matrix A with entries in Q. We can write it as a matrix $T^r \bar{A}$, where $\bar{A} \in \mathrm{GL}(n, R)$. Since R is a principal ideal domain, we can reduce the matrix $\bar{A}$ to diagonal form so that $A = \bar{C}_1 \bar{D} \bar{C}_2$, where $\bar{C}_i \in G(R)$, and $\bar{D}$ is the diagonal matrix $\mathrm{diag}[T^{r_1}, \ldots, T^{r_n}]$. Now we can define a one-parameter subgroup of G by

$$\lambda(t) = \mathrm{diag}[t^{r_1}, \ldots, t^{r_n}].$$

Then λ represents the double coset of the point $A \in G(Q)$ as asserted. □

9.3 The proof

Let us prove Theorem 9.1. We have already proved the necessity of the conditions. First of all, by replacing L with a sufficiently high tensor power, we can place ourselves in the following situation: G acts on a projective space $\mathbb{P}^n$ by means of a linear representation $\rho : G \to \mathrm{GL}_{n+1}$, X is a G-invariant closed subvariety of $\mathbb{P}^n$. We have to prove the following.

Let $x \in X$ and $x \in X \setminus X^{\mathrm{s}}(L)$. Then there exists $\lambda \in \mathcal{X}(G)^*$ such that $\mu^L(x, \lambda) \geq 0$. Moreover, if $x \in X^{\mathrm{us}}(L)$ then there exists $\lambda \in \mathcal{X}(G)^*$ such that $\mu^L(x, \lambda) > 0$.

From now on we drop L from the notation $\mu^L(x, \lambda)$, remembering that $L = i^*(\mathcal{O}_{\mathbb{P}^n}(n+1))$.

Assume $\mu(x, \lambda) < 0$ for all $\lambda \in \mathcal{X}(G)^*$. We have to show that $x \in X^{\mathrm{s}}$. Suppose $x \notin X^{\mathrm{s}}$. Choose a point x^* over x. Then the map $a : G \to V = \mathbb{A}^{n+1}, g \mapsto g \cdot x^*$, is not proper. In fact, if it is proper, $G \cdot x^*$ is closed and the fibre of a over x^* is proper over k (Exercise 9.4). Since the fibre is a closed subvariety of an affine variety, it must consist of finitely many points (Exercise 9.3). This easily implies that G_x is finite and $G \cdot x$ is closed, so that x is a stable point, contradicting the assumption. By the valuative criterion of properness, there exists an R-point of V which, viewed as a Q-point of V, has an inverse image under $a_Q : G(Q) \to V(Q)$ but does not arise from any R-point of G. In other words, there exists an element $g \in G(Q) \setminus G(R)$ such that $g \cdot x^* \in V(R) = R^{n+1}$. By Lemma 9.1 we can write $g = g_1[\lambda]g_2$, where $g_1, g_2 \in G(R)$, and $[\lambda] \in G(Q)$ which comes from a one-parameter subgroup λ. Let $\bar{g}_2$ be the image of g_2 under the "reduction" homomorphism $G(R) \to G(k)$ corresponding to the natural homomorphism $R \to k, \sum_i a_i T^i \mapsto a_0$. We can write

$$\bar{g}_2^{-1} g_1^{-1} g = (\bar{g}_2^{-1}[\lambda]\bar{g}_2)\bar{g}_2^{-1} g_2.$$

The expression in the parentheses is a Q-point of G defined by a one-parameter subgroup $\lambda' = \bar{g}_2^{-1}\lambda\bar{g}_2$ of G. Choose a basis $(e_0, \ldots, e_n)$ in k^{n+1} such that the action of $\lambda'(\mathbb{G}_m)$ is diagonalized. That is, we may assume that

$$\lambda'(t) \cdot e_i = t^{r_i} e_i, \quad i = 0, \ldots, n.$$

This is equivalent to

$$[\lambda'] \cdot e_i = T^{r_i} e_i, \quad i = 0, \ldots, n.$$

Thus, if we write $x^* = x_0^* e_0 + \cdots + x_n^* e_n$, we obtain

$$(\bar{g}_2^{-1} g_1^{-1} g \cdot x^*)_i = ([\lambda'] \cdot (\bar{g}_2^{-1} g_2 \cdot x^*))_i = T^{r_i} (\bar{g}_2^{-1} g_2 \cdot x^*)_i.$$

Since $g \cdot x^* \in R^{n+1}$, this tells us that

$$(\bar{g}_2^{-1} g_2 \cdot x^*)_i = T^{-r_i} (\bar{g}_2^{-1} g_1^{-1} g \cdot x^*)_i \in T^{-r_i} R. \tag{9.5}$$

This implies that $r_i \geq 0$ if $x_i^* \neq 0$. In fact, the element $\bar{g}_2^{-1}g_2$ is reduced to the identity modulo (T), hence $(\bar{g}_2^{-1}g_2 \cdot x^*)_i$ modulo (T) are constants equal to x_i^*. On the other hand, they are equal to $T^{-r_i}a_i$ modulo (T) for some $a_i \in R$. This of course implies that $r_i \geq 0$ if $x_i^* \neq 0$.

Recalling our definition of $\mu(x, \lambda')$ we see that $\mu(x, \lambda') \geq 0$. This contradiction shows that $x \in X^{\mathrm{s}}$ if $\mu(x, \lambda) < 0$ for all λ.

Assume now that $\mu(x, \lambda) \leq 0$ for all λ. We have to show that $x \in X^{\mathrm{ss}}$. If x is unstable, $0 \in \overline{G \cdot x^*}$ and hence we can choose $g \in G(K) \setminus G(R)$ such that $g \cdot x^* \in R^{n+1}$ is reduced to zero modulo (T) (this follows immediately from the proof of the valuative criterion of properness). Therefore the left-hand side of (9.5) belongs to $T^{-r_i+1}R$ and hence we get $r_i > 0$ if $x_i^* \neq 0$. Thus $\mu(x, \lambda') > 0$. This contradiction proves the theorem.

9.4 The weight polytope

Recall from Chapter 5 that a linear representation of a torus $T = \mathbb{G}_m^r$ in a vector space V splits into the direct sum of eigensubspaces

$$V = \bigoplus_{\chi \in \mathcal{X}(T)} V_\chi,$$

where

$$V_\chi = \{v \in V : t \cdot v = \chi(t)v\}.$$

Also recall from Chapter 5 that there is a natural identification between the sets $\mathcal{X}(T)$ and $\mathbb{Z}^r$ which preserves the natural structures of abelian groups on both sets. We define the *weight set* of the representation space V by setting

$$\mathrm{wt}(V) = \{\chi \in \mathcal{X}(T) : V_\chi \neq \{0\}\}.$$

This is a finite subset of $\mathbb{Z}^r$. Its convex hull in $\mathbb{R}^r$ is called the *weight polytope* and is denoted by $\overline{\mathrm{wt}(V)}$. Let us choose a basis of V which is the sum of the bases of the *weight spaces* $V_\chi, \chi \in \mathrm{wt}(V)$. In this basis our representation is defined by a homomorphism $\rho : T \to \mathrm{GL}_n$ given by a formula

$$\rho((t_1, \ldots, t_r)) = \begin{pmatrix} \mathbf{t}^{\mathbf{m}_1} & 0 & \ldots & \ldots & 0 \\ 0 & \mathbf{t}^{\mathbf{m}_2} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 & \mathbf{t}^{\mathbf{m}_n} \end{pmatrix}, \tag{9.6}$$

where we use the vector notation for a monomial $\mathbf{t}^{\mathbf{m}} = t_1^{m_1} \cdots t_r^{m_r}$.

Now let $\lambda : \mathbb{G}_m \to T$ be a one-parameter subgroup of T. It is given by a formula $t \mapsto (t^{a_1}, \ldots, t^{a_r})$ for some $\mathbf{a} = (a_1, \ldots, a_r) \in \mathbb{Z}^r$. Composing the representation ρ with λ we have a representation $\rho \circ \lambda : \mathbb{G}_m \to \mathrm{GL}_n$ given by the formula

$$t \mapsto \begin{pmatrix} t^{\mathbf{a}\cdot\mathbf{m}_1} & 0 & \ldots & \ldots & 0 \\ 0 & t^{\mathbf{a}\cdot\mathbf{m}_2} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 & t^{\mathbf{a}\cdot\mathbf{m}_n} \end{pmatrix}. \tag{9.7}$$

Let $x \in \mathbb{P}(V)$ with $x^* = \sum_\chi v_\chi, v_\chi \in V_\chi$. We define the *weight set* of x by setting

$$\text{wt}(x) = \{\chi \in \mathcal{X}(G) : v_\chi \neq 0\}. \tag{9.8}$$

We define the *weight polytope* of x by setting

$$\overline{\text{wt}(x)} = \text{convex hull of wt}(x) \text{ in } \mathcal{X}(G) \otimes \mathbb{R} \cong \mathbb{R}^n. \tag{9.9}$$

If we choose coordinates in V as in (9.7) and write $x^* = (\alpha_1, \ldots, \alpha_n)$ then

$$\text{wt}(x) = \{\mathbf{m}_i : \alpha_i \neq 0\}.$$

Since $\lambda(t) \cdot x^* = (t^{\mathbf{a}\cdot\mathbf{m}_1}\alpha_1, \ldots, t^{\mathbf{a}\cdot\mathbf{m}_2}\alpha_n)$, we obtain that

$$\mu^L(x, \lambda) = \min\{\mathbf{a} \cdot \mathbf{m}_i : \alpha_i \neq 0\} = \min_{\chi \in \text{wt}(x)} \langle \lambda, \chi \rangle.$$

(Recall that the natural bilinear pairing $(\lambda, \chi) \to \langle \lambda, \chi \rangle$ between $\mathcal{X}(T)^*$ and $\mathcal{X}(T)$ is defined by the composition $\chi \circ \lambda \in \mathcal{X}(\mathbb{G}_m) = \mathbb{Z}$. When we identify $\mathcal{X}(T)^*$ and $\mathcal{X}(T)$ with $\mathbb{Z}^r$, it corresponds to the usual dot-product.)

Example 9.2. Let T be the subgroup of diagonal matrices in GL_n. Consider its natural representation in $V = k^n$. Then $\text{wt}(V) = \{e_1, \ldots, e_n\}$, where e_i are the unit basis vectors. Each e_i corresponds to the character $\chi_i : \text{diag}[t_1, \ldots, t_n] \mapsto t_i$. The weight space V_{χ_i} is the coordinate axis ke_i. The weight polytope of V is the standard simplex

$$\Delta_n = \Big\{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_i \leq 1, \sum_{i=1}^n x_i = 1\Big\}.$$

The weight set of a point $x \in \mathbb{P}^{n-1}$ with projective coordinates $(a_1, \ldots, a_n)$ is the set $\{e_i : a_i \neq 0\}$. Its weight polytope is the subsimplex $\{x \in \Delta_n : x_i \neq 0\}$. If λ is given by $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ corresponding to $t \mapsto \text{diag}[t^{m_1}, \ldots, t^{m_n}]$, then

$$\mu^{\mathcal{O}_{\mathbb{P}^{n-1}}(1)}(x, \lambda) = \min\{m_i : \alpha_i \neq 0\}.$$

Clearly, one can always find $\mathbf{m}$ such that this number is positive, so all points are unstable.

In the case when G is a torus we can restate Theorem 9.1 in the following way.

Theorem 9.2. *Let G be a torus and let L be an ample G-linearized line bundle on a projective G-variety X. Then*

$$\begin{aligned} x \in X^{\mathrm{ss}}(L) &\quad \Leftrightarrow \quad 0 \in \overline{\mathrm{wt}(x)}, \\ x \in X^{\mathrm{s}}(L) &\quad \Leftrightarrow \quad 0 \in \mathrm{interior}\big(\overline{\mathrm{wt}(x)}\big). \end{aligned}$$

Proof. We use a well-known fact from the theory of convex sets. Let Δ be a closed convex subset of $\mathbb{R}^n$. For any point $a \in \mathbb{R}^n \setminus \mathrm{interior}(\Delta)$ (resp. $a \in \mathbb{R}^n \setminus \Delta$) there exists an affine function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that $\phi(a) \le 0$ (resp. $\phi(a) < 0$), and $\phi(\Delta) \subset \mathbb{R}_{\ge 0}$. Moreover, the proof of this fact shows that one can choose ϕ with integral coefficients if Δ is the convex hull of a set of points with integral coordinates. We refer for the proofs to any textbook on convex sets (see for example [84]). The result follows. $\square$

Now let G be any reductive group acting linearly on a projective variety $X \subset \mathbb{P}^n$, and L be the restriction to X of some positive tensor power of $\mathcal{O}_{\mathbb{P}^n}(1)$. We know that any one-parameter subgroup of G has its image in a maximal torus T of G, and hence can be considered as a one-parameter subgroup of T. Now, applying Theorem 9.1, we obtain

$$X^{\mathrm{ss}}(L) = \bigcap_{\text{maximal tori } T} X_T^{\mathrm{ss}}(L_T),$$

$$X^{\mathrm{s}}(L) = \bigcap_{\text{maximal tori } T} X_T^{\mathrm{s}}(L_T).$$

Here T runs over the set of all maximal tori of G, and the subscript T indicates the restriction of the action (and the linearization) to T.

Let us fix one maximal torus T. Then for any other maximal torus T', we can find $g \in G$ such that $gT'g^{-1} = T$. From the preceding chapter we know that x is semi-stable (resp. stable) with respect to $\lambda(\mathbb{G}_m)$ if and only if $0 \notin \overline{\lambda(\mathbb{G}_m) \cdot x^*}$ (resp. $\lambda(\mathbb{G}_m) \cdot x^*$ is closed and the stabilizer of x^* in $\lambda(\mathbb{G}_m)$ is finite). It immediately follows that this property is satisfied if and only if $g \cdot x$ is semi-stable (resp. stable) with respect to $g\lambda g^{-1}(\mathbb{G}_m)$. This implies that

$$x \in X_{T'}^{\mathrm{ss}}(L_{T'}) \Leftrightarrow g \cdot x \in X_T^{\mathrm{ss}}(L_T),$$

and similarly for stable points. Putting these together we obtain

Theorem 9.3. *Let T be a maximal torus in G. Then*

$$\begin{aligned} x \in X^{\mathrm{ss}}(L) &\Leftrightarrow \forall g \in G,\ g\cdot x \in X_T^{\mathrm{ss}}(L_T), \\ x \in X^{\mathrm{s}}(L) &\Leftrightarrow \forall g \in G,\ g\cdot x \in X_T^{\mathrm{s}}(L_T). \end{aligned}$$

9.5 Kempf-stability

To finish this chapter we give a very nice necessary condition for a point to be unstable in terms of its isotropy subgroup. This is a result due to G. Kempf which is very important in applications to construction of various moduli spaces in algebraic geometry. Let $X \subset \mathbb{P}(V)$, where G acts on X via a linear representation in V. Suppose $x \in X$ is unstable. Let v be its representative in V. We know that there is a one-parameter subgroup $\lambda : \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t)\cdot v = 0$. We call λ a *destabilizing one-parameter subgroup* of x. Among all destabilizing one-parameter subgroups of x we want to consider those for which $\mu(x,\lambda)$ is maximal. Since $\mu(x,\lambda^d) = d\mu(x,\lambda)$, we should first normalize $\mu(x,\lambda)$ by dividing it by $\parallel \lambda \parallel$ and show that the maximum is defined. Here $\parallel \lambda \parallel$ means the Euclidean norm in $\mathbb{R}^n$ if we choose to identify $\mathcal{X}(T)^*$ with $\mathbb{Z}^r$; of course, the image of λ could belong to different maximal tori, so we have to proceed more carefully. First we can fix one maximal torus T. For any $\lambda \in \mathcal{X}(G)^*$ we can find $g \in G$ such that $\lambda' = g^{-1}\cdot\lambda\cdot g$ belongs to $\mathcal{X}(T)^*$. Then we can set $\parallel \lambda \parallel = \parallel \lambda' \parallel$. However, we have to check that this definition does not depend on the choice of g as above; equivalently, we have to check that $\parallel \lambda \parallel = \parallel \lambda' \parallel$ if $g^{-1}\cdot T\cdot g = T$ (i.e., g belongs to the normalizer $N_G(T)$ of T in G). The quotient group $N_G(T)/T$ is called the *Weyl group* of G. It is a finite group which acts linearly on $\mathcal{X}(T)^*$. If $G = \mathrm{GL}_n$ and T is the subgroup of diagonal matrices, we easily check that $W = N_G(T)/T$ can be represented by the permutation matrices. By conjugation, W acts on T by permutation of the diagonal entries and hence it acts on $\mathcal{X}(T)^* = \mathbb{Z}^n$ by permutation of the coordinates. In particular, $\parallel \lambda \parallel$ is W-invariant. In general we choose a norm $\parallel \lambda \parallel$ on $\mathcal{X}(T)^*$ which is W-invariant; this is always possible since W is finite. This solves our problem of defining $\parallel \lambda \parallel$ for any $\lambda \in \mathcal{X}(G)^*$. So we set

$$\nu_x(\lambda) = \frac{\mu(x,\lambda)}{\parallel \lambda \parallel}.$$

For any $\lambda \in \mathcal{X}(G)^*$ we define

$$P(\lambda) = \Big\{ g \in G : \lim_{t\to 0} \lambda(t)\cdot g\cdot \lambda(t)^{-1} \text{ exists in } G \Big\}.$$

Lemma 9.2. *$P(\lambda)$ is a subgroup of G which contains a Borel subgroup. Moreover, for any $g \in P(\lambda)$,*

$$\lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} \in Z_G(\lambda) := \{h \in G : h\lambda(t)h^{-1} = \lambda(t), \forall t \in k^*\}.$$

Proof. Again we prove this only for $G = \mathrm{GL}_n$. Without loss of generality we may assume that λ is a one-parameter subgroup of the group of diagonal matrices and is given by $\lambda(t) = \mathrm{diag}[t^{m_1}, \ldots, t^{m_n}]$. By a further change of basis we may also assume that $m_1 \le \cdots \le m_n$. Let $g = (a_{ij})$. We have

$$\lambda(t)g\lambda(t)^{-1} = (t^{m_i - m_j} a_{ij}).$$

The limit exists if and only if $a_{ij} = 0$ when $m_i < m_j$. Thus $g \in P(\lambda)$ if and only if $a_{ij} = 0$ whenever $i > j$ and $m_i \ne m_j$. It is easy to see that $P(\lambda)$ is a subgroup; it contains the group B of upper triangular matrices and is equal to this group if $m_1 < \cdots < m_n$. Now the limits $\lim_{t\to 0} \lambda(t) \cdot g \cdot \lambda(t)^{-1}, g \in P(\lambda)$, form a set of matrices $(a_{ij}) \in P(\lambda)$ such that $a_{ij} = 0$ if $m_i > m_j$. It is immediately checked that this is the subgroup $Z_G(\lambda)$. □

Lemma 9.3. *For any $g \in P(\lambda)$,*

$$\mu(x, g^{-1}\lambda g) = \mu(x, \lambda).$$

Proof. We have, for any $g \in P(\lambda)$,

$$\begin{aligned} \lim_{t\to 0}(g^{-1}\lambda(t)g) \cdot x &= \lim_{t\to 0}(g^{-1}\lambda(t)g\lambda(t)^{-1}) \cdot \lambda(t) \cdot x \\ = \lim_{t\to 0} g^{-1}(\lambda(t)g\lambda(t)^{-1})(\lambda(t) \cdot x) &= g^{-1} \lim_{t\to 0}(\lambda(t)g\lambda(t)^{-1}) \cdot y, \end{aligned}$$

where $y = \lim_{t\to 0} \lambda(t) \cdot x$. It is easy to see that $\mu(x, \lambda) = \mu(\lim_{t\to 0} \lambda(t) \cdot x, \lambda)$ (see Exercise 9.2(iv)). Therefore, putting $h = \lim_{t\to 0}(\lambda(t)g\lambda(t)^{-1})$, we obtain

$$\mu(x, g^{-1}\lambda g) = \mu(g^{-1}h \cdot y, g^{-1}\lambda g).$$

Now

$$\mu(g^{-1}h \cdot y, g^{-1}\lambda g) = \mu(h \cdot y, \lambda) = \mu(y, h^{-1}\lambda h) = \mu(y, \lambda) = \mu(x, \lambda).$$

Here we use that h centralizes λ and $\mu(x, g^{-1}\lambda g) = \mu(g \cdot x, \lambda)$ (see Exercise 9.2 (i)). This proves the assertion. □

Definition. The *flag complex* of G is the set $\Delta(G)$ of one-parameter subgroups of G modulo the following equivalence relation:

$$\lambda_1 \sim \lambda_2 \Leftrightarrow \exists n_1, n_2 \in \mathbb{Z}_{>0}, g \in P(\lambda_1) \quad \text{such that} \quad \lambda_2^{n_1} = g^{-1}\lambda_2^{n_2} g.$$

It follows from Lemma 9.2 that the function $\nu_x(\lambda)$ is well-defined as a function on $\Delta(G)$. Also the function $\lambda \mapsto P(\lambda)$ is well-defined on $\Delta(G)$. Now the idea is to find a maximum of $\nu_x : \Delta(G) \to \mathbb{R}$. It is achieved at a point $[\lambda]$ representing the one-parameter subgroup which is "most responsible" for the instability of x. The existence of such a point λ was conjectured by J. Tits and was proven by G. Kempf ([60]) and G. Rousseau ([97]). The idea is to show that ν_x is strictly convex on the set of points in $\Delta(G)$ representing destabilizing subgroups of x and achieves a maximum on this set.

Theorem 9.4. *There exists a one-parameter subgroup $\lambda_x \in \mathcal{X}(G)^*$ such that*

$$\nu_x(\lambda_x) = \max\{\nu_x(\lambda) : \lambda \in \mathcal{X}(G)^*\}.$$

All such subgroups represent the same point in $\Delta(G)$.

Definition. A one-parameter subgroup $\lambda \in \mathcal{X}(G)^*$ is called *adapted* for the point $x \in X^{\mathrm{us}}(L)$ if it satisfies the assertion of the preceding theorem.

Let $\Lambda(x)$ be the set of adapted one-parameter subgroups of x. It is an equivalence class representing one point $\delta(x) \in \Delta(G)$. We can assign to it the unique parabolic subgroup $P(\delta)$ which we denote by $P(x)$. Of course we have to remember that all of these objects depend on the linearization of the action.

Corollary 9.1. *Assume x is unstable. Then*

$$G_x \subset P(x).$$

Proof. For any $g \in G_x$ and $\lambda \in \Lambda(x)$ we have $g^{-1}\lambda g \in \Lambda(x)$. Indeed

$$\mu(x, \lambda) = \mu(g \cdot x, \lambda) = \mu(x, g^{-1}\lambda g).$$

By Theorem 9.4, we must have $P(g^{-1}\lambda g) = P(\lambda)$. It follows from the definition that $P(g^{-1}\lambda g) = g^{-1}P(\lambda)g$. However, it is known that the normalizer of a parabolic subgroup is equal to the subgroup. $\square$

Corollary 9.2. *Assume G is semisimple (e.g. $G = \mathrm{SL}_n$) and G_x is not contained in any proper parabolic subgroup of G. Then x is semi-stable with respect to any linearization.*

Proof. We use that $P(x) \neq G$ if G is semisimple. Otherwise there is an adapted one-parameter subgroup which belongs to the center of G. $\square$

In fact, one can strengthen the preceding corollary by showing that $G \cdot x^*$ is closed in V if G_x is not contained in any proper parabolic subgroup of G. This is due to Kempf ([60]). To prove it he considers a closed orbit $G \cdot y^*$in $\overline{\mathrm{O}(x^*)}$ and proves the existence of a one-parameter subgroup λ with $\lim_{t\to 0} \lambda(t) \cdot x^* \in G \cdot y^*$. Next he defines the set of adapted subgroups with this property for which the limit is reached the fastest. These subgroups define a unique proper parabolic subgroup and G_x is contained in this subgroup.

Definition. $x \in X \subset \mathbb{P}(V)$ is called *Kempf-stable* if $G \cdot x^*$ is closed in V.

This definition is obviously independent of the choice of $x^* \in V$ representing x. Note that

$$\text{stability} \Longrightarrow \text{Kempf-stability} \Longrightarrow \text{semi-stability}.$$

Indeed, if $G \cdot x$ is closed in X^{ss} then $G \cdot x^*$ is obviously closed in $V \setminus \mathcal{N}(G; V)$ (otherwise the image in $\mathbb{P}(V)$ of a point in the closure belongs to the closure of $G \cdot x$ in X^{ss}). Also $G \cdot x^*$ is closed in V since otherwise a point in its closure belongs to the null-cone and hence any invariant polynomial will vanish at x^*. Now if x is Kempf-stable, the point x^* cannot belong to the null-cone. If it does, we can find a one-parameter subgroup λ such that $\lim_{t\to 0} \lambda(t) \cdot x^* = 0$. But then 0 must belong to $G \cdot x^*$, which is absurd since $\{0\}$ is an orbit.

Thus we can generalize Corollary 9.2 to obtain:

Corollary 9.3. *Assume G is semisimple and G_x is not contained in any proper parabolic subgroup of G. Then x is Kempf-stable.*

Example 9.3. This is intended for the reader with some knowledge of the theory of abelian varieties (see [74]). Let A be an abelian variety of dimension g over an algebraically closed field k and let L be an ample divisor on A. One defines the subgroup $K(L)$ of A which consists of all points $a \in A$ such that $t_a^*(L) \cong L$. Here t_a denotes the translation map $x \mapsto x + a$. Although L is obviously $K(L)$-invariant, it does not admit a $K(L)$-linearization. However, one defines a certain extension group $\mathcal{G}(L) \to K(L)$ with kernel isomorphic to $\mathbb{G}_m$, with respect to which L admits a linearization. Of course, the subgroup $\mathbb{G}_m$ of $\mathcal{G}(L)$ acts trivially on A. The group $\mathcal{G}(L)$ is called the *theta group* of L. The linear representation of $\mathcal{G}(L)$ in $H^0(A, L)$ is irreducible. As an abstract group $K(L)$ is isomorphic to $K(D) = \mathbb{Z}^g/D\mathbb{Z}^g \oplus \mathbb{Z}^g/D\mathbb{Z}^g$, where $D = \mathrm{diag}[d_1, \ldots, d_g], d_1 | \cdots | d_g$, is the type of the polarization of L. For example, when $L = M^{\otimes n}$, where M is a principal polarization, we have $K(L) = A_n$, the group of n-torsion points, and $K(L) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$. The vector space $H^0(A, L)$ is isomorphic to the vector space $k[\mathbb{Z}^g/D\mathbb{Z}^g]$ of k-valued functions on the finite abelian group $\mathbb{Z}^g/D\mathbb{Z}^g$, and the representation of $\mathcal{G}(L)$ on this space is called the *Schrödinger representation*. If we assume that $d_1 \geq 3$, then L is very ample and can be used to define a $\mathcal{G}(L)$-equivariant embedding of A in $\mathbb{P}(H^0(A, L)^*)$. Let us now consider an abelian variety with polarization of type D and level structure as a triple (A, L, ϕ), where A and L are as above, and $\phi : K(L) \cong K(D)$ is an isomorphism of abelian groups. Each such triple defines a point $h_{(A,L,\phi)}$ in the Hilbert scheme of closed subschemes in $\mathbb{P}_D = \mathbb{P}(k[\mathbb{Z}^g/D\mathbb{Z}^g]^*)$. We say that two triples (A, L, ϕ) and (A', L', ϕ') are isomorphic if there exists an isomorphism of abelian varieties $f : A \to A'$ such that $f^*(L') = L$ and $\phi \circ f = \phi'$. It is easy to see from this definition that $(A, L, \phi) \cong (A', L', \phi')$ if and only if $h_{(A,L,\phi)} = g \cdot h_{(A',L',\phi')}$ for some projective transformation of $\mathbb{P}_D$. One can show that there is an irreducible component X of the Hilbert scheme which contains the points $h_{(A,L,\phi)}$. Since the space $\mathbb{P}_D$ corresponds to an irreducible representation $V_D = k[\mathbb{Z}^g/D\mathbb{Z}^g]^*$ of the group $K(D)$, the isotropy subgroup of $h_{(A,L,\phi)}$ (equal to $K(D)$) is not contained in any proper parabolic subgroup of $\mathrm{GL}(V_D)$ (see Exercise 9.8). Thus $h_{(A,L,\phi)}$ is a Kempf-stable point in X. It is also a stable point since its isotropy subgroup is finite. The set of points in X corresponding to smooth schemes is an open subset U of X, and is also a $\mathrm{GL}(V_D)$-invariant subset contained in X^{s}. Thus we can consider the geometric quotient $U/\mathrm{GL}(V_D)$ which is a fine moduli scheme for abelian varieties with polarization of type D and a level structure.

Bibliographical notes

Most of the material of this chapter is taken from [75]. Our function $\mu^L(x, \lambda)$ differs by a minus sign from the one studied in Mumford's book [75]. The numerical criterion of stability goes back to D. Hilbert ([47]) who introduced it for the description of the null-cone for the action of SL_n on the space of homogeneous polynomials.

One can give a criterion of stability in terms of the *moment map* $m : \mathbb{P}(V) \to \mathrm{Lie}(K)$, where K is a maximal compact subgroup of G ($\mathrm{SU}(n)$ if $G = \mathrm{SL}(n, \mathbb{C})$). It is defined by the formula $m(v) = \| v \|^{-2} \, \mathrm{d}p_v(1)$, where, for any $g \in G$, $p_v(g) = \| g \cdot v \|^2$. Here we fixed a K-invariant hermitian norm $\| \ \|$ in V. The criterion states that x is semi-stable if and only if 0 belongs to the closure of the moment map image $m(G \cdot x)$ of the orbit of x (see [61]). For more information about the relationship between GIT and the theory of moment maps we refer to [63] and Chapter 8 of the new edition of Mumford's book.

One can consider $\mu^L(x, \lambda)$ as a function in L. One can also get rid of the dependence on λ by showing that the function $M^L(x) = \sup_{\lambda \in \mathcal{X}(G)^*} \frac{\mu^L(x,\lambda)}{\|\lambda\|}$ is well-defined and can be extended to a function $l \mapsto M^l(x)$ on the vector space $l \in \text{Pic}^G(X) \otimes \mathbb{R}$. These functions are used in [23] to define walls and chambers in the vector space $\text{Pic}^G(X) \otimes \mathbb{R}$ which play an important role in the theory of variation of GIT quotients.

A recent book of S. Mukai ([72]) discusses applications of invariant theory to construction of various modili spaces in algebraic geometry. Other books on this topic are [75] and [82].

Exercises

9.1 An algebraic group G is called *diagonalizable* if $\mathcal{O}(G)$ is generated as k-algebra by the characters $\phi : G \to \mathbb{G}_m$ considered as regular functions on G. Prove that a torus is a diagonalizable group and that every connected diagonalizable group is isomorphic to a torus. Give examples of nonconnected diagonalizable groups.

9.2 Check the following properties of the function $\mu^L(x, \lambda)$:

(i) $\mu(g \cdot x, \lambda) = \mu(x, g^{-1}\lambda g)$ for any $g \in G, \lambda \in \mathcal{X}(G)^*$;

(ii) for any $x \in X, \lambda \in \mathcal{X}(G)^*$, the map $\text{Pic}^G(X) \to \mathbb{Z}$ defined by the formula $L \mapsto \mu^L(x, \lambda)$ is a homomorphism of groups;

(iii) if $f : X \to Y$ is a G-equivariant morphism of G-varieties, and $L \in \text{Pic}^G(Y)$, then $\mu^{f^*(L)}(x, \lambda) = \mu^L(f(x), \lambda)$;

(iv) $\mu^L(x, \lambda) = \mu^L(\lim_{t \to 0} \lambda(t) \cdot x, \lambda)$.

9.3 Prove that an affine variety over a field k is proper if and only if it is a finite set of points.

9.4 Prove that a fibre of a proper map is a proper variety. Give an example of a nonproper map such that all its fibres are proper varieties.

9.5 Prove that G acts properly on $X^{\text{s}}(L)$ (i.e., the map $\Psi : G \times X^{\text{s}}(L) \to X^{\text{s}}(L) \times X^{\text{s}}(L)$ is proper).

9.6 Let T be an r-dimensional torus acting linearly on a projective space $\mathbb{P}^n$. Show that $\mathrm{Pic}^T(\mathbb{P}^n) \cong \mathbb{Z}^{r+1}$ and the set of $L \in \mathrm{Pic}^T(\mathbb{P}^n)$ such that $(\mathbb{P}^n)^{\mathrm{ss}}(L) \neq \emptyset$ is a finitely generated semigroup of $\mathbb{Z}^{r+1}$.

9.7 In the notation of Exercise 8.6 from Chapter 8, find the sets $X^{\mathrm{ss}}(L)$ and $X^{\mathrm{s}}(L)$ by using the numerical criterion of stability.

9.8 Suppose x is Kempf-stable. Show that its isotropy group G_x is a reductive subgroup of G. [Hint: Use the following fact: if H is a closed subgroup of G with G/H affine then H is reductive.]

9.9 Let $X = \mathbb{P}(M_n)$ be the projective space associated to the space of square matrices of size n. Consider the action of the group SL_n on X defined by conjugation of matrices. Using the numerical criterion of stability find the sets of unstable and stable points.

9.10 Let $X \subset \mathbb{P}(V)$ and let G act on X via its linear representation. Consider the flag complex $\Delta(G)$. For any point $x \in X$ let $C(x) = \{\delta \in \Delta(G) : \nu_x(\delta) > 0\}$. Show that this set is convex.

Chapter 10

Projective hypersurfaces

10.1 Nonsingular hypersurfaces

Let $G = \mathrm{SL}_{n+1}$ act linearly on $\mathbb{A}^{n+1}$ in the natural way. This action defines an action of G on the subspace $k[Z_0, \ldots, Z_n]_d \subset \mathcal{O}(\mathbb{A}^{n+1})$ of homogeneous polynomials of degree $d > 0$. We view the latter as the affine space $\mathbb{A}^N$, where $N = \binom{n+d}{d}$. A point of the projective space

$$\mathrm{Hyp}_d(n) := \mathbb{P}(k[Z_0, \ldots, Z_n]_d) \cong \mathbb{P}^{N-1}$$

is called a *hypersurface* of degree d in $\mathbb{P}^n$. For each nonzero $F \in k[Z_0, \ldots, Z_n]_d$ we denote the corresponding hypersurface by $V(F)$. When F is an irreducible polynomial, it can be identified with the set of zeros of F in $\mathbb{P}^n$, which is an irreducible closed subvariety of $\mathbb{P}^n$ of dimension $n - 1$. In general, $V(F)$ can be viewed as the union of irreducible subvarieties of dimension $n-1$ taken with multiplicities. In this chapter we shall try to describe the sets of semi-stable and stable points for this action. Note that there is no choice for a nontrivial linearization, since $\mathrm{Pic}(\mathbb{P}^{N-1}) \cong \mathbb{Z}$ and $\mathcal{X}(G) = \{1\}$; we must take $L = \mathcal{O}_{\mathbb{P}^{N-1}}(1)$.

Let

$$C_d(n) = \mathrm{Hyp}_d(n) /\!/ \mathrm{SL}_{n+1}.$$

This is a normal unirational variety. According to a classical result of Jordan and Lie, the group of projective automorphisms of an irreducible hypersurface of degree $d \geq 3$ is finite (see a modern proof in [87]). This implies that SL_{n+1} acts on an open nonempty subset with finite stabilizer groups. By Corollary 6.2,

$$\dim C_d(n) = \dim \mathrm{Hyp}_d(n) - \dim \mathrm{SL}_{n+1} = \binom{n+d}{d} - (n+1)^2. \tag{10.1}$$

Let n be arbitrary. Recall that a hypersurface $V(F) \in \mathrm{Hyp}_d(n)$ defines a nonsingular variety if and only if the equations

$$F = 0, \quad \frac{\partial F}{\partial T_i} = 0, \quad i = 0, \dots, n,$$

have no common zeros. Note that, by the Euler formula,

$$dF = \sum_{i=0}^{n} T_i \frac{\partial F}{\partial T_i},$$

So if char(k) does not divide d, the first equation can be eliminated. Let D be the resultant of the polynomials $\partial F/\partial T_i$. It is a homogeneous polynomial of degree $(n+1)(d-1)^n$ in the coefficients of the form F. It is called the *discriminant* of F. Its value at F is equal to zero if and only if the $\partial F/\partial T_i$ have a common zero in $\mathbb{P}^n$. Since the latter property is independent of the choice of coordinates, the hypersurface $V(D) \subset \mathrm{Hyp}_d(n)$ is invariant with respect to the action of $G = \mathrm{SL}_{n+1}$. This means that for any $g \in G$ we have $g^*(D) = \phi(g)D$ for some $\phi(g) \in k^*$. One immediately verifies that the function $g \mapsto \phi(g)$ is a character of SL_{n+1}. Since the latter is a simple group, its group of characters is trivial. This implies that $\phi(g) = 1$ for all g, and hence D is an invariant polynomial. Since D does not vanish on the set of nonsingular hypersurfaces of degree d prime to the characteristic, we obtain

Theorem 10.1. *Assume* char(k) *is prime to* d*. Any nonsingular hypersurface is a semi-stable point of* $\mathrm{Hyp}_d(n)$.

If $d > 2$, one can replace “semi-stable” with “stable”. This follows from the previously observed fact that, under these assumptions, the group of projective automorphisms of a nonsingular hypersurface is finite.

Example 10.1. Assume $d = 2$ and char$(k) \neq 2$. Then $\mathrm{Hyp}_2(n)$ is the space of quadrics. The space $k[T_0, \dots, T_n]_2$ is the space of quadratic forms

$$F = \sum_{i,j=0}^{n} a_{ij} T_i T_j,$$

or equivalently, the space of symmetric matrices

$$B = (b_{ij})_{i,j=0,\dots,n}, \quad b_{ii} = 2a_{ii}, b_{ij} = b_{ji} = a_{ij}, i \neq j.$$

A quadric $V(F)$ is nonsingular if and only if the rank of the corresponding matrix is equal to $n+1$. The determinant function on $k[T_0,\ldots,T_n]_2$ is the resultant R from above. Thus all nonsingular quadrics are semi-stable. We know that by a linear change of variables every quadratic form can be reduced to the sum of squares $X_0^2+\cdots+X_r^2$, where the number r is equal to the rank of the matrix B from above. In our situation we are allowed to use only linear transformations with determinant 1 but since we are considering homogeneous forms only up to a multiplicative factor, the result is the same. We have exactly n orbits for the action of SL_r on $\mathrm{Hyp}_2(n)$; each is determined by the rank of the corresponding nonzero quadratic form. In fact any invariant nonzero homogeneous polynomial vanishes on an invariant subvariety of codimension 1 in $\mathrm{Hyp}_2(d)$, which must consist of all orbits except the unique open one representing nondegenerate quadratic forms. By Hilbert's Nullstellensatz, this invariant polynomial must be a power of the discriminant of the quadratic form. The stabilizer of the quadratic form $T_0^n+\cdots+T_n^2$ is the special orthogonal group SO_{n+1}. Since it is of positive dimension (if $n>0$), there are no stable points.

10.2 Binary forms

Let us consider the case $n=1$. The elements of the space $k[Z_0,Z_1]_d$ are binary forms of degree d. The corresponding hypersurfaces can be viewed as finite subsets of points in $\mathbb{P}^1$ taken with multiplicities (or, equivalently, as effective divisors $D=\sum n_x x$ on $\mathbb{P}^1$). Let

$$F=\sum_{i=0}^{d} a_i Z_0^{d-i} Z_1^i \in K[Z_0,Z_1]_d.$$

Let T be the maximal torus of SL_2 which consists of diagonal matrices and is equal to the image of the one-parameter group

$$\lambda(t)=\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}.$$

Let us first investigate the stability of $H=V(F)$ with respect to T. For this we will follow the last section of the preceding chapter. We have to compute the weight set $\mathrm{wt}(H)$. We have

$$\lambda(t)\cdot(a_0,\ldots,a_d)=(a_0t^d,a_1t^{d-2},\ldots,a_dt^{-d}).$$

The weight set is a subset of the set

$$S = \{-d, -d+2, \ldots, d-2, d\} \subset \mathbb{Z} = \mathcal{X}(T).$$

Let α_{min} (resp. α_{max}) be the smallest (resp. largest) element of this set.

Obviously, $\alpha_{min} = -d + 2i$, where Z_0^i is the maximum power of Z_0 which divides F. Similarly, $\alpha_{max} = d - 2i$, where Z_1^i is the maximum power of Z_1 which divides F.

By Theorem 9.2, we know that H is semi-stable (resp. stable) with respect to T if and only if

$$\alpha_{min} \leq 0 \leq \alpha_{max} \quad \text{(resp. } \alpha_{min} < 0 < \alpha_{max}\text{).} \tag{10.2}$$

This can be interpreted as follows:

H is semi-stable (resp. properly stable) with respect to T if and only if the points $(0,1)$ and $(1,0)$ are zeros of H of multiplicity $\leq d/2$ (resp. $< d/2$).

From this we easily deduce

Theorem 10.2. $\mathrm{Hyp}_d(1)^{\mathrm{ss}}$ *(resp.* $\mathrm{Hyp}_d(1)^{\mathrm{s}}$*) is equal to the set of hypersurfaces with no roots of multiplicity* $> d/2$ *(resp.* $\geq d/2$*).*

Proof. Suppose H is semi-stable and has a root $(z_0, z_1) \in \mathbb{P}^1$ of multiplicity $> d/2$. Let $g \in G$ take this point to the point $(1,0)$. Then $H' = g \cdot H$ has the point $(1,0)$ as a root of multiplicity $> d/2$. This shows that H' is unstable with respect to T. Hence H is unstable with respect to G, contradicting the assumption. Conversely, assume H has no roots of multiplicity $> d/2$ and is unstable. Then there exists a maximal torus T' with respect to which H is unstable. Let $gT'g^{-1} = T$ for some $g \in G$. Then $g \cdot H$ is unstable with respect to T. But then it has one of the points $(1,0)$ or $(0,1)$ as a root of multiplicity $> d/2$. Thus H has $g^{-1} \cdot (1,0)$ and $g^{-1} \cdot (0,1)$ as a root of multiplicity $> d/2$.

A similar argument proves the assertion about stability. □

Corollary 10.1. *Assume d is odd. Then*

$$\mathrm{Hyp}_d(1)^{\mathrm{ss}} = \mathrm{Hyp}_d(1)^{\mathrm{s}}.$$

Now assume d is even and let $H \in \mathrm{Hyp}_d(1)^{\mathrm{ss}} \setminus \mathrm{Hyp}_d(1)^{s}$. This means that H has a root of multiplicity $d/2$ but no roots of multiplicity greater than $d/2$. Consider the fibre of the projection $\mathrm{Hyp}_d(1)^{\mathrm{ss}} \to \mathrm{Hyp}_d(1)^{\mathrm{ss}}/\!\!/G$ containing H. Since our categorical quotient is good, the fibre contains a unique closed orbit. H

belongs to this orbit if and only if its stabilizer is of positive dimension. Assume H belongs to this orbit. Since any group element stabilizing H stabilizes its set of roots, and it is easy to see that any subset of $\mathbb{P}^1$ consisting of more than two points has a finite stabilizer. Thus, H must have only two roots. Since one of these roots is of multiplicity $d/2$, the other one is also of multiplicity $d/2$. Since any two-point sets on $\mathbb{P}^1$ are projectively equivalent, this tells us that

$$\mathrm{Hyp}_d(1)^{\mathrm{ss}} \setminus \mathrm{Hyp}_d(1)^{\mathrm{s}} = G \cdot H_0,$$

where H_0 is given by the equation $(Z_0 Z_1)^{d/2} = 0$. In particular,

$$\mathrm{Hyp}_d(1)^{\mathrm{ss}} /\!\!/ G \setminus \mathrm{Hyp}_d(1)^{\mathrm{s}}/G = \{x_0\},$$

where the single point x_0 represents the orbit of H_0.

The variety $C_d(1) := \mathrm{Hyp}_d(1)^{\mathrm{ss}} /\!\!/ G$ is an irreducible normal projective variety of dimension $d-3$: by construction of the categorical quotient,

$$C_d(1) = \mathrm{Projm}(\mathrm{Pol}(\mathrm{Pol}_d(k^2))^{\mathrm{SL}_2}).$$

So it can be explicitly computed if we know the algebra of invariant polynomials on the space of binary forms of degree d.

Let us consider some special cases with small d.

If $d = 1$ we have $\mathrm{Hyp}_1(1)^{\mathrm{ss}} = \emptyset$. If $d = 2$ we have $\mathrm{Hyp}_1(2)^{\mathrm{s}} = \emptyset$ and $\mathrm{Hyp}_2(1)^{\mathrm{ss}}$ consists of subsets of two distinct points in $\mathbb{P}^1$. There is only one orbit of such subsets.

The set $\mathrm{Hyp}_3(1)^{\mathrm{ss}}$ consists of three distinct points in $\mathbb{P}^1$. By a projective transformation they can be reduced to the points $\{0, 1, \infty\}$. So the variety $C_d(1)$ is again one point. This also agrees with the fact that $\mathrm{Pol}(\mathrm{Pol}_3(k^2))^{\mathrm{SL}_2} = k[D]$, where D is the discriminant invariant (see Exercise 2.6).

The set $\mathrm{Hyp}_4(1)^{\mathrm{s}}$ consists of subsets of four distinct points in $\mathbb{P}^1$ and the set $\mathrm{Hyp}_4(1)^{\mathrm{ss}}$ consists of closed subsets $V(F)$ where F has at most double roots. Since $\mathrm{Hyp}_4(1)^s$ is an open Zariski subset of the projective space $\mathbb{P}^4$ (see Exercise 10.1), and the fibres of the projection $\mathrm{Hyp}_4(1)^{\mathrm{s}} \to \mathrm{Hyp}_4(1)^{\mathrm{s}}/G$ are of dimension 3, we obtain that $C_4(1)$ is a normal, hence nonsingular, curve. Since it is obviously unirational, it must be isomorphic to $\mathbb{P}^1$. The image of the set of semi-stable but not properly stable points is one point. If we consider the map

$$\pi : \mathrm{Hyp}_4(1)^{\mathrm{ss}} \to C_4(1) \cong \mathbb{P}^1$$

as a rational function on $\mathrm{Hyp}_4(1)^{\mathrm{s}}$ then we can find its explicit expression as a rational function $R(a_0, \ldots, a_4)$ in the coordinates of a binary form. To do this we have first to find the algebra of invariants $\mathrm{Pol}(\mathrm{Pol}_4(k^2))^{\mathrm{SL}_2}$. We already know one invariant, the catalecticant

$$T = a_0a_2a_4 - a_0a_3^2 + 2a_1a_2a_3 - a_1^2a_4 - a_2^3$$

(see Example 1.4). Its bracket expression is $(12)^2(13)^2(23)^2$. Another invariant is of degree 2:

$$S = a_0a_4 - 4a_1a_3 + 3a_2^2.$$

Its bracket expression is $(12)^4$. One can show that any other invariant must be a polynomial in S and T. We will prove this in the next chapter. This agrees with the fact that $C_1^4 = \mathbb{P}^1$. The discriminant D of a quartic polynomial is an invariant whose bracket expression is equal to $(12)^2(13)^2(14)^2(23)^2(24)^2(34)^2$. It is a polynomial of degree 6 in the coefficients a_i and we have

$$D = S^3 - 27T^2.$$

Thus the rational function

$$R(a_0, \ldots, a_4) = \frac{S^3}{S^3 - 27T^2} \tag{10.3}$$

is invariant with respect to SL_2 and defines a regular map from $\mathrm{Hyp}_4(1)^{\mathrm{s}}$ to $\mathbb{A}^1$. This is the geometric quotient map. The map

$$\mathrm{Hyp}_4(1)^{\mathrm{ss}} \to \mathbb{P}^1, \quad (a_0, \ldots, a_4) \mapsto (S^3 - 27T^2, S^3)$$

is the categorical quotient map. Its fibre over $(0, 1) = \infty$ is equal to the union of orbits of binary forms of degree 4 with double roots (up to a nonzero scalar factor). The only closed orbit in this fibre is represented by $V(Z_0^2Z_1^2)$.

Consider the special case when $F = T_0(T_1^3 + aT_0^2T_1 + bT_0^3)$. If $\mathrm{char}(k) \neq 3$ then each orbit contains a representative of such a form. The value of R on $V(F)$ is equal to

$$j = \frac{a^3}{4a^3 + 27b^2}.$$

The expression in the denominator is the discriminant of the cubic polynomial $x^3 + ax + b$. The reader familiar with the theory of elliptic curves will immediately recognize this function; it is the *absolute invariant* j of the elliptic curve given in the Weierstrass form

$$y^2 = x^3 + ax + b.$$

This coincidence is not accidental. The equation above describes an elliptic curve as a double cover of $\mathbb{P}^1$ branched over four points: the infinity point and the three roots of the equation $x^3 + ax + b = 0$. In other words they are the zeros of the binary form $T_0(T_1^3 + aT_0^2T_1 + bT_0^3)$. Two elliptic curves are isomorphic if and only if the corresponding sets of four points on $\mathbb{P}^1$ are in the same orbit with respect to the action of SL_2.

Let $d = 5$. The algebra of invariants

$$A = \mathrm{Pol}(\mathrm{Pol}_5(k^2))^{\mathrm{SL}_2}$$

can be computed explicitly (see [28]). Let us write a general binary quintic in the form

$$f = at_0^5 + 5bt_0^4t_1 + 10ct_0^3t_1^2 + 10dt_0^2t_1^3 + 5et_0t_1^4 + ft_1^5$$

(we assume that $\mathrm{char}(k) \neq 5$). Then A is generated by the following invariants:

$$\begin{aligned}
I_4 &= (ae - 4bd + 3c^2)(bf - 4ce + 3d^2) - (af - 3be + 2cd)^2,\\
I_8 &= a^2b^2e^2f^2 - 2a^3e^5 - 2b^5f^3 + 27b^4e^4,\\
I_{12} &= b^2e^2(a^2b^2e^2f^2 - 4a^3e^5 - 4b^5f^3 + 18ab^3e^3f - 27b^4e^4),\\
I_{18} &= (a^3e^5 - b^5f^3)[(af - 5be)(a^3e^5 + b^5f^3) - 10a^2b^3e^3f^2\\
&\quad + 90ab^4e^3f^2 - 216b^5e^5].
\end{aligned}$$

There is also one basic relation between these invariants which expresses I_{18}^2 as a polynomial $F(I_4, I_8, I_{12})$ in invariants I_4, I_8, and I_{12}. We will consider A as a graded algebra whose grading is defined by the natural grading of $\mathrm{Pol}(\mathrm{Pol}_5(k^2))$ with the degree divided by 2. It follows that there is an isomorphism of graded algebras

$$A \cong k[T_0, T_1, T_2, T_3]/(T_3^2 - F(T_0, T_1, T_2)),$$

where $k[T_0, T_1, T_2, T_3]$ is graded by setting

$$\deg T_0 = 2, \deg T_1 = 4, \deg T_2 = 6, \deg T_3 = 9,$$

and F is a weighted homogeneous polynomial. Let $A^{(2)}$ be the subalgebra of A generated by elements of even degree. Then $A^{(2)}$ is generated by homogeneous elements of even degree T_0, T_1, T_2. Since T_3^2 can be expressed as a polynomial in T_0, T_1, T_2, we see that $A^{(2)}$ is isomorphic to the graded polynomial algebra $k[T_0, T_1, T_2]$. This implies that

$$C_5(1) \cong \mathrm{Projm}(A) \cong \mathrm{Projm}(A^{(2)}) \cong \mathbb{P}(2, 4, 6) \cong \mathbb{P}(1, 2, 3).$$

In particular $C_5(1)$ is a rational surface.

Note that the discriminant Δ of a binary quintic can be expressed via the basic invariants as follows:

$$\Delta = I_4^8 - 128 I_8.$$

This shows that the locus of orbits of binary quintics with a double root is equal to $V(T_0^2 - 128T_1) \subset \mathbb{P}(1,2,3)$ and hence is isomorphic to $\mathbb{P}(1,3) \cong \mathbb{P}^1$.

Let $d = 6$. We will use the explicit description of the algebra of invariants $A = \text{Pol}(\text{Pol}_6(k^2))^{\text{SL}_2}$ due to A. Clebsch ([12]). For a modern treatment see [55]. A is generated by invariants $I_2, I_4, I_6, I_{10}, I_{15}$, where the subscript denotes the degree. The only relation between the basic invariants is

$$I_{15}^2 = F(I_2, I_4, I_6, I_{10}),$$

for some polynomial F. We will consider A as a graded algebra whose grading is defined by the natural grading of $\text{Pol}(\text{Pol}_6(k^2))$. It follows that there is an isomorphism of graded algebras

$$A \cong k[T_0, T_1, T_2, T_3, T_4]/(T_4^2 - F(T_0, T_1, T_2, T_3)),$$

where $k[T_0, T_1, T_2, T_3, T_4]$ is graded by setting

$$\deg T_0 = 2, \deg T_1 = 4, \deg T_2 = 6, \deg T_3 = 10, \deg T_4 = 15,$$

and F is a weighted homogeneous polynomial. Arguing as in the preceding example, we see that

$$C_6(1) \cong \text{Projm}(k[T_0, T_1, T_2, T_3]) \cong \mathbb{P}(1,2,3,5).$$

In particular $C_6(1)$ is a rational three-dimensional variety.

Note that the invariant I_{10} is the discriminant of a binary sextic, so it vanishes on the locus of binary sextics with a double root. The complement of this locus in $C_6(1)$ represents reduced divisors of degree 6 in $\mathbb{P}^1$. It is isomorphic to the moduli space $\mathcal{M}_2$ of genus 2 curves. The isomorphism is defined similarly by assigning to a genus 2 curve the six branch points of its canonical degree 2 map to $\mathbb{P}^1$. So we obtain that $\mathcal{M}_2$ is isomorphic to the open subset $D(T_4)$ of $\mathbb{P}(1,2,3,5)$ where the last coordinate T_4 is not equal to zero. Since each point in this subset is represented by a point (t_0, t_1, t_2, t_3) in $\mathbb{A}^4$ with $t_3 = 1$, it follows from the definition of weighted projective space that

$$\mathcal{M}_2 \cong \mathbb{A}^3/(\mathbb{Z}/5),$$

where a generator of the cyclic group $\mathbb{Z}/5$ acts on $\mathbb{A}^3$ by the formula

$$(t_0, t_1, t_2) \mapsto (\eta t_0, \eta^2 t_1, \eta^3 t_2), \quad \eta = \exp(2\pi\mathrm{i}/5).$$

The image of the origin is the unique singular point of $\mathcal{M}_2$. It represents the isomorphism class of the hyperelliptic curve corresponding to the binary quintic $t_0(t_0^5 + t_1^5)$. It admits an automorphism of order 5.

Finally observe that the locus $V(T_4)$ of binary sextics with a multiple root and $C_5(1)$ are both isomorphic to $\mathbb{P}(1, 2, 3)$.

10.3 Plane cubics

Let $n = 2$ and $d = 3$. Every homogeneous form of degree 3 in three variables (a *ternary cubic*) can be written in the form:

$$F = a_1T_0^3 + a_2T_0^2T_1 + a_3T_0^2T_2 + a_4T_0T_1^2 + a_5T_0T_1T_2$$

$$+a_6T_0T_2^2 + a_7T_1^3 + a_8T_1^2T_2 + a_9T_1T_2^2 + a_{10}T_2^3.$$

Now let us recall the classification of plane cubic curves. First of all it is easy to list all reducible curves. They are of the following types:

(1) the union of an irreducible conic and a line intersecting it at two distinct points;

(2) the union of an irreducible conic and its tangent line;

(3) the union of three nonconcurrent lines;

(4) the union of three concurrent lines;

(5) the union of two lines, one of them double;

(6) one triple line.

Since all nonsingular conics are projectively equivalent to the conic $C : T_0T_2 + T_1^2 = 0$ and the group of projective automorphisms of the conic C acts transitively on the set of tangents to C or on the set of lines intersecting C transversally, we obtain that any curve of type (1) or (2) is projectively equivalent to the curve

(1) $(T_0T_2 + T_1^2)T_1 = 0$,

(2) $(T_0T_2 + T_1^2)T_0 = 0$,

respectively. Since the group of projective transformation of $\mathbb{P}^2$ acts transitively on the set of k lines with $k \leq 4$, we obtain that any curve of type (3–6) is projectively equivalent to the curve given by the equation

(3) $T_0T_1T_3 = 0$,

(4) $T_0^2T_1 + T_0T_1^2 = 0$,

(5) $T_0^3 + T_0^2T_1 = 0$,

(6) $T_0^3 = 0$,

respectively. Now let us assume that F is irreducible. First let us assume that C is nonsingular. Choose a system of coordinates such that the point $(0,0,1)$ is an inflection point and $T_0 = 0$ is the equation of the tangent line at this point. It is known that any plane curve contains at least one inflection point. Then we can write the equation as

$$T_2^2T_0 + T_2L_2(T_0,T_1) + L_3(T_0,T_1) = 0,$$

where L_2 is a form of degree 2 and L_3 is a form of degree 3. Since the line $T_0 = 0$ intersects the curve at one point, we easily see that the coefficient of L_2 at T_1^2 is equal to zero. Thus in affine coordinates $X = T_1/T_0, Y = T_2/T_0$, the equation takes the form

$$Y^2 + aYX + bY + dX^3 + eX^2 + fX + g = 0. \tag{10.4}$$

Obviously $d \neq 0$, so after scaling we may assume $d = 1$.

Assume char$(k) \neq 2$. Replacing Y with $Y + \frac{a}{2}aX + \frac{b}{2}$, we may assume that $a = b = 0$. If char$(k) \neq 3$, by a change of variables $X \to X + \frac{e}{3}$, we may assume that $e = 0$. Thus, we obtain the *Weierstrass equation* of a nonsingular plane cubic:

$$\begin{aligned} Y^2 + X^3 + aX + b &= 0, \ \text{char}(k) \neq 2,3, && (10.5)\\ Y^2 + aYX + bY + X^3 + cX + d &= 0, \ \text{char}(k) = 2, && (10.6)\\ Y^2 + X^3 + aX^2 + bX + c &= 0, \ \text{char}(k) = 3. && (10.7) \end{aligned}$$

The condition that the curve is nonsingular is expressed by $\Delta \neq 0$, where Δ is the discriminant defined by

$$\Delta = \begin{cases} 4a^3 + 27b^2, & \text{if char}(k) \neq 2,3,\\ a^3b^3 + b^4 + a^4(abc + c^3 + a^2d), & \text{if char}(k) = 2,\\ b^3 + (b^2 - ac)a^2, & \text{if char}(k) = 3. \end{cases} \tag{10.8}$$

Two curves are isomorphic if and only if their *absolute invariants*

$$j = \begin{cases} a^3/\Delta, & \text{if char}(k) \neq 2,3; \\ a^{12}/\Delta, & \text{if char}(k) = 2; \\ a^6/\Delta, & \text{if char}(k) = 3. \end{cases} \tag{10.9}$$

are equal.

Now suppose C is singular. We may choose $(0,0,1)$ to be the singular point. Then the equation is of the form

$$T_2L_2(T_0,T_1) + L_3(T_0,T_1) = 0. \tag{10.10}$$

By a linear transformation of variables T_0, T_1 we reduce L_2 to one of two forms: $L_2 = T_0^2$ or $L_2 = T_0T_1$. Consider the first case. The singular point is a cusp; the equation is

$$T_2T_0^2 + aT_0^3 + bT_0^2T_1 + cT_0T_1^2 + dT_1^3 = 0.$$

Replacing T_2 with $T_2 + aT_0 + bT_1$, we may assume that $a = b = 0$. Since the curve is irreducible we have $d \neq 0$; by scaling we may assume that $d = 1$ and $c = 0$ or 1.

If $\text{char}(k) = 3$, we see that there are two orbits of cuspidal curves, represented by the equations

$$T_2T_0^2 + T_1^3 = 0 \quad \text{and} \quad T_2T_0^2 + T_0T_1^2 + T_1^3 = 0.$$

All nonsingular points of the first curve are inflection points. The second curve does not have nonsingular inflection points.

If $\text{char}(k) \neq 3$, then the curve has only one inflection point $(1, -\frac{c}{3}, -\frac{2c}{27})$ with tangent line given by $T_2 + c(\frac{1}{27}T_0 + \frac{1}{3}T_1) = 0$. Now change the coordinates in such a way that $(1,0,0)$ is the unique nonsingular inflection point, the line $T_2 = 0$ is the tangent line at this point and the singular point is $(0,0,1)$. Then, the equation reduces to the form

$$T_2T_0^2 + T_1^3 = 0.$$

Now we consider the case of nodal curves (when the quadratic form L_2 in (10.10) is equal to T_0T_1) so that the equation is

$$T_2T_0T_1 + aT_0^3 + bT_0^2T_1 + cT_0T_1^2 + dT_1^3 = 0.$$

Changing T_2 to $T_2 + bT_0 + cT_1$ we reduce the equation to the form

$$T_2T_0T_1 + aT_0^3 + dT_1^3 = 0.$$

Clearly, $a, d \neq 0$, so by scaling, we reduce the equation to the form

$$T_2T_0T_1 + T_0^3 + T_1^3 = 0.$$

We leave it to the reader to find a projective isomorphism between this curve and the curve

$$T_2^2T_0 + T_1^2(T_1 + T_0) = 0,$$

if char$(k) \neq 2$.

Summarizing, we get the following list of equations of irreducible plane curves (up to projective transformation):

char$(k) \neq 2, 3$:

(7) nonsingular cubic

$$T_2^2T_0 + T_1^3 + aT_1T_0^2 + bT_0^3 = 0, \quad 4a^3 + 27b^2 \neq 0;$$

(8) nodal cubic

$$T_2^2T_0 + T_1^2(T_1 + T_0) = 0;$$

(9) cuspidal cubic:

$$T_2^2T_0 + T_1^3 = 0.$$

char$(k) = 3$:

(7) nonsingular cubic

$$T_2^2T_0 + T_1^3 + aT_1^2T_0 + bT_1T_0^2 + cT_0^3 = 0, \quad b^3 + (b^2 - ac)a^2 \neq 0;$$

(8) nodal cubic

$$T_0T_1T_2 + T_0^3 + T_1^3 = 0;$$

(9) cuspidal cubic:

$$T_2^2T_0 + T_1^3 = 0, \quad \text{or} \quad T_2^2T_0 + T_1^2(T_1 + T_2) = 0.$$

char$(k) = 2$:

(7) nonsingular cubic

$$T_2^2T_0 + aT_1T_2T_0 + bT_2T_0^2 + T_1^3 + cT_1T_0^2 + dT_0^3 = 0,$$

where $a^3b^3 + b^4 + a^4(abc + c^2 + a^2d) \neq 0$;

(8) nodal cubic

$$T_0T_1T_2 + T_0^3 + T_1^3 = 0;$$

(9) cuspidal cubic

$$T_2^2T_0 + T_1^3 = 0.$$

Let T be the diagonal maximal torus in SL_3. It consists of matrices of the form

$$t = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_1^{-1}t_2^{-1} \end{pmatrix}.$$

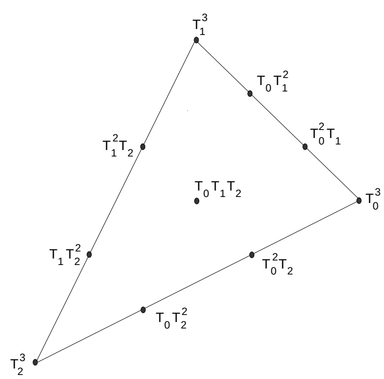
The standard torus $\mathbb{G}_m^2$ acts on $V = \mathrm{Pol}_3(k^3)$ via its natural homomorphism $\mathbb{G}_m^2 \to \mathrm{SL}_3, (t_1, t_2) \mapsto \mathrm{diag}(t_1, t_2, (t_1t_2)^{-1})$. For each monomial $T_0^aT_1^bT_2^c, a + b + c = 3$, we have

$$(t_1, t_2) \cdot T_0^aT_1^bT_2^c = t_1^{a-c}t_2^{b-c}T_0^aT_1^bT_2^c.$$

Thus each monomial $T_0^aT_1^bT_2^c$ belongs to the eigensubspace $V_{\chi_{a,b}}$, where $\chi_{a,b}$ is the character of $\mathbb{G}_m^2$ defined by the vector $(a-c, b-c) = (2a+b-3, 2b+a-3)$. It is easy to see that $V_{\chi_{a,b}}$ is one-dimensional and is spanned by the monomial $T_0^aT_1^bT_2^c$. Thus

$$\mathrm{wt}(V) = \{(2a + b - 3, 2b + a - 3) \in \mathbb{Z}^2 : a, b \geq 0, a + b \leq 3\}.$$

It is a set of 10 lattice points in $\mathbb{R}^2$:



Suppose $V(F)$ is unstable with respect to T. Then the origin lies outside of the convex hull of $\mathrm{wt}(F)$. It is easy to see that this is possible only if $\mathrm{wt}(F)$ consists of lattice points on one edge of the triangle plus one point nearest to the edge but not the interior point. After permuting the coordinates we may assume that

$$F = a_1T_0^3 + a_2T_0^2T_1 + a_3T_0^2T_2 + a_4T_0T_1^2 + a_7T_1^3.$$

It is clear that $(0, 0, 1)$ is a singular point of $V(F)$. In affine coordinates $X = T_0/T_2, Y = T_1/T_2$, the equation looks like

$$F = a_1X^3 + a_2X^2Y + a_3X^2 + a_4XY^2 + a_7Y^3.$$

From this we see that the singular point is not an ordinary double point.

It follows from the above classification of plane cubic curves that the following curves are unstable:

(us1) irreducible cuspidal curve (two orbits if char$(k) = 3$);

(us2) the union of an irreducible conic and its tangent line;

(us3) the union of three concurrent lines;

(us4) the union of two lines, one of them double;

(us5) one triple line.

By looking at the equations of the remaining curves and drawing their weight sets we see that any nonsingular cubic is stable and any singular curve not from the above list is semi-stable. Note that it is enough to check the numerical criterion only for one fixed torus. In fact, the property of being nonsingular or have at most ordinary double points is independent of the chosen coordinates. Thus we have the following list of semi-stable points:

(ss1) nonsingular cubic (stable point);

(ss2) irreducible nodal curve;

(ss3) the union of an irreducible conic and a line intersecting it at two distinct points;

(ss4) the union of three non-concurrent lines.

Consider the quotient map

$$\pi : \mathrm{Hyp}_3(2)^{\mathrm{ss}} \to \mathrm{Hyp}_3(2)^{\mathrm{ss}}/\!\!/\mathrm{SL}_3.$$

The dimension of its fibres containing stable curves is equal to 8 ($= \dim \mathrm{SL}_3$). Note that in the process of the previous analysis, we found that curves of types (ss1), (ss2) and (ss3), each form a single orbit represented by the curves

$$T_0T_1T_2 + T_0^3 + T_1^3 = 0, \quad T_0T_1T_2 + T_1^3 = 0, \quad T_0T_1T_2 = 0,$$

respectively. Moreover the curves of types (ss2) and (ss3) have stabilizer of positive dimension. In fact the torus $\lambda(\mathbb{G}_m)$, where $\lambda(t) = (t, 1, t^{-1})$, stabilizes the second curve, and the maximal diagonal torus stabilizes the third curve. This shows that the orbits of curves of types (ss2) and (ss3) are of dimension ≤ 7. Thus they lie in the closure of some orbit of dimension 8. It cannot be a stable orbit, hence the only possible case is that it is the orbit of curves of type (ss1). Hence this orbit is nether closed nor stable.

Since $\mathrm{Hyp}_3(2)$ is of dimension 9, we obtain $\dim \mathrm{Hyp}_3(2)^{\mathrm{ss}}/\!\!/\mathrm{SL}_3 = 1$. It is a normal projective unirational curve, hence we find that

$$\mathrm{Hyp}_3(2)^{\mathrm{ss}}/\!\!/\mathrm{SL}_3 \cong \mathbb{P}^1.$$

Since there is only one closed semi-stable but not stable orbit, namely the set of three non-concurrent lines, we obtain

$$\mathrm{Hyp}_3(2)^{\mathrm{s}}/\mathrm{SL}_3 \cong \mathbb{A}^1.$$

It is easy to see that the orbit of the curve $T_0T_1T_2 = 0$ is of dimension 6. In the same fibre we find two other orbits: of nodal irreducible cubics (of dimension 8) and of curves of type (ss2) (of dimension 7). The second orbit lies in the closure of the first one, and the closed orbit lies in the closure of the second one.

If $\mathrm{char}(k) \neq 3$, we have five unstable orbits: irreducible cuspidal cubics (of dimension 8), curves of type (us2) (of dimension 6), of type (us3) (of dimension 5), of type (us4) (of dimension 4), and of type (us5) (of dimension 2). It is easy to see that the orbit of type (us$i$) lies in the closure of the orbit of type (us(i-1)).

If $\mathrm{char}(k) = 3$ we have two unstable orbits of type (us1), and four other unstable orbits lying in the closure of these two orbits.

One can give the explicit formula for the quotient map similar to (10.3). In characteristic $\neq 2, 3$, it can be given by the following rational function J in the coefficients a_i (see [100], p. 189–192):

$$J = \frac{16S^3}{T^2 + 64S^3},$$

where

$$
\begin{aligned}
S &= abcm - (bca_2a_3 + cab_1b_3 + abc_1c_2) - m(ab_3c_2 + bc_1a_3 + ca_2b_1) \\
&- m^4 + 2m^2(b_1c_1 + c_2a_2 + a_3b_3) - 3m(a_2b_3c_1 + a_3b_1c_2) \\
&+ (ab_1c_2^2 + ac_1b_3^2 + ba_2c_1^2 + bc_2a_3^2 + cb_3a_2^2 + ca_3b_1^2) \\
&- (b_1^2c_1^2 + c_2^2a_2^2 + a_3^2b_3^2) + (c_2a_2a_3b_3 + a_3b_3b_1c_1 + b_1c_1c_2a_2),
\end{aligned}
$$

$$
\begin{aligned}
T &= a^2b^2c^2 - 6abc(ab_3c_2 + bc_1a_3 + ca_2b_1) + 12abcm(b_1c_1 + c_2a_2 + a_3b_3) \\
&+ 36m^2(bca_2a_3 + cab_1b_3 + abc_1c_2) - 3(a^2b_3^2c_2^2 + b^2c_1^2a_3^2 + c^2a_2^2b_1^2) \\
&+ 4(a^2bc_2^3 + a^2cb_3^3 + b^2ca_3^3 + b^2ac_1^3 + c^2ab_1^3 + c^2ba_2^3) \\
&- 24m(bcb_1a_3^2 + bcc_1a_2^2 + cac_2b_1^2 + caa_2b_3^2 + aba_3c_2^2 + abb_3c_1^2) \\
&- 12(bcc_2a_3a_2^2 + bcb_3a_2a_3^2 + cac_1b_3b_1^2 + caa_3b_1b_3^2 + abb_1c_2c_1^2 + aba_2c_1c_2^2) \\
&+ 6abca_3b_1c_2 + 12m^2(ab_1c_2^2 + ac_1b_3^2 + ba_2c_1^2 + bc_2a_3^2 + cb_3a_2^2 + ca_3b_1^2) \\
&- 20abcm^3 - 60m(ab_1b_3c_1c_2 + bc_1c_2a_2a_3 + ca_2a_3b_1b_3) \\
&+ 12m(aa_2b_3c_2^2 + aa_3c_2b_3^2 + bb_3c_1a_3^2 + bb_1a_3c_1^2 + cc_1a_2b_1^2 + cc_2b_1a_2^2) \\
&+ 6(ab_3c_2 + bc_1a_3 + ca_2b_1)(a_2b_3c_1 + a_3b_1c_2) - 6b_1c_1c_2a_2a_3b_3 \\
&+ 24(ab_1b_3^2c_1^2 + ac_1c_2^2b_1^2 + bc_2c_1^2a_2^2 + ba_2a_3^2c_2^2 + ca_3a_2^2b_3^2 + cb_3b_1^2a_3^2) \\
&- 12(aa_2b_1c_2^3 + aa_3c_1b_3^3 + bb_3c_2a_3^3 + bb_1a_2c_1^3 + cc_1a_3b_1^3 + cc_2b_3a_2^3) \\
&- 8m^6 + 24m^4(b_1c_1 + c_2a_2 + a_3b_3) - 36m^3(a_2b_3c_1 + a_3b_1c_2) \\
&+ 36m(a_2b_3c_1 + a_3b_1c_2)(b_1c_1 + c_2a_2 + a_3b_3) + 8(b_1^3c_1^3 + c_2^3a_2^3 + a_3^3b_3^3) \\
&- 12(b_1^2c_1^2c_2a_2 + b_1^2c_1^2a_3b_3 + c_2^2a_2^2a_3b_3 + c_2^2a_2^2b_1c_1 + a_3^2b_3^2b_1c_1 + a_3^2b_3^2c_1a_2) \\
&- 12m^2(b_1c_1c_2a_2 + c_2a_2a_3b_3 + a_3b_3b_1c_1) - 24m^2(b_1^2c_1^2 + c_2^2a_2^2 + a_3^2b_3^2) \\
&+ 18(bcb_1c_1a_2a_3 + cac_2a_2b_3b_1 + aba_3b_3c_1c_2) - 27(a_2^2b_3^2c_1^2 + a_3^2b_1^2c_2^2) \\
&+ 6abca_2b_3c_1 - 12m^3(ab_3c_2 + bc_1a_3 + ca_2b_1).
\end{aligned}
$$

Here we use the following dictionary between our notation of coefficients and Salmon's:

$$(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}) = (a, 3a_2, 3a_3, 3b_1, 6m, 3c_1, b, 3b_3, 3c_2, c).$$

In fact the algebra $\mathrm{Pol}(\mathrm{Pol}_3(k^3))^{\mathrm{SL}_3}$ is freely generated by S and T. If one evaluates S and T on the curve given in the Weierstrass form from above, we obtain

$$S = \frac{a}{27}, \quad T = \frac{4b}{27}.$$

In this special case the value of the function J is equal to

$$J = \frac{a^3}{(4a^3 + 27b^2)}.$$

This is the absolute invariant of the elliptic curve. Note that we arrived at the same function by studying the orbits of binary quartics.

10.4 Cubic surfaces

Consider the case $d = 3, n = 3$. It corresponds to cubic surfaces in $\mathbb{P}^3$. The algebra of invariants $\mathrm{Pol}(\mathrm{Pol}_3(k^4))^{SL_4}$ was computed by G. Salmon and A. Clebsch ([99]). It is generated by invariants $I_8, I_{16}, I_{24}, I_{32}, I_{40}, I_{100}$, where the subscript indicates the degree. The square of the last invariant is expressed as a polynomial in the first five invariants. In analogy with the case $(d, n) = (6, 1)$, we find that

$$C_3(3) \cong \mathbb{P}(1, 2, 3, 4, 5).$$

In particular, $C_3(3)$ is a rational variety. The invariant I_{32} corresponding to the variable T_3 with weight 4 is the discriminant. Thus we obtain the following isomorphism for the moduli space $\mathcal{M}_{\text{cubic}}$ of nonsingular cubic surfaces:

$$\mathcal{M}_{\text{cubic}} \cong \mathbb{A}^4/(\mathbb{Z}/4\mathbb{Z}),$$

where a generator of the cyclic group $\mathbb{Z}/4$ acts on $\mathbb{A}^4$ by the formula

$$(t_1, t_2, t_3, t_4) \mapsto (\eta t_1, \eta^2 t_2, \eta^3 t_3, \eta t_4), \quad \eta = \exp(2\pi \mathrm{i}/4).$$

The unique singular point of $\mathcal{M}_{\text{cubic}}$ corresponds to the following cubic surface (see [81]):

$$T_1(tT_0^2 + T_1T_2 + T_1T_3) + T_2T_3(T_2 + \tfrac{(t^2+1)^2}{4t^2}T_3) = 0,$$

where $t = 1 + \sqrt{2}$. The automorphism group of this surface is isomorphic to the dihedral group of order 8.

The subvariety of $C_3(3)$ defined by the equation $T_3 = 0$ is isomorphic to $\mathbb{P}(1, 2, 3, 5)$. Recall that the latter is isomorphic to $C_6(1)$; this is not an accident. If a point of $C_6(1)$ represents six distinct points in $\mathbb{P}^1$, we consider the Veronese map to identify them with six points on a nonsingular conic in $\mathbb{P}^2$. Then the linear system of cubics through these points defines a rational map from $\mathbb{P}^2$ to $\mathbb{P}^3$. Its image is a singular cubic representing a point of $C_3(3)$. The singular point of this cubic is the image of the conic. Thus we see that the moduli space $\mathcal{M}_2$ is isomorphic to an open subset of the hypersurface $T_3 = 0$ in $C_3(3)$.

The following are the other values of (d, n) for which the analysis of stability has been worked out:

$$\begin{aligned}(d,n) \quad &= \quad (2,4),(2,5),(3,3)([75]),(2,6)([105]) \\ &\quad\quad (3,4)([106]),(3,5)([2],\ [124]).\end{aligned}$$

Bibliographical Notes

The examples of explicit computation of the the quotient spaces $C_d(n)$ given in this lecture have been known since the nineteenth century (see [30], [38], [98]). The other known cases are $(n, d) = (1, 7), (1, 8)$ (see [36], [35] and also [109], [20]). A modern proof of the completeness of the Clebsch-Salmon list of fundamental invariants of cubic surfaces was given by Beklemishev ([4]). These are probably the only examples where one can compute the spaces $C_d(n)$ explicitly. In fact, one can show that the number of generators of the algebra of invariants on the space of homogeneous polynomials of degree d grows very rapidly with d (see [90]).

It is conjectured that all the spaces $C_d(n)$ are rational varieties. In the case of binary forms, this was proven by F. Bogomolov and P. Katsylo ([5]). The spaces $C_d(2)$ are known to be rational only in some cases (see [58], [59], [108] and also a survey of results on rationality in [21]).

Exercises

10.1 Show that $\mathrm{Hyp}_d(1) \cong \mathbb{P}^d$. Desribe the sets of semi-stable and stable points as subsets of $\mathbb{P}^d$.

10.2 Let $(a_i, b_i), i = 1, 2, 3, 4$, be four distinct roots of a binary quartic F. Let $[ij]$ denote the determinant of the matrix with columns $(a_i, b_i), (a_j, b_j)$. The expression $r = [12][34]/[13][24]$ is called the *cross-ratio* of the four points. Prove that two binary quartics define the same orbit in $\mathrm{Hyp}_4(1)$ if and only if the corresponding cross-ratios coincide after we make some permutations of the roots.

10.3 Let X be the complement of the quartic $V(D)$ in $\mathbb{P}^3$, where D is the discriminant of a binary cubic form. Show that X is isomorphic to a homogeneous space SL_2/H, where H is a subgroup of order 12.

10.4 Show that there are exactly two orbits in $\mathrm{Hyp}_4(1)^{\mathrm{s}}$ with non-trivial stabilizer. Show that the closures of these orbits in $\mathrm{Hyp}_4(1)$ are given by the equations $S = 0$ and $T = 0$, where S, T are the polynomials of degree 2 and 3 defined in section 10.2.

10.5 Show that $\mathrm{Hyp}_4(1)^{\mathrm{us}}$ is isomorphic to a surface of degree 6 in $\mathbb{P}^4$. Its singular set is isomorphic to a Veronese curve of degree 4.

10.6 Construct a rational map from $C_d(1)$ to $C_{d+1}(1)$ whose image is equal to the locus of zeroes of the discriminant invariant. Describe the points of indeterminacy of this map and its inverse.

10.7 Find the orbits of the binary quintics which correspond to singular points of $C_5(1)$.

10.8 Find the group of projective automorphisms of a nonsingular cubic curve (you may assume that $\mathrm{char}(k) \neq 2, 3$).

10.9 Find all projective automorphisms of an irreducible cuspidal cubic.

10.10 Perform the analysis of stability in the case $(d, n) = (3, 3)$ and compare the result with the answer in [75].

10.11 Prove that nonsingular quadrics are semi-stable in all characteristics.

10.12 Show that a plane curve of degree d is unstable if it has a singular point of multiplicity $> 2d/3$.

Chapter 11

Configurations of linear subspaces

11.1 Stable configurations

In the last two chapters, for typographical reasons, we denote the Grassmannian $\mathrm{Gr}(r+1, n+1)$ of r-dimensional linear projective subspaces in $\mathbb{P}^n$ by $\mathrm{Gr}_{r,n}$. The group $G = \mathrm{SL}_{n+1}$ acts naturally on $\mathrm{Gr}_{r,n}$ via its linear representation in k^{n+1}. In this lecture we investigate the stability for the diagonal action of G on the variety

$$X_{\mathbf{r},n} = \prod_{i=1}^{m} \mathrm{Gr}_{r_i,n},$$

where $\mathbf{r} = (r_1, \ldots, r_m)$. First we have to describe the possible linearizations of this action.

Lemma 11.1.

$$\mathrm{Pic}^G(\mathrm{Gr}_{r,n}) \cong \mathrm{Pic}(\mathrm{Gr}_{r,n}) \cong \mathbb{Z}.$$

A generator of this group is the line bundle $\mathcal{O}_{\mathrm{Gr}_{r,n}}(1)$ corresponding to a hyperplane section in the Plücker embedding of $\mathrm{Gr}_{r,n}$ in $\mathbb{P}(\Lambda^{r+1}(k^{n+1})) = \mathbb{P}^N$, $N = \binom{n+1}{r+1} - 1$.

Proof. We will represent a point $W \in \mathrm{Gr}_{r,n}$ as a matrix

$$A = \begin{pmatrix} a_{00} & a_{01} & \ldots & a_{0n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r0} & a_{r1} & \ldots & a_{rn} \end{pmatrix}.$$

Its rows form a basis of W. The Plücker coordinates $p_{i_0\ldots i_r}$ of W are the maximal minors of this matrix formed by the columns $A_{i_0},\ldots,A_{i_r}$. The open subset of $\mathrm{Gr}_{r,n}$ with $p_{12\ldots r+1}\neq 0$ is the affine space $\mathbb{A}^{(r+1)(n-r)}$. The restriction of any $L\in \mathrm{Pic}(\mathrm{Gr}_{r,n})$ to this open subset is trivial, so L is isomorphic to the line bundle associated to a divisor equal to a multiple of a hyperplane section. Since any line bundle admits a unique linearization with respect to SL_{n+1}, the assertion follows. □

We use the notation $Z_{i_0,\ldots,i_r}$ to denote the projective coordinates in $\mathbb{P}^N$ (we order them lexicographically). The value of this coordinate at any $W\in\mathrm{Gr}_{r,n}$ is equal to the Plücker coordinate $p_{i_0\ldots i_r}$ of W. Since $\mathrm{Gr}_{r,n}$ is not contained in a linear subspace of $\mathbb{P}^N$, the restriction map

$$\Gamma(\mathbb{P}^N,\mathcal{O}_{\mathbb{P}^N}(1))\to\Gamma(\mathrm{Gr}_{r,n},\mathcal{O}_{\mathrm{Gr}_{r,n}}(1))$$

is injective. One can also show that it is surjective.

For any vector $\mathbf{k}=(k_1,\ldots,k_m)\in\mathbb{Z}^m$ we define a line bundle on $X_{\mathbf{r},n}$

$$L_{\mathbf{k}}=\bigotimes_{i=1}^m \mathrm{pr}_i^*(\mathcal{O}_{\mathrm{Gr}_{r_i,n}}(1)^{\otimes k_i}),$$

where $\mathrm{pr}_i: X_{\mathbf{r},n}\to\mathrm{Gr}_{r_i,n}$ is the i-th projection. It follows from Lemma 11.1 that any line bundle on X is isomorphic to $L_{\mathbf{k}}$ for some $\mathbf{k}$ (use [46], p. 292). Since each pr_i is an SL_{n+1}-equivariant morphism, $L_{\mathbf{k}}$ admits a canonical SL_{n+1}-linearization. Thus

$$\mathrm{Pic}^{\mathrm{SL}_{n+1}}(X_{\mathbf{r},n})\cong\mathbb{Z}^m.$$

Also $L_{\mathbf{k}}$ is ample if and only if all k_i are positive. In fact, if some tensor power of $L_{\mathbf{k}}$ defines a closed embedding $X_{\mathbf{r},n}\to\mathbb{P}^M$, then the restriction of $L_{\mathbf{k}}$ to any subvariety isomorphic to a factor is an ample line bundle. But it is obvious that this restriction is isomorphic to $\mathcal{O}_{\mathrm{Gr}_{r,n}}(1)^{\otimes k_i}$. The latter is ample if and only if $k_i>0$. Conversely, any $L_{\mathbf{k}}$ with positive $\mathbf{k}$ (meaning that all k_i's are positive) is very ample. It defines a projective embedding of $X_{\mathbf{r},n}$ which is equal to the composition

$$X_{\mathbf{r},n}\to(\mathbb{P}^N)^m\to\prod_{i=1}^m\mathbb{P}^{\binom{N+k_i}{N}-1}\to\mathbb{P}^{\prod_{i=1}^m\binom{N+k_i}{N}-1},$$

where the first map is the product of the Plücker embeddings, the second map is the product of the Veronese embeddings, and the last map is the Segre map.

Now we are ready to describe semi-stable and stable configurations of linear projective subspaces

$$\mathcal{W} = (W_1, \dots, W_m) \in X_{\mathbf{r},n}.$$

Theorem 11.1. *Let* $\mathbf{k} = (k_1, \dots, k_m) \in \mathbb{Z}_+^m$*. Then* $\mathcal{W} \in X_{\mathbf{r},n}^{ss}(L_{\mathbf{k}})$ *(resp.* $\in (X_{\mathbf{r},n})^s(L_{\mathbf{k}})$*) if and only if for any proper linear subspace* W *of* $\mathbb{P}^n$

$$(n+1)\sum_{j=1}^{m} k_j[\dim(W_j \cap W) + 1] \le (\dim W + 1)\sum_{i=1}^{m} k_i(r_i + 1)$$

(resp. the strict inequality holds).

Proof. Let T be the maximal diagonal torus in SL_{n+1}. Each one-parameter subgroup of T is defined by $\lambda(t) = \mathrm{diag}[t^{q_0}, \dots, t^{q_n}]$, where $q_0 + \cdots + q_n = 0$. By permuting coordinates we may assume that

$$q_0 \ge q_1 \ge \cdots \ge q_n. \tag{11.1}$$

Suppose $\mathcal{W} = (W_1, \dots, W_m)$ is semi-stable. Let $\overline{E}_s, s = 0, \dots, n,$ be the linear space spanned by the unit vectors $e_0, \dots, e_s$ and let E_s be the corresponding projective subspace. For any $W \in \mathrm{Gr}_{r,n}$ and any integer $j, 0 \le j \le r$, there is a unique integer ν_j for which

$$\dim(W \cap E_{\nu_j}) = j, \quad \dim(W \cap E_{\nu_j - 1}) = j - 1.$$

To see this we list the numbers $a_s = \dim(W \cap E_s), s = 0, \dots, n,$ and observe that $0 \le a_s - a_{s-1} \le 1, a_n = r,$ since each E_{s-1} is a hyperplane in E_s and $E_n = \mathbb{P}^n$. Then we see that each j occurs among these numbers and we define ν_j to be the first s with $a_s = j$.

With this notation we can represent W by a matrix A of the form

$$A = \begin{pmatrix} a_{00} & \dots & a_{0\nu_0} & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ a_{10} & \dots & \dots & a_{1\nu_1} & 0 & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{r_i 0} & \dots & \dots & \dots & \dots & \dots & a_{r\nu_r} & 0 & \dots & 0 \end{pmatrix}, \tag{11.2}$$

where $a_{j\nu_j} \ne 0$ for all j. It is clear from viewing the maximal minors of this matrix that $p_{i_0 \dots i_r}(W_i) = 0$ if $i_j > \nu_j$ for any value of j and $p_{\nu_0 \dots \nu_r}(W) \ne 0$.

Now we notice that the projective coordinates of $\mathcal{W} = (W_1, \dots, W_m)$ in the embedding defined by the line bundle $L_{\mathbf{k}}$ are equal to the product of m monomials of degree k_i in the Plücker coordinates of W_i. Since for each λ as in (11.1) we have

$$p_{i_0\dots i_r}(\lambda(t)\cdot W) = t^{q_{i_0}+\cdots+q_{i_r}} p_{i_0\dots i_r}(W),$$

it is easy to see that

$$\mu^{L_{\mathbf{k}}}(\mathcal{W},\lambda) = \sum_{i=1}^m k_i\Big(\sum_{j=0}^{r_i} q_{\nu_j^{(i)}}\Big).$$

Here $\nu_0^{(i)},\dots,\nu_{r_i}^{(i)}$ are defined for each $W_i, i = 1,\dots,m,$ as in the above. Using that $\dim(W_i\cap E_j) - \dim(W_i\cap E_{j-1}) = 0$ if $j\neq \nu_j^{(i)}$, we can rewrite the previous sum as follows:

$$\begin{aligned}
\mu^{L_{\mathbf{k}}}(\mathcal{W},\lambda) &= \sum_{i=1}^m k_i\Big(\sum_{j=0}^{n} q_j\Big(\dim(W_i\cap E_j)-\dim(W_i\cap E_{j-1})\Big)\Big)\\
&= \sum_{i=1}^m k_i\Big((r_i+1)q_n + \sum_{j=0}^{n-1}(\dim(W_i\cap E_j)+1)(q_j-q_{j+1})\Big)\\
&= q_n\sum_{i=1}^m k_i(r_i+1) + \sum_{j=0}^{n-1}\Big(\sum_{i=1}^m k_i(\dim(W_i\cap E_j)+1)(q_j-q_{j+1})\Big).
\end{aligned}$$

Since we want this number to be non-positive (resp. negative) for all λ, we can take the special one-parameter subgroup λ_s given by

$$q_0 = \cdots = q_s = n-s, q_{s+1} = \cdots = q_n = -(s+1), 0\le s\le n-1.$$

It is easy to see that any λ satisfying (11.1) is a positive linear combination of such one-parameter subgroups. Plugging in these values of q_j, we find

$$-\sum_{i=1}^m k_i(r_i+1)(s+1) + (n+1)\Big(\sum_{i=1}^m k_i(\dim(W_i\cap E_s)+1)\Big) \le 0 \ \ (\text{resp.} < 0). \tag{11.3}$$

Since any s-dimensional linear subspace of $\mathbb{P}^n$ is projectively equivalent to E_s, we obtain the necessary condition for semi-stability or stability stated in the theorem. It is also sufficient. In fact, if it is satisfied but $(W_1,\dots,W_m)$ is not semi-stable, we can find some $\lambda\in\mathcal{X}^*(\mathrm{SL}_{n+1})$ such that $\mu^{L_{\mathbf{k}}}(\mathcal{W},\lambda) > 0$. By choosing appropriate coordinates, we may assume that $\lambda\in\mathcal{X}(T)^*$ and satisfies (11.1). Then we write λ as a positive linear combination of λ_s's to obtain that $\mu^{L_{\mathbf{k}}}(\mathcal{W},\lambda_s) > 0$ for some s. Then the above computations show that (11.2) does not hold, contradicting our assumption. $\square$

Corollary 11.1. *Assume that the numbers* $\sum_{i=1}^{m} k_i(r_i+1)$ *and* $n+1$ *are coprime. Then*

$$X^{ss}_{\mathbf{r},n}(L_{\mathbf{k}}) = X^{s}_{\mathbf{r},n}(L_{\mathbf{k}}).$$

Let us rewrite Theorem 11.1 in the case where all r_i and k_i are equal (in this case the linearization is called *democratic*). We set

$$\begin{aligned} X^{\mathrm{ss}}_{r^m,n} &= X^{\mathrm{ss}}_{r^m,n}(L_{k^m}), \quad X^{\mathrm{s}}_{r^m,n} = X^{\mathrm{s}}_{r^m,n}(L_{k^m}), \\ P^m_{r,n} &= X^{\mathrm{ss}}_{r^m,n}/\!\!/\mathrm{SL}_{n+1}. \end{aligned}$$

Corollary 11.2.

$$\mathcal{W} \in X^{ss}_{r^m,n} \Leftrightarrow \sum_{i=1}^{m}(\dim(W_j \cap W)+1) \le (\dim W + 1)\frac{m(r+1)}{n+1},$$

for any proper subspace W *of* $\mathbb{P}^n$*. Also,*

$$\mathcal{W} \in X^{s}_{r^m,n} \Leftrightarrow \sum_{i=1}^{m}(\dim(W_j \cap W)+1) < (\dim W + 1)\frac{m(r+1)}{n+1}$$

for any proper subspace W *of* $\mathbb{P}^n$*.*

Let us consider some examples.

Example 11.1. Let $n = 1, \mathbf{k} = 1^m$. Taking W to be a point, we get that W can be equal to at most $m/2$ points among $\mathcal{W} = (p_1, \ldots, p_m) \in (\mathbb{P}^1)^m$ if $\mathcal{W}$ is semi-stable with respect to L_{1^m}. This is similar to the stability criterion for a binary form of degree n. This is not surprising, since $\mathrm{Hyp}_m(1) = (\mathbb{P}^1)^m/\Sigma_m$ and L_{1^m} is equal to the inverse image of $\mathcal{O}(1)$ under the projection $(\mathbb{P}^1)^m \to \mathrm{Hyp}_m(1)$. Note that if we change L_{1^m} to $L_{\mathbf{k}}$, where $k_1 + \cdots + k_{m-1} < k_m$, we get that $(p_1, \ldots, p_1, p_m)$ is semi-stable.

Example 11.2. Let us take $n = 2, r_i = 0, \mathbf{k} = (1, \ldots, 1)$. Then

$$\begin{aligned} &(p_1, \ldots, p_m) \quad \text{is semistable} \Leftrightarrow \text{no point is repeated more than} \\ &m/3 \quad \text{times and no more than} \quad 2m/3 \quad \text{points are on a line}. \end{aligned}$$

Semi-stability coincides with stability when 3 does not divide m.

For instance, let us take $n = 6$. Then stable sextuples of points are all distinct and have at most three collinear. On the other hand, semi-stable but not stable sextuples have either two coinciding points or four collinear points among them. It is easy to see that minimal closed orbits of semi-stable but not stable points are represented by sextuples $(p_1, \dots, p_6)$, where $p_i = p_j$ for some $i \neq j$ with the remaining four points on a line. Among them there are special orbits $\mathrm{O}_{ij,kl,st}$ corresponding to the sextuples with $p_i = p_j, p_k = p_l, p_s = p_t$, where $\{1, \dots, 6\} = \{i, j\} \sqcup \{k, l\} \sqcup \{s, t\}$. So $X^{\mathrm{ss}}_{1^6,2} /\!\!/ G$ is a four-dimensional variety, and $(X^{\mathrm{ss}}_{1^6,2} /\!\!/ G) \setminus (X^{\mathrm{s}}_{1^6,2}/G)$ is isomorphic to the union of 15 curves C_{ij} each isomorphic to $X^{\mathrm{ss}}_{1^4,1} /\!\!/ \mathrm{SL}_2 \cong \mathbb{P}^1$. Each curve C_{ij} contains three points $P_{ij,kl,mn}$ represented by the orbits $\mathrm{O}_{ij,kl,mn}$. Each point $P_{ij,kl,mn}$ lies on three curves C_{ij}, C_{kl} and C_{mn}.

Let us consider the subset Z of $X^{\mathrm{s}}_{1^6,2}$ of sextuples $(p_1, \dots, p_6)$ such that there exists an irreducible conic containing the points $p_1, \dots, p_6$. Since all irreducible conics are projectively equivalent, the orbit space $X^{\mathrm{s}}_{1^6,2}/\mathrm{SL}_3$ is isomorphic to the orbit space $((\mathbb{P}^1)^6)^{\mathrm{s}}/\mathrm{SL}_2$ of sextuples of distinct points on $\mathbb{P}^1$. However, as we will see later, its closure in $P^6_{0,2} = ((\mathbb{P}^2)^6)^{\mathrm{ss}} /\!\!/ \mathrm{SL}_3$ is not isomorphic to $P^6_{0,1} = ((\mathbb{P}^1)^6)^{\mathrm{ss}} /\!\!/ \mathrm{SL}_2$.

Example 11.3. Let us take $r = 1, n = 3, \mathbf{k} = (1, \dots, 1)$. Then we are dealing with sequences $(l_1, \dots, l_m)$ of lines in $\mathbb{P}^3$. Let us apply the criterion of semi-stabilty, taking W to be first a point, then a line, and finally a plane. In the first case we obtain

$$\#\{i : W \in W_i\} \leq m/2;$$

that is, no more than $m/2$ lines intersect at one point.

Taking W to be a line, we obtain

$$2\#\{i : W = W_i\} + \#\{i : W_i \neq W, W \cap W_i \neq \emptyset\} \leq m;$$

in particular, no more than $m/2$ lines coincide and no more than $m - 2t$ lines W_i intersect a line W_j which is repeated t times.

Finally, taking W to be a plane, we get

$$2\#\{i : W_i \subset W\} + \#\{i : W_i \not\subset W\} \leq 3m/2;$$

that is, no more than $m/2$ lines are coplanar.

For example, there are no stable points if $m \leq 4$. This follows from the fact that for any four lines in $\mathbb{P}^3$ there is a line intersecting all of them. There are no semi-stable points for $m = 1$. If $m = 2$, a pair of lines is semi-stable if and only if they don't intersect. It is easy to see that by a projective transformation a pair of skew lines is reduced to the two lines given by the equations $x_0 = x_1 = 0$ and $x_2 = x_3 = 0$. Thus we have one orbit. Similarly, if $m = 3$ we get one semi-stable orbit represented by the lines $x_0 = x_1 = 0,\ x_2 = x_3 = 0$, and $x_0 + x_2 = x_1 + x_3 = 0$. If $m = 4$, the formula for the dimension of the quotient space gives us that $\dim X^{\mathrm{ss}}/\!\!/G = 1 + \dim G_x$, where G_x is the stabilizer of a generic point in X^{ss}. In our case $\dim G_x > 0$ since there are no stable orbits. It is easy to see that $\dim G_x = 1$ (use that there is a unique quadric Q through the first three lines, and the fourth line is determined by two points of intersection with the quadric; the subgroup of the automorphisms of the quadric which fix two points and three lines in one ruling is isomorphic to $\mathbb{G}_m$). We will show later, by explicit computation of invariants, that

$$P^4_{1,3} = X^{\mathrm{ss}}_{1^4,3}/\!\!/\mathrm{SL}_4 \cong \mathbb{P}^2. \tag{11.4}$$

Let us give a geometric reason why this can be true. For any four skew lines in general position, there exist two lines which intersect them all (they are called *transversals*). This is a classical fact which can be proven as follows. Consider the unique quadric Q through the first three lines l_1, l_2, l_3. They belong to one ruling of lines on Q. The fourth line l_4 intersects Q at two points q_1, q_2. The two transversals are the lines from the other ruling of Q which pass through q_1, q_2. If the fourth line happens to be tangent to Q, so that $q_1 = q_2$, we get only one transversal. Now let t_1, t_2 be the two transversals. Then we have two ordered sets of four points on $\mathbb{P}^1$:

$$(p_1, p_2, p_3, p_4) = (l_1 \cap t_1, l_2 \cap t_1, l_3 \cap t_1, l_4 \cap t_1),$$

$$(p'_1, p'_2, p'_3, p'_4) = (l_1 \cap t_2, l_2 \cap t_2, l_3 \cap t_2, l_4 \cap t_2).$$

This defines a rational map

$$P^4_{1,3} - \to (P^4_{0,1} \times P^4_{0,1})/\Sigma_2 \cong (\mathbb{P}^1 \times \mathbb{P}^1)/\Sigma_2 \cong \mathbb{P}^2.$$

The proof that this map extends to an isomorphism consists of the study of how this construction can be extended to degenerate configurations.

11.2 Points in $\mathbb{P}^n$

Let us consider configurations of m points in $\mathbb{P}^n$. We have

Theorem 11.2. *Let $\mathcal{P} = (p_1, \dots, p_m) \in (\mathbb{P}^n)^m$. Then*

$$\mathcal{P} \in ((\mathbb{P}^n)^m)^{\mathrm{ss}}(L_{\mathbf{k}}) \quad \textit{(resp. } \mathcal{P} \in ((\mathbb{P}^n)^m)^{\mathrm{s}}(L_{\mathbf{k}})\textit{)}$$

if and only if for every proper linear subspace W of $\mathbb{P}^n$

$$\sum_{i, p_i \in W} k_i \leq \frac{\dim W + 1}{n+1}\Big(\sum_{i=1}^m k_i\Big)$$

(resp. the strict inequality holds).

In particular, if all $k_i = 1$, the last condition can be rewritten in the form

$$\#\{i : p_i \in W\} \leq \frac{\dim W + 1}{n+1} m \quad \text{(resp. } < \text{)}.$$

Corollary 11.3.

$$((\mathbb{P}^n)^m)^{\mathrm{ss}}(L_{\mathbf{k}}) \neq \emptyset \quad \Leftrightarrow \quad \forall i = 1, \dots, m, \;\; (n+1)k_i \leq \sum_{i=1}^m k_i,$$
$$((\mathbb{P}^n)^m)^{\mathrm{s}}(L_{\mathbf{k}}) \neq \emptyset \quad \Leftrightarrow \quad \forall i = 1, \dots, m, \;\; (n+1)k_i < \sum_{i=1}^m k_i.$$

Proof. If $m \leq n$, the left-hand side is empty and the assertion is obviously true in this case. We assume that $m > n$. Let

$$((\mathbb{P}^n)^m)^{\mathrm{gen}} = \Big\{(p_1, \dots, p_m) : \text{ each subset of } n+1 \text{ points spans } \mathbb{P}^n\Big\}.$$

This is an open nonempty subset of $(\mathbb{P}^n)^m$. We know that $((\mathbb{P}^n)^m)^{\mathrm{ss}}(L_{\mathbf{k}})$ is an open subset. So if it is not empty it has nonempty intersection with $((\mathbb{P}^n)^m)^{\mathrm{gen}}$. If we take a set of points $\mathcal{P} = (p_1, \dots, p_m)$ in the intersection, we obtain, since no two points p_i coincide, $(n+1)k_i \leq \sum_{i=1}^m k_i$ for each $i = 1, \dots, m$. Conversely, if this condition is satisfied then each point $\mathcal{P} = (p_1, \dots, p_m) \in ((\mathbb{P}^n)^m)^{\mathrm{gen}}$ is semi-stable with respect to $L_{\mathbf{k}}$. In fact, each subspace W of dimension s contains at most $s+1$ points p_i. Hence

$$\sum_{i, p_i \in W} k_i \leq (\dim W + 1) \max\{k_i : i = 1, \dots, m\} \leq \frac{\dim W + 1}{n+1}\Big(\sum_{i=1}^m k_i\Big).$$

This proves the assertion about the semi-stability. We prove the second assertion similarly. $\square$

Let

$$\Delta_{n,m} = \Big\{x = (x_1, \ldots, x_m) \in \mathbb{R}^m : \sum_{i=1}^m x_i = n+1, 0 \le x_i \le 1, i = 1, \ldots, m\Big\}.$$

This is called an $(m-1)$-dimensional *hypersimplex* of type n. One can restate the preceding corollary in the following form. Consider the cone over $\Delta_{n,m}$ in $\mathbb{R}^{m+1}$

$$C\Delta_{n,m} = \{(x, \lambda) \in \mathbb{R}^m \times \mathbb{R}_+ : x \in \lambda\Delta_{n,m}\}.$$

We have the injective map

$$\mathrm{Pic}^{\mathrm{SL}_{n+1}}((\mathbb{P}^n)^m) \to \mathbb{R}^{m+1}, L_{\mathbf{k}} \mapsto \Big(k_1, \ldots, k_m, (n+1)^{-1}\sum_{i=1}^m k_i\Big),$$

which allows us to identify $\mathrm{Pic}^{\mathrm{SL}_{n+1}}((\mathbb{P}^n)^m)$ with a subset of $\mathbb{R}^{m+1}$. We have

$$\mathrm{Pic}^{\mathrm{SL}_{n+1}}((\mathbb{P}^n)^m) \cap C\Delta_{n,m} = \Big\{L \in \mathrm{Pic}^{\mathrm{SL}_{n+1}}((\mathbb{P}^n)^m) : ((\mathbb{P}^n)^m)^{\mathrm{ss}}(L) \neq \emptyset\Big\}.$$

In fact, if the first m coordinates of a point $x \in \mathbb{R}^{m+1}$ from the left-hand side are all positive, this follows immediately from Corollary 11.3. Suppose some of the first coordinates of x are equal to zero, say the first t coordinates. Then $L_{\mathbf{k}} = \mathrm{pr}^*(L'_{\mathbf{k}})$, where $\mathrm{pr} : (\mathbb{P}^n)^m \to (\mathbb{P}^n)^{m-t}$ is the projection to the last $m-t$ factors, and $\mathbf{k}' = (k_{t+1}, \ldots, k_m)$. By applying Corollary 11.3 to $L'_{\mathbf{k}}$, we obtain that $((\mathbb{P}^n)^{m-t})^{\mathrm{ss}}(L'_{\mathbf{k}}) \neq \emptyset$. It is easy to see that

$$((\mathbb{P}^n)^m)^{\mathrm{ss}}(L_{\mathbf{k}}) = \mathrm{pr}^{-1}(((\mathbb{P}^n)^{m-t})^{\mathrm{ss}}(L'_{\mathbf{k}}))$$

and we have a commutative diagram

$$\begin{array}{ccc}
((\mathbb{P}^n)^m)^{\mathrm{ss}}(L_{\mathbf{k}}) & \xrightarrow{\mathrm{pr}} & ((\mathbb{P}^n)^{m-t})^{\mathrm{ss}}(L'_{\mathbf{k}}) \\
\downarrow & & \downarrow \\
((\mathbb{P}^n)^m)^{\mathrm{ss}}(L_{\mathbf{k}})/\!\!/\mathrm{SL}_{n+1} & \xrightarrow{\overline{\mathrm{pr}}} & ((\mathbb{P}^n)^{m-t})^{\mathrm{ss}}(L'_{\mathbf{k}})/\!\!/\mathrm{SL}_{n+1}
\end{array}$$

where the vertical arrows are quotient maps and the map $\overline{\mathrm{pr}}$ is an isomorphism.

Note that the relative boundary of the convex cone $C\Delta_{n,m}$ consists of points with one of the first m coordinates equal to zero, and of points $(x, \lambda) \in \mathbb{R}^{m+1}$ satisfying $(n+1)x_i = \lambda$ for some $i, 0 \le i \le m$. The intersection of the latter part of the boundary with $\mathrm{Pic}^{\mathrm{SL}_{n+1}}((\mathbb{P}^n)^m)$ consists of line bundles $L_{\mathbf{k}}$ such that $(n+1)k_i = \sum_{i=1}^{m} k_i$ for some i. This shows that all points from $((\mathbb{P}^n)^m)^{\mathrm{gen}}$ are semi-stable but not stable (with respect to $L_{\mathbf{k}}$). Since the set of stable points must be open, it must be empty.

Observe that $\mathcal{P} \in ((\mathbb{P}^n)^m)^{\mathrm{ss}}(L_{\mathbf{k}}) \setminus ((\mathbb{P}^n)^m)^{\mathrm{s}}(L_{\mathbf{k}})$ if and only if there exists a subspace W of dimension $d, 0 \le d \le n-1$, such that

$$(n+1)\sum_{p_i \in W} k_i = (\dim W + 1)\sum_{i=1}^{m} k_i.$$

This is equivalent to the condition that $L_{\mathbf{k}}$ belongs to the hyperplane

$$H_{I,d} := \Big\{(x_1,\ldots,x_m,\lambda) \in \mathbb{R}^m : \sum_{i\in I} x_i = \lambda d\Big\},$$

where I is a nonempty subset of $\{1,\ldots,m\}$. Let C be a connected component of $C\Delta_{n,m} \setminus \bigcup_{I,d} H_{I,d}$ (called a *chamber*). One can show that any two line bundles from the same chamber have the same set of semi-stable points. Suppose $L_{\mathbf{k}}$ belongs to some $H_{I,d}$ and does not lie on other hyperplanes $H_{J,d'}$. Then there are two chambers C_-, C_+ with common boundary $H_{I,d}$. We have a commutative diagram

$$\begin{array}{ccc} ((\mathbb{P}^n)^m)^{\mathrm{s}}(C_+)/\mathrm{SL}_{n+1} & \dashrightarrow & ((\mathbb{P}^n)^m)^{\mathrm{s}}(C_-)/\mathrm{SL}_{n+1} \\ \searrow & & \swarrow \\ & ((\mathbb{P}^n)^m)^{\mathrm{ss}}(L_{\mathbf{k}})/\!\!/\mathrm{SL}_{n+1} & \end{array}$$

Here $((\mathbb{P}^n)^m)^{\mathrm{s}}(C_\pm)$ means that we define the stability with respect to any $L_{\mathbf{k}}$ from $C_\pm$. The corner maps are birational morphisms, and the upper arrow is a birational map (a *flip*). We refer the reader to [23] for more general and precise results on this subject.

The spaces

$$P_n^m := P_{0,n}^m = ((\mathbb{P}^n)^m)^{\mathrm{ss}}(L_{1^m})/\!\!/\mathrm{SL}_{n+1}$$

can be described explicitly in a few cases. It follows from the construction of the quotient that

$$P_n^m = \mathrm{Projm}\Big(\bigoplus_{d\ge 0}\Gamma((\mathbb{P}^n)^m, L_{1^m}^{\otimes d})^{\mathrm{SL}_{n+1}}\Big) = \mathrm{Projm}\Big(\bigoplus_{d\ge 0}(\mathrm{Pol}_d(V^*)^{\otimes m})^{\mathrm{SL}_{n+1}}\Big),$$

where $\mathbb{P}^n = \mathbb{P}(V)$. Let us denote the graded algebra $\bigoplus_{d\geq 0}(\mathrm{Pol}_d(V^*)^{\otimes m})^{\mathrm{SL}_{n+1}}$ by R_n^m.

The First Fundamental Theorem tells us how to compute generators of the graded algebra R_n^m. We have

$$(R_n^m)_d = \mathrm{Pol}(\mathrm{Mat}_{n+1,m})^{\mathrm{SL}_{n+1}}_{d^m, w^{n+1}}. \tag{11.5}$$

Thus the space is generated by standard tableau functions μ_τ of size $(n+1)\times w$, degree d with $w = \frac{md}{n+1}$.

Remark 11.1. Note that the symmetric group Σ_m acts naturally on P_n^m, via permuting the factors. It acts on the graded algebra R_n^m via its action on the columns of matrices of size $(n+1)\times m$. The quotient P_n^m/Σ_m is the moduli space of (unordered) sets of m-points in $\mathbb{P}^n$. In the special case $n=1$, an unordered set of m-points is the set of zeros of a binary form of degree m. Recall that, by the First Fundamental Theorem, we have an isomorphism

$$\mathrm{Pol}_m(\mathrm{Pol}_d(V))^{\mathrm{SL}(V)} \cong (\mathrm{Pol}(\mathrm{Mat}_{n+1,m})^{\mathrm{SL}_{n+1}}_{d^m,w^{n+1}})^{\Sigma_m}.$$

In view of (11.5) we obtain an isomorphism

$$(R_n^m)_d^{\Sigma_m} \cong \mathrm{Pol}_m(\mathrm{Pol}_d(V))^{\mathrm{SL}(V)}.$$

Now, if we use Hermite Reciprocity (Theorem 5.6), we get an isomorphism

$$\phi_m : (R_n^m)_d^{\Sigma_m} \cong \mathrm{Pol}_d(\mathrm{Pol}_m(V))^{\mathrm{SL}(V)}. \tag{11.6}$$

It can be shown (see Remark 5.2) that the isomorphisms ϕ_m define an isomorphism of graded algebras

$$\left(\bigoplus_{d=0}^{\infty}(R_n^m)_d\right)^{\Sigma_m} \cong \bigoplus_{d=0}^{m}\mathrm{Pol}_d(\mathrm{Pol}_m(V))^{\mathrm{SL}(V)}.$$

The projective spectrum of the left-hand side is the variety P_1^m/Σ_m. The projective spectrum of the right-hand side is the variety $\mathrm{Hyp}_m(1)/\!\!/\mathrm{SL}_2$. Thus

$$P_1^m/\Sigma_m \cong C_m(1).$$

Example 11.4. Let us start with the case $n=1, m=4$. Then the degree 1 piece of (R_1^4) is spanned by the two functions $[12][34]$ and $[13][24]$. The value of the ratio $r = [12][34]/[13][24]$ on the set (p_1,p_2,p_3,p_4) defined by the coordinate matrix

$$A = \begin{pmatrix} a_0 & b_0 & c_0 & d_0 \\ a_1 & b_1 & c_1 & d_1 \end{pmatrix}$$

is equal to

$$r(p_1,p_2,p_3,p_4) = \frac{(a_0b_1-a_1b_0)(c_0d_1-c_1d_0)}{(a_0c_1-a_1c_0)(b_0d_1-b_1d_0)}.$$

This is called the *cross-ratio* of four ordered points. Two distinct ordered quadruples of points in $\mathbb{P}^1$ are projectively equivalent if and only if they have the same cross-ratio. If we choose coordinates in the form $(1,x_i), i=1,\ldots,4$, assuming that none of the points is the infinity point, we obtain

$$r(p_1,p_2,p_3,p_4) = \frac{(x_2-x_1)(x_4-x_3)}{(x_3-x_1)(x_4-x_2)}.$$

If $p=(0,1,\infty,x)=((1,0),(1,1),(0,1),(1,x))$ we get

$$r(0,1,\infty,x) = 1-x.$$

Note that the cross-ratio of four distinct points never takes the values $0,1,\infty$. The quadruples (p_1,p_2,p_3,p_4) go to 0 if $p_1=p_2$ or $p_3=p_4$. The only closed orbit in the fibre over 0 consists of configurations with $p_1=p_2, p_3=p_4$. Similarly, one describes the fibres over 1 and ∞. It is easy to see that the graded algebra R_1^4 is equal to $k[[12][34],[13][24]]$ and hence is isomorphic to the polynomial algebra $k[x,y]$ (prove this by following the next example). The permutation group Σ_4 acts on this algebra as follows:

$$\begin{aligned}(12)=(34) \quad &: \quad x\mapsto -x, \quad y\mapsto y-x,\\ (23) \quad &: \quad x\mapsto y, \quad y\mapsto x.\end{aligned}$$

This easily implies that

$$\mathrm{Pol}(\mathrm{Pol}_4(k^2))^{\mathrm{SL}_2} \cong k[x,y]^{\Sigma_4} = k[A,B],$$

where

$$A = x^2-xy+y^2, \quad B = -2x^3+3xy^2-2y^3+3x^2y.$$

Using (11.6) we can identify (up to a constant factor) these invariants with the invariants S and T from section 10.2 of Chapter 10.

Example 11.5. Let $n=1, m=5$. The computations here are more involved than in the case $m=6$ which we will discuss in the next example. Here we only sketch a proof that the space P_1^5 is isomorphic to a Del Pezzo surface $\mathcal{D}_5$ of degree 5 isomorphic to the blow-up of $\mathbb{P}^2$ with center at four points p_1,p_2,p_3,p_4

no three of which are on a line. The linear system of conics defines a morphism $f : \mathcal{D}_5 \to \mathbb{P}^1$. Its fibres are conics through the four points p_i. There are three singular fibres corresponding to three reducible conics. There are four sections of f corresponding to the exceptional curves E_i blown up from the points p_i. Let us construct a map $\Phi : \mathcal{D}_5 \to P_1^5$. If $x \in \mathcal{D}_5$ lies on a nonsingular fibre F, we consider the fibre as $\mathbb{P}^1$ and assign to x the orbit $\Phi(x)$ of the five points $(E_1 \cap F, \ldots, E_4 \cap F, x)$ in $\mathbb{P}^1$. If x lies on a singular fibre, say on the proper transform l of the line l_{12} passing through the points p_1, p_2 we assign to x the orbit of $(E_1 \cap l, E_2 \cap l, a, a, x)$, where a is the inverse image of the point $l_{12} \cap l_{34}$. If $x = a$ we assign to it the unique orbit of $(0, 0, 1, 1, \infty)$. Note that under this assignment the fibration map f corresponds to the natural map $P_1^5 \to P_1^4$ defined by the projection $(x_1, x_2, x_3, x_4, x_5) \mapsto (x_1, x_2, x_3, x_4)$. The three points in $\mathbb{P}^1$ over which the fibre is singular are the three special orbits of (a, a, b, b), (a, b, a, b) and (a, b, b, a). The section E_i corresponds to the set of orbits of $(x_1, x_2, x_3, x_4, x_5)$, where $x_5 = x_i$.

Example 11.6. Let $n = 1, m = 6$. A standard tableau of degree d and size $2 \times 3d$ is given by a table

$$\begin{bmatrix} a_1^1 & a_2^2 \\ a_2^1 & a_3^2 \\ a_3^1 & a_4^2 \\ a_4^1 & a_5^2 \\ a_5^1 & a_6^2 \end{bmatrix}, \tag{11.7}$$

where we use the notation from section 2.4. We have

$$\begin{aligned} &\mid a_1^1 \mid = \mid a_6^2 \mid = d, \quad \mid a_i^1 \mid + \mid a_i^2 \mid = d, \quad 2 \le i \le 5, \\ &\mid a_2^1 \mid + \mid a_3^1 \mid + \mid a_4^1 \mid + \mid a_5^1 \mid = 2d. \end{aligned}$$

Set

$$l_2 = \mid a_2^1 \mid, \ l_3 = \mid a_3^1 \mid, \ l_4 = \mid a_4^1 \mid .$$

These numbers satisfy the following inequalities:

$$\begin{aligned} 0 &\le l_2, l_3, l_4 \le d, \quad d \le l_2 + l_3 + l_4 \le 2d, \\ d &\le 2l_2 + l_3, \quad 2d \le 2l_2 + 2l_3 + l_4. \end{aligned}$$

The last two inequalities say that each row consists of two different numbers, so that

$$d + \mid a_2^1 \mid \ \ge \ \mid a_2^2 \mid + \mid a_3^2 \mid, \quad d + \mid a_2^1 \mid + \mid a_3^1 \mid \ \ge \ \mid a_2^2 \mid + \mid a_3^2 \mid + \mid a_4^2 \mid .$$

Setting $x = l_2, y = l_2+l_3, z = l_2+l_3+l_4$, we obtain that our tableau is completely determined by a vector (x, y, z) satisfying

$$\begin{aligned} 0 &\leq x \leq d, \quad 0 \leq y - x \leq d, \quad 0 \leq z - y \leq d, \\ d &\leq x + y, \quad y + z \geq 2d, \quad d \leq z \leq 2d. \end{aligned}$$

When $0 \leq y \leq d$ these inequalities are equivalent to

$$y \geq x \geq d - y, \quad 2d - y \leq z \leq y + d.$$

This gives $\sum_{i=d/2}^{d}(2i - d + 1)^2$ solutions. When $2d \geq y \geq d$ we have $y \leq z \leq 2d$ which gives $\sum_{i=d}^{2d}(2d - i + 1)^2$ solutions. Summing up, we get

$$\dim(R_1^6)_d = \frac{1}{2}(d^3 + 3d^2 + 4d) + 1.$$

Thus the Hilbert function of the graded ring R_1^6 is equal to

$$\sum_{d=0}^{\infty}(\tfrac{1}{2}(d^3 + 3d^2 + 4d) + 1)t^d = \frac{1 - t^3}{(1 - t)^5}.$$

This suggests that P_1^6 is isomorphic to a cubic hypersurface in $\mathbb{P}^4$. This is true. First of all we have the following generators of R_1^6:

$$\begin{aligned} t_0 &= [12][34][56], \quad t_1 = [13][24][56], \quad t_2 = [12][35][46], \\ t_3 &= [13][25][46], \quad t_4 = [14][25][36]. \end{aligned}$$

For every $(i, j) \neq (0, 3), (0, 4)$, the product $t_i t_j$ is a standard tableau function from $(R_1^6)_2$. Applying the straightening algorithm, we find

$$\begin{aligned} t_0 t_3 &= -[12][13][23][45][46][56] + t_1 t_2, \\ t_0 t_4 &= [12][14][24][35][36][56] - t_1 t_2 + t_0 t_1 + t_0 t_2 - t_0^2. \end{aligned}$$

So the standard monomials

$$y_1 = [12][13][23][45][46][56], \quad y_2 = [12][14][24][35][36][56]$$

can be expressed as polynomials of degree 2 in the t_i. Counting the number of standard tableau functions of size 2×6, we find that $(R_1^6)_2 = (R_1^6)^2$. In fact, we have $(R_1^6)_n = (R_1^6)^n$ for any n. If we take a tableau function $\mu_{a,b,c,k}$ corresponding to tableau (11.7) with $l_1 = a, l_2 = b, l_3 = c$, we can write it as

$$\mu_{a,b,c,k} = \begin{cases} t_0^{2k-2a-b-c} t_1^{k-c} t_2^c y_2^{a+b-k} & \text{if } a+b \geq k, \\ t_0^{2k-2a-b-c} t_1^{2a+b-k} t_2^{2a+2b+c-2k} y_1^{k-b-a} & \text{if } a+b \leq k, \end{cases}$$

whenever $2a+b+c \leq 2k$, and similarly

$$\mu_{a,b,c,k} = \begin{cases} t_1^{2k-2a-b-c} t_2^{k-a} t_3^{a+c-k} t_4^{a+b-k} & \text{if } a+b \geq k, a+c \geq k, \\ t_1^{k-b} t_1^c t_3^{2a+b+c-2k} y_2^{k-c-a} & \text{if } a+c \leq k, \\ t_1^{k-c} t_1^b t_3^{2a+b+c-2k} y_1^{k-b-a} & \text{if } a+b \leq k, \end{cases}$$

whenever $2a+b+c \leq 2k$. It is easy to verify that

$$t_3 y_2 = t_1 t_2 t_4$$

which gives us the cubic relation

$$t_1 t_2 t_3 - t_3 t_0 t_4 + t_3 t_1 t_2 + t_3 t_0 t_1 + t_3 t_0 t_2 - t_3 t_0^2 = 0.$$

Let

$$F_3 = T_1 T_2 T_3 - T_3 T_0 T_4 + T_3 T_1 T_2 + T_3 T_0 T_1 + T_3 T_0 T_2 - T_3 T_0^2.$$

There is a surjective homomorphism of the graded algebras

$$k[T_0, T_1, T_2, T_3, T_4]/(F_3(T_0, T_1, T_2, T_3, T_4)) \to R_1^6$$

and comparing the Hilbert functions we see that it is bijective. Thus $P_1^6 = \text{Projm}(R_1^6)$ is isomorphic to the cubic hypersurface $F_3(T_0, T_1, T_2, T_3, T_4) = 0$. If we change the variables,

$$\begin{aligned} Z_0 &= 2T_0 - T_1 - T_2 + T_3 + T_4, \quad Z_1 = T_1 - T_2 - T_3 + T_4, \\ Z_2 &= -T_1 + T_2 - T_3 + T_4, \quad Z_3 = T_1 + T_2 - T_3 - T_4, \\ Z_4 &= -T_1 - T_2 + T_3 - T_4, \quad Z_5 = -2T_0 + T_1 + T_2 + T_3 - T_4, \end{aligned}$$

we obtain that P_1^6 can be given by the equations

$$\sum_{i=0}^{5} Z_i = 0, \quad \sum_{i=0}^{5} Z_i^3 = 0$$

in $\mathbb{P}^5$ which manifest the Σ_6-symmetry. The cubic hypersurface defined by these equations is called the *Segre cubic primal*. It contains 10 nodes (the maximum possible number for a cubic hypersurface in $\mathbb{P}^4$) and 15 planes. The nodes correspond to the minimal closed orbits of semi-stable but not stable points. The singular points can be indexed by the subsets $\{i,j,k\}$ of $\{1,\dots,6\}$. For example, $p_{123} = (1,1,1,-1,-1,-1)$. The planes correspond to the orbits of sextuples with two coinciding points. They have equations of the form $Z_i + Z_j = Z_k + Z_l = Z_m + Z_n = 0$, where $\{i,j,k,l,m,n\} = \{1,\dots,6\}$. Each plane contains four singular points. Each point is contained in 6 planes. The blow-up of the plane at the four points is naturally isomorphic to P_1^5 (see Exercise 11.7).

Example 11.7. Let $n = 2$ and $m = 6$. Again we take $\mathbf{k} = (1,\dots,1)$ and try to compute the graded algebra R_2^6 explicitly. We skip the computations ([25], p.17) and give only the results. First we compute the Hilbert function of the graded algebra R_2^6:

$$\sum_{k=0}^{\infty} \dim(R_2^6)_k t^k = \frac{1-t^4}{(1-t)^5(1-t^2)}.$$

This suggests that R_2^6 is generated by five elements of degree 1 and one element of degree 2 with a relation of degree 4. We have the following.

Generators:

degree 1

$$t_0 = [123][456], t_1 = [124][356], t_2 = [125][346], t_3 = [134][256], t_4 = [135][246];$$

degree 2

$$t_5 = [123][145][246][356] - [124][135][236][456].$$

Relation:

$$t_5^2 + t_5(t_2t_3 + t_1t_4 + t_0t_1 + t_0t_4 + t_0t_2 + t_0t_3 + t_0^2)$$
$$+t_0t_1t_4(t_0 + t_1 + t_2 + t_3 + t_4).$$

This shows that P_2^6 is isomorphic to a hypersurface of degree 4 in the weighted projective space $\mathbb{P}(1,1,1,1,1,2)$ given by the equation

$$\begin{aligned} F_4 &= T_5^2 + T_5(-T_2T_3 + T_1T_4 + T_0T_1 + T_0T_4 - T_0T_2 - T_0T_3 - T_0^2) \\ &\quad +T_0T_1T_4(-T_0 + T_1 - T_2 - T_3 + T_4) = 0. \end{aligned}$$

If $\mathrm{char}(k) \neq 2$ this can be transformed into the equation

$$\begin{aligned} F_4 &= T_5^2 + (-T_2T_3 + T_1T_4 + T_0T_1 + T_0T_4 - T_0T_2 - T_0T_3 - T_0^2)^2 \\ &\quad +4T_0T_1T_4(-T_0 + T_1 - T_2 - T_3 + T_4) = 0. \end{aligned}$$

The equation is again symmetric with respect to a linear representation of Σ_6 in the variables $T_0, \ldots, T_4$ (but not with respect to the standard permutation representation in k^5). The quartic hypersurface V_4 in $\mathbb{P}^4$ given by the equation

$$\begin{aligned} &(-T_2T_3 + T_1T_4 + T_0T_1 + T_0T_4 - T_0T_2 - T_0T_3 - T_0^2)^2 \\ &+4T_0T_1T_4(-T_0 + T_1 - T_2 - T_3 + T_4) = 0 \end{aligned}$$

is called the *Segre quartic primal* (or *Igusa quartic*). It corresponds to the relation

$$[123][145][246][356] - [124][135][236][456] = 0.$$

If we fix the points $p_1, \ldots, p_5$ and vary p_6 we see that this is of degree 2 in the coordinates of p_6 and vanishes when $p_6 = p_i$ for some $i = 1, \ldots, 5$. Thus it describes the conic through the points $p_1, \ldots, p_5$ and expresses the condition that the six points are on a conic. Using the equation $F_4 = 0$, we can exhibit P_2^6 as a double cover of $\mathbb{P}^4$ branched along the Segre quartic hypersurface. In other words, there is an involution on P_2^6 whose fixed points are the sextuples lying on a conic. This is the *self-association involution*. We have a remarkable isomorphism, the *association isomorphism*:

$$a : P_n^m \cong P_{m-n-2}^m.$$

It is defined by the isomorphism of the graded algebras $R_n^m \to R_{m-n-2}^m$ defined on tableau functions by replacing each determinant $[i_1, \ldots, i_{n+1}]$ with the determinant $[j_1, \ldots, j_{m-n-1}]$, where $\{j_1, \ldots, j_{m-n-1}\} = \{1, \ldots, m\} \setminus \{i_1, \ldots, i_{n+1}\}$. In the case $m = 2n + 2$, we get an involutive automorphism of the algebra R_n^{2n+2} which defines the self-association involution of the variety P_n^{2n+2}. We refer to [25] and [27] for the details and for some geometric interpretations of the association isomorphism.

11.3 Lines in $\mathbb{P}^3$

Let us give an algebraic proof of the existence of the isomorphism (11.4). Recall that $\mathrm{Gr}_{1,3}$ is isomorphic to a nonsingular quadric in $\mathbb{P}^5$. Its automorphism group is the complex projective orthogonal group $\mathrm{PO}(6) = \mathrm{O}(6)/(\pm 1)$. The natural action of SL_4 on $\mathrm{Gr}_{1,3}$ defines an injective homomorphism from $\mathrm{PSL}(4)$ to $\mathrm{PO}(6)$. Counting the dimensions we see that the image is the connected component of the identity of the group $\mathrm{PO}(6)$. It is the subgroup $\mathrm{PO}(6)^+$ whose elements are represented by orthogonal matrices with determinant 1. Now the analysis of stability for lines in $\mathbb{P}^3$ shows that a semi-stable configuration of lines, considered as an ordered set of points in $\mathbb{P}^5$, is semi-stable with respect to the action of SL_4 in $\mathbb{P}^5$. Thus $P_{1,3}^m$ is a closed subset of the quotient $((\mathbb{P}^5)^m)^{\mathrm{ss}}/\!\!/\mathrm{O}^+(6)$. The latter can be computed using the First and the Second Fundamental Theorem of invariant theory for the orthogonal group. The symmetric bilinear form on the space $\Lambda^2(k^4) \cong k^6$ defined by the Grassmannian quadric is the wedge product. If V is a vector space equipped with a nondegenerate symmetric bilinear form $\langle v, w\rangle$, then the algebra of polynomial invariants of $\mathrm{O}(V)$ in the space $V^{\oplus m}$ is generated by the functions $[ij]$ defined by $[ij](v_1, \ldots, v_m) = \langle v_i, v_j\rangle$ (see Exercise 2.9, or [123]). This algebra is equal to the algebra of invariants for $\mathrm{O}(V)^+$ unless $m \geq \dim V$, when additional invariants are the basic invariants for $\mathrm{SL}(V)$, i.e., the bracket functions. For $m < \dim V$, there are no relations between the basic invariants. Now

$$(\mathbb{P}(V)^m)^{\mathrm{ss}}/\!\!/\mathrm{O}^+(V) \cong \bigoplus_{d=0}^{\infty} \Gamma(\mathbb{P}(V)^m, L_{1^m}^d)^{\mathrm{O}^+(V)} \cong \bigoplus_{d=0}^{\infty} (\mathrm{Pol}_d(V)^{\otimes m})^{\mathrm{O}^+(V)}.$$

As we saw in Chapter 2, elements of $\mathrm{Pol}_d(V)^{\otimes m}$ are polynomial functions on $V^{\oplus m}$ which are homogeneous of degree d in each factor. Thus the space of invariants $(\mathrm{Pol}_d(V)^{\otimes m})^{\mathrm{O}^+(V)}$ is spanned by monomials $[i_1 j_1] \ldots [i_s j_s]$ in $[ij]$ such that each index $a = 1, \ldots, m$ appears among $i_1, \ldots, i_s, j_1, \ldots, j_s$ exactly d times. In our case $m = 4$ we have 10 basic invariants $[ij]$. For $d = 1$ we have three monomials $[ij][kl]$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. For $d > 2$, we have products of these three monomials plus additionally the monomials which contain one of the monomials $[ii]$ as its factor. Now observe that the restriction of the function $[ii]$ to the subset of points in $\mathbb{P}(V)$ lying on the quadric $Q : \langle v, v\rangle = 0$ is obviously zero. Thus, the restriction of the algebra

$$\bigoplus_{d=0}^{\infty} \Gamma(\mathbb{P}(V)^4, L_{1^4}^d)^{\mathrm{O}^+(V)}$$

to Q^4 is freely generated by $[12][34], [13][24], [14][23]$. Its projective spectrum is $\mathbb{P}^2$.

Note that a similar computation can be made in the case $m = 5$ and $m = 6$ (see [119]). In the case $m = 6$, the algebra

$$\bigoplus_{d=0}^{\infty} \Gamma((\mathrm{Gr}_{1,3})^6, L_{1^6}^d)^{\mathrm{O}^+(6)}$$

is generated by the 15 functions $p_{ij,kl,mn} = [ij][kl][mn]$, where $\{i,j,k,l,m,n\} = \{1,2,3,4,5,6\}$, and the determinant function $D = [123456]$. The square of D^2 is the determinant of the Gram matrix $([ij]_{1\le i,j\le 6})$ and hence can be expressed as a polynomial in the $p_{ij,kl,mn}$. The subalgebra generated by the functions $p_{ij,kl,mn}$ is isomorphic to the projective coordinate algebra of a certain nine-dimensional toric variety Y (see the next chapter), so that $\mathbb{P}^6_{1,3}$ is isomorphic to a double cover of Y branched along a hypersurface defined by the equation $D = 0$. The locus of sextuples of lines defined by this hypersurface coincides with the locus of self-polar sextuples, i.e., the sextuples $(l_1,\ldots,l_6)$ for which there exists a nondegenerate quadric in $\mathbb{P}^3$ such that the set of the polar lines $(l_1^\perp,\ldots,l_6^\perp)$ is projectively equivalent to $(l_1,\ldots,l_6)$. Note the remarkable analogy with the structure of the variety P_2^6, where the analog of the polarity involution is the association involution.

Bibliographical notes

The stability criterion for configurations of linear spaces (with respect to the democratic linearization) was first given by Mumford ([75], Chapter 3). He also proved that the quotient map for stable configurations of points in $\mathbb{P}^n$ is a principal fibration of the group SL_{n+1}. The generalization of the criterion to the case of arbitrary linearization is straighforward. The cross-ratio invariant is as classical as can be. Examples 11.6 and 11.7 are taken from [25]. They go back to Coble [13] who found a beautiful relationship between the moduli spaces of points in $\mathbb{P}^n$ and classical geometry. The book [25] gives a modern exposition of some of the results of Coble. The invariants of lines in $\mathbb{P}^3$ are discussed in the book of Sturmfels ([115]). The algebra of SL_{n+1}-invariants on the tensor product of the projective coordinate algebras of four Grassmannians $\mathrm{Gr}_{r_i,n+1}, i = 1,2,3,4$ was studied by R. Howe and R. Huang ([52], [50]). They show that this ring is isomorphic to a polynomial algebra. In the case when $n+1 = 2r_1 = \cdots = 2r_4$ this was first proved by H. W. Turnbull ([118]). Note that the GIT quotient $X_{\mathbf{r},n}(L_{\mathbf{k}})/\!\!/\mathrm{SL}_{n+1}$ considered in this chapter is isomorphic to the projective spectrum of a subalgebra of the algebra of invariants in the tensor product of the projective coordinate algebras of the Grassmannians; so one needs additional work to compute the quotients. One can also describe all orbits of four lines in $\mathbb{P}^3$ (see [22]). The moduli spaces of five and six lines in $\mathbb{P}^3$ and their relationship to the classical algebraic geometry are discussed in the Ph. D. thesis of D. Vazzana ([119], [120]).

The rationality of the configuration spaces P_n^m of points is obvious. It is not known whether the spaces $X_{\mathbf{r},n}^{\mathrm{ss}}/\!\!/\mathrm{SL}_{n+1}$ are rational in general. This is known for lines in $\mathbb{P}^3$ ([125]) and, more generally, in the case when $(r_1+1,\ldots,r_m+1,n+1) \leq 3$ (see [102]).

Exercises

11.1 Prove that the orbit of $p = (p_1,\ldots,p_m)$ in $((\mathbb{P}^n)^m)^{ss}(L_{\mathbf{k}})$ is closed but not stable if and only if there exists a partition of $\{1,\ldots,m\}$ into subsets $J_s, s = 1,\ldots,r$, such that for any s one can find a proper subspace W_s of $\mathbb{P}^n$ such that $\sum_{i\in J_s, p_i\in W_s} k_i = (\dim W_s+1)(\sum_{i=1}^m k_i)/(n+1)$.

11.2 For what $\mathbf{k}$ is the quotient $((\mathbb{P}^1)^5)^{\mathrm{ss}}(L_{\mathbf{k}})$ isomorphic to $\mathbb{P}^2$?

11.3 Draw a picture of the hypersimplex $\Delta_{1,4}$ and describe the chambers of the cone $C\Delta_{1,4}$.

11.4 Consider the action of the permutation group Σ_4 on P_1^4 and show that the kernel of this action is isomorphic to the group $(\mathbb{Z}/2\mathbb{Z})^2$. Find the orbits whose stabilizers are of order strictly larger than 4. Compute the corresponding cross-ratio.

11.5 Prove that the algebra R_1^5 can be generated by six elements of degree 5 satisfying five linearly independent quadric relations.

11.6 Show that each projection $\pi : (\mathbb{P}^n)^m \to (\mathbb{P}^n)^{m-1}$ defines a rational map $\bar{\pi} : P_n^m \to P_n^{m-1}$.

(i) Find the points of indeterminacy of $\bar{\pi}$.

(ii) Show that $\bar{\pi}$ is a regular map if $(n+1,m) = 1$.

(iii) Construct $m-1$ rational sections $P_n^{m-1} \to P_n^m$ of $\bar{\pi}$.

11.7 Find the equation (in terms of functions $[ij]$) of the closure of the locus of quadruples of lines in $\mathbb{P}^3$ which have only one transversal line.

11.8 Prove that the closure of the locus of $(W_1,\ldots,W_5) \in \mathrm{Gr}_{1,3}^5$ which admit a common transversal line is of codimension 1. Find its equation in terms of functions $[ij]$.

11.9 Show that $\mathrm{Gr}_{r,n}$ is a homogeneous space isomorphic to G/P, where $G = \mathrm{SL}_{n+1}$ and P is its parabolic subgroup of matrices (a_{ij}) with entries $a_{ij} = 0$ for $r+1 < i \leq n+1, 0 \leq j \leq r+1$.

11.10 Consider the action of SL_2 on $\mathbb{P}^3$ via its linear representation in k^4 equal to the direct sum of the two standard two-dimensional representations of SL_2. Find stable and semi-stable points of the diagonal action of SL_2 on $X = \mathbb{P}^3 \times \mathbb{P}^3$ with respect to the line bundle $L_{1,1}$. Using the Fundamental Theorem of Invariant Theory show that $X^{\mathrm{ss}}/\mathrm{SL}_2 \cong \mathbb{P}^3$.

11.11 Find stable and semi-stable points in $X = (\mathbb{P}^3)^2 \times \mathrm{Gr}_{1,3}^3$ with respect to the group SL_4 and linearization L_{1^5} (three lines and two points in $\mathbb{P}^3$).

11.12 Prove that

(i) the Segre cubic primal V_3 is isomorphic to the image of $\mathbb{P}^3$ under the rational map to $\mathbb{P}^4$ given by the linear system of quadrics through five points $p_1, \ldots, p_5$ in general position;

(ii) the nodes of V_3 are the images of the lines ℓ_{ij} joining two points p_i, p_j,

(iii) the planes of V_3 are the images of the planes π_{ijk} through three points p_i, p_j, p_k,

(iv) the blowing up $\widetilde{\mathbb{P}^3}$ at the points $p_1, \ldots, p_5$ is a resolution of singularities of V_3 with inverse image of each node isomorphic to $\mathbb{P}^1$.

11.13 Let V_4 be the Segre quartic primal in $\mathbb{P}^4$. We use the notation from the preceding exercise. Prove that

(i) V_4 is isomorphic to the image of $\mathbb{P}^3$ under the rational map $\Phi : \mathbb{P}^3 - \to \mathbb{P}^4$ given by the linear system of quartics which pass through the points $p_1, \ldots, p_5$ with multiplicity 2 and contain the 10 lines ℓ_{ij},

(ii) V_4 contains 15 double lines, each line is intersected by three other double lines (find the meaning of the double lines and the corresponding points of intersection in terms of the quotient $((\mathbb{P}^2)^6)^{\mathrm{ss}} /\!\!/ \mathrm{SL}_3$),

(iii) the double lines are the images of the planes π_{ijk} under the rational map Φ,

(iv) the blowing up $\widetilde{\mathbb{P}^3}$ at the points $p_1, \ldots, p_5$ followed by the blowing up of the proper transforms of the lines ℓ_{ij} is a resolution of singularities of V_4,

(v) V_4 is isomorphic to the dual hypersurface of the Segre cubic primal V_3.

11.14 Describe the orbits of SL_4 in its diagonal action on $\mathrm{Gr}_{1,3}^4$. Match the minimal orbits of semi-stable points with points in $\mathbb{P}^2$.

Chapter 12

Toric varieties

12.1 Actions of a torus on an affine space

In this chapter we consider an interesting class of algebraic varieties which arise as categorical quotients of some open subsets of affine space. These varieties are generalizations of the projective spaces and admit a very explicit description in terms of some combinatorial data of convex geometry. In algebraic geometry they are often used as natural ambient spaces for embeddings of algebraic varieties and for compactifying moduli spaces. In combinatorics of convex polyhedra they have served as a powerful tool for proving some of the fundamental conjectures in the subject.

Let $T = \mathbb{G}_m^r$ act linearly on $\mathbb{A}^n$ by the formula

$$(t_1, \dots, t_r) \cdot (z_1, \dots, z_n) = (\mathbf{t}^{\mathbf{a}_1} z_1, \dots, \mathbf{t}^{\mathbf{a}_n} z_n),$$

where

$$\mathbf{a}_j = (a_{1j}, \dots, a_{rj}) \in \mathbb{Z}^r,\ \mathbf{t}^{\mathbf{a}_j} = t_1^{a_{1j}} \cdots t_r^{a_{rj}}.$$

As always we will identify the group $\mathcal{X}(T)$ with $\mathbb{Z}^r$ so that we consider the vectors $\mathbf{a}_j$ as characters of T. Since Pic$(\mathbb{A}^n)$ is trivial and $\mathcal{O}(\mathbb{A}^n)^* = k^*$, we have a natural isomorphism (see Chapter 5)

$$\mathrm{Pic}^T(\mathbb{A}^n) \cong \mathcal{X}(T) \cong \mathbb{Z}^r.$$

Let us fix $\mathbf{a} = (\alpha_1, \dots, \alpha_r) \in \mathbb{Z}^r$ and denote by $L_{\mathbf{a}}$ the corresponding linearized line bundle. It is the trivial line bundle $\mathbb{A}^n \times \mathbb{A}^1$ with the linearization defined by the formula

$$t \cdot (z, w) = (t \cdot z, t^{\mathbf{a}} w).$$

We identify its sections with polynomials $F \in k[Z_1, \ldots, Z_n]$. A polynomial F defines an invariant section of some nonnegative tensor power $L_{\mathbf{a}}^{\otimes d}$ if

$$F(t^{\mathbf{a}_1} Z_1, \ldots, t^{\mathbf{a}_n} Z_n) = t^{d\mathbf{a}} F(Z_1, \ldots, Z_n).$$

Here $t = (t_1, \ldots, t_r)$ are independent variables. It is clear that F belongs to $H^0(\mathbb{A}^n, L_{\mathbf{a}}^{\otimes d})^T$ if and only if F is equal to a linear combination of monomials $Z^{\mathbf{m}}$ such that $m_1\mathbf{a}_1 + \cdots + m_n\mathbf{a}_n = d\mathbf{a}$, or, equivalently,

$$A \cdot \mathbf{m} = d\mathbf{a}.$$

Let S be the set of nonnegative integral solutions of the system

$$(A| - \mathbf{a}) \cdot \begin{pmatrix} \mathbf{m} \\ d \end{pmatrix} = 0, \tag{12.1}$$

where the matrix of coefficients is obtained from A by adding to it one more column formed by the vector $-\mathbf{a}$.

The set of real nonnegative solutions of a linear system of equations forms a *convex polyhedral cone*. By definition, this is a subset of $\mathbb{R}^n$ given by a system of linear inequalities

$$\mathbf{c}_1 \cdot \mathbf{x} \geq 0, \ldots, \mathbf{c}_s \cdot \mathbf{x} \geq 0. \tag{12.2}$$

Obviously any linear equation $\mathbf{c} \cdot \mathbf{x} = 0$ can be considered as a pair of inequalities $(-\mathbf{c}) \cdot \mathbf{x} \leq 0, \mathbf{c} \cdot \mathbf{x} \leq 0$. A convex polyhedral cone is called a *rational convex polyhedral cone* if the vectors $\mathbf{c}_i$ can be chosen from $\mathbb{Q}^n$ (or equivalently from $\mathbb{Z}^n$). For every polyhedral cone σ one can define the dual cone:

$$\check{\sigma} = \{\mathbf{y} \in \mathbb{R}^n : \mathbf{x} \cdot \mathbf{y} \geq 0, \forall \mathbf{x} \in \sigma\}.$$

It is equal to the convex hull of the rays $\mathbb{R}_{\geq 0}\mathbf{c}_1, \ldots, \mathbb{R}_{\geq 0}\mathbf{c}_s$. It can be shown that the dual of a rational convex polyhedral cone is a rational convex polyhedral cone. We have

$$\check{\check{\sigma}} = \sigma.$$

This shows that any rational polyhedral cone can be defined as a convex hull of a finite set of positive rays spanned by vectors in $\mathbb{Z}^n$.

So we see that the set of vectors $(\mathbf{m}, d) \in \mathbb{Z}_{\geq 0}^{n+1}$ satisfying the system of linear equations (12.1) is equal to a set of the form $\sigma \cap \mathbb{Z}^{n+1}$ for some rational convex polyhedral cone σ in $\mathbb{R}^{n+1}$. Now we use

Lemma 12.1. *(P. Gordan) Let C be a rational convex polyhedral cone in $\mathbb{R}^n$. Then $C \cap \mathbb{Z}^n$ is a finitely generated submonoid of $\mathbb{Z}^n$.*

Proof. Let C be spanned by some vectors $v_1, \ldots, v_k$. The set

$$K = \Big\{ \sum_i x_i v_i \in \mathbb{R}^n : 0 \leq x_i \leq 1 \Big\}$$

is compact and hence its intersection with $\mathbb{Z}^n$ is finite. Let $\{w_1, \ldots, w_n\}$ be this intersection. This obviously includes the vectors v_i. We claim that this set generates the monoid $\mathcal{M} = C \cap \mathbb{Z}^n$. In fact we can write each $m \in \mathcal{M}$ in the form $m = \sum_i (x_i + m_i) v_i$, where m_i is a nonnegative integer and $0 \leq x_i \leq 1$. Thus $m = (\sum_i x_i v_i) + \sum_i (m_i v_i)$ is the sum of some vector w_j and a positive linear combination of vectors v_i. This proves the assertion. $\square$

For any commutative monoid $\mathcal{M}$ we denote by $k[\mathcal{M}]$ its monoid algebra. This is the k-linear space freely generated by elements of $\mathcal{M}$ with the multiplication law given on the generators by the monoid multiplication. If $\mathcal{M} = \mathbb{Z}^n$ we can identify $k[\mathcal{M}]$ with the algebra of Laurent polynomials $k[Z_1^{\pm 1}, \ldots, Z_n^{\pm 1}]$ by assigning to each $\mathbf{m} = (m_1, \ldots, m_n)$ the monomial $Z^{\mathbf{m}}$. If $\mathcal{M}$ is a submonoid of $\mathbb{Z}^n$ we identify $k[\mathcal{M}]$ with the subalgebra of $k[Z_1^{\pm 1}, \ldots, Z_n^{\pm 1}]$ which is generated by monomials $Z^m, m \in \mathcal{M}$.

Now we can easily construct a natural isomorphism of graded algebras

$$\bigoplus_{d \geq 0} \Gamma(\mathbb{A}^n, L_{\mathbf{a}}^{\otimes d})^T \cong k[S] = \bigoplus_{d \geq 0} k[S_d], \tag{12.3}$$

where S is the monoid of nonnegative vectors $\mathbf{m}$ which satisfy (12.1) for some $d \geq 0$, and $k[S_d]$ is the linear span of the subset $S_d \subset S$ of monomials $Z^{\mathbf{m}}$ with $A \cdot \mathbf{m} = d\mathbf{a}$. By Gordan's Lemma, $k[S]$ is a finitely generated graded algebra. Its homogeneous part of degree d is $k[S_d]$.

Let $k[S]_{>0}$ be the ideal $\bigoplus_{d>0} k[S]_d$. It can be generated by monomials and we choose a minimal set of monomial generators $Z^{\mathbf{m}_1}, \ldots, Z^{\mathbf{m}_s}$. For each $\mathbf{m}_j = (m_{1j}, \ldots, m_{nj})$ let $I_j := \{i : m_{ij} \neq 0\}$. For each subset I of $\{1, \ldots, n\}$ let $Z_I = \prod_{i \in I} Z_i$. Obviously, the open sets $D(Z^{\mathbf{m}_j}) = \mathbb{A}^n \setminus \{Z^{\mathbf{m}_j} = 0\}$ and $D(Z_{I_j}) = \mathbb{A}^n \setminus \{Z_{I_j} = 0\}$ coincide. By definition of semi-stability

$$(\mathbb{A}^n)^{\mathrm{ss}}(L_{\mathbf{a}}) = \bigcup_{j=1}^{s} D(Z_{I_j}).$$

For any $j = 1, \dots, s$, let

$$R_j = \mathcal{O}(D(Z_{I_j}))^T = \left\{ \frac{F(Z)}{(Z_{I_j})^p} : p \geq 0, F(Z) \in (Z_{I_j})^p k[S_0] \right\}, \tag{12.4}$$

where

$$S_0 = \{\mathbf{m} \in \mathbb{Z}^n_{\geq 0} : A \cdot \mathbf{m} = 0\}. \tag{12.5}$$

We know that the categorical quotient is obtained by gluing together the affine algebraic varieties X_j with $\mathcal{O}(X_j) \cong R_j$. We will now describe these rings and their gluing in terms of certain combinatorial structures.

12.2 Fans

Let $\mathbb{Z}^n \to \mathbb{Z}^r$ be the map given by the matrix A, and let M be its kernel. It is a free abelian group of rank $l = n - \text{rank}(A)$. Let

$$(\mathbb{Z}^n)^* \to N = M^* \tag{12.6}$$

be the map given by the restriction of linear functions to M. Let $(e_1^*, \dots, e_n^*)$ be the dual basis of the standard basis $(e_1, \dots, e_n)$ of $\mathbb{Z}^n$, and let $\bar{e}_1^*, \dots, \bar{e}_n^*$ be the images of these vectors in M^*. For each I_j let σ_j be the convex cone in the linear space

$$N_{\mathbb{R}} := N \otimes \mathbb{R} \cong \mathbb{R}^l$$

spanned by the vectors $\bar{e}_i^*, i \notin I_j$.

More explicitly, let $B = (b_{ij})$ be the matrix of size $l \times n$ whose rows are formed by a basis $(v_1, \dots, v_l)$ of M. If we choose to identify N with $\mathbb{Z}^l$ by means of the dual basis $(v_1^*, \dots, v_l^*)$, then

$$\bar{e}_i^* = \sum_{j=1}^{l} b_{ji} v_j^*, \quad i = 1, \dots, n.$$

This shows that σ_j is spanned in $\mathbb{R}^l = N_{\mathbb{R}}$ by the columns B_i of B with $i \notin I_j$.

Lemma 12.2. *Let R_j be as in (12.4). Then*

$$R_j \cong k[\check{\sigma}_j \cap M].$$

Proof. Obviously R_j is isomorphic to $k[\mathcal{M}]$, where

$$\mathcal{M} = \Big\{ m \in M : m + p \sum_{i \in I_j} e_i \in \mathbb{Z}^n_{\geq 0} \quad \text{for some } p \geq 0 \Big\}.$$

For each $i \in I_j$,

$$\bar{e}_i^* \Big(m + p \sum_{i \in I_j} e_i \Big) = \bar{e}_i^*(m) = m_i \geq 0 \Leftrightarrow m \in \mathcal{M}.$$

On the other hand

$$m \in \check{\sigma}_j \Leftrightarrow \bar{e}_i^*(m) \geq 0, \forall i \in I.$$

□

Lemma 12.3. *Let Σ be the set of convex cones $\sigma_j, j = 1, \ldots, s$. For any $\sigma, \sigma' \in \Sigma$, $\sigma \cap \sigma'$ is a face of both σ and σ'.*

Proof. Let $I = I_a, J = I_b$. We want to show that $\sigma_a \cap \sigma_b$ is a common face of σ_a and σ_b. Recall that a face of a convex set σ is the intersection of σ with a hyperplane such that σ lies in one of the two halfspaces defined by the hyperplane. We know that $\mathcal{O}(D(Z_I Z_J))^T$ is equal to the localization $\mathcal{O}(D(Z_I))^T_{Z^{\mathbf{c}}}$, where $\mathbf{c} = (c_1, \ldots, c_n) \in M$ and $c_i = 0$ for $i \notin I \cup J$. Considering $\mathbf{c}$ as a linear function on M^* we have

$$\mathbf{c}(\bar{e}_i^*) = e_i^*(\mathbf{c}) = 0 \quad \text{for } i \notin I \cup J.$$

This shows that $\mathbf{c}$ is identically zero on $\sigma_a \cap \sigma_b$. On the other hand, it follows from Lemma 12.2 that $\mathbf{c}$ is nonnegative on σ_a and on σ_b. This proves the assertion. □

Definition. A finite collection $\Sigma = \{\sigma_i\}_{i \in I}$ of rational convex polyhedral cones in $\mathbb{R}^n$ such that $\sigma_i \cap \sigma_j$ is a common face of σ_i and σ_j is called a *fan*.

In a coordinate-free approach one replaces the space $\mathbb{R}^n$ by any real linear space V of finite dimension, then chooses a lattice N in V, i.e., a finitely generated abelian subgroup of the additive group of V with $N \otimes \mathbb{R} = V$, and considers N-rational convex polyhedral cones, i.e., cones spanned by a finite subset of N. Then an N-*fan* Σ is a finite collection of N-rational polyhedral cones in V satisfying the property from the above definition. A version of this definition includes in the fan all faces of all cones $\sigma \in \Sigma$.

Let $M = N^*$ be the dual lattice in the dual space V^*. By Gordan's Lemma, for each $\sigma \in \Sigma$ the algebra $A_\sigma = k[\check{\sigma} \cap M]$ is finitely generated. Let $X_\sigma = \mathrm{Spm}(A_\sigma)$

be the affine variety with $\mathcal{O}(X_\sigma)$ isomorphic to $k[\check{\sigma} \cap M]$. Since for any $\sigma, \sigma' \in \Sigma, \sigma \cap \sigma'$ is a face in both cones, we obtain that $k[(\sigma \check{\cap} \sigma') \cap M]$ is a localization of each algebra A_σ and A'_σ. This shows that Spm$(k[(\sigma \check{\cap} \sigma') \cap M])$ is isomorphic to an open subset of X_σ and X'_σ. This allows us to glue together the varieties X_σ to obtain a separated (abstract) algebraic variety. It is denoted by X_Σ and is called the *toric variety* associated to the fan Σ. It is not always a quasi-projective algebraic variety.

By definition X_Σ has a cover by open affine subsets U_σ isomorphic to X_σ. Since each algebra A_σ is a subalgebra of $k[M] \cong k[Z_1^{\pm 1}, \ldots, Z_l^{\pm 1}]$ we obtain a morphism $(\mathbb{G}_m)^l \to X_\Sigma$. It is easy to see that this morphism is $(\mathbb{G}_m)^l$-equivariant if one considers the action of $(\mathbb{G}_m)^l$ on itself by left translations and on X_Σ by means of the $\mathbb{Z}^r$-grading of each algebra A_σ. If no cone $\sigma \in \Sigma$ contains a linear subspace, the morphism $(\mathbb{G}_m)^l \to X_\Sigma$ is an isomorphism onto an open orbit. In general, X_Σ always contains an open orbit isomorphic to a factor group of $(\mathbb{G}_m)^l$. All toric varieties X_Σ are normal and, of course, rational.

Keeping our old notations we obtain

Theorem 12.1. *Let $(\mathbb{Z}^n)^* \to M^*$ be the transpose of the inclusion map $M \to \mathbb{Z}^n$ and let N be its image. Let Σ be the N-fan formed by the cones $\sigma_j, j = 1, \ldots, s$. Then*

$$(\mathbb{A}^n)^{\mathrm{ss}}(L_{\mathbf{a}})/\!\!/T \cong X_\Sigma.$$

Recall that a cone in a linear space V is called simplicial if it is spanned by a part of a basis of V. A fan is called *simplicial* if each $\sigma \in \Sigma$ is simplicial. The geometric significance of this property is given by the following result, the proof of which can be found in [32].

Lemma 12.4. *A fan Σ is simplicial if and only if each affine open subset $U_\sigma, \sigma \in \Sigma$, is isomorphic to the product of a torus and the quotient of an affine space by a finite abelian group.*

In our situation, we have

Proposition 12.1. *Let X_Σ be the toric variety $(\mathbb{A}^n)^{\mathrm{ss}}(L_{\mathbf{a}})/\!/\!\!/T$. Assume the kernel of the action homomorphism $T \to \mathrm{Aut}(\mathbb{A}^n)$ is finite. Then Σ is simplicial if and only if*

$$(\mathbb{A}^n)^{\mathrm{ss}}(L_{\mathbf{a}}) = (\mathbb{A}^n)^{\mathrm{s}}(L_{\mathbf{a}}).$$

Proof. Assume some $\sigma \in \Sigma$ is not simplicial. We have to show that there exists a semi-stable but not stable point. Let $\bar{e}_i^*, i \notin I$, be the spanning vectors of σ. Since σ is not simplicial, $\sum_{i \notin I} c_i \bar{e}_i^* = 0$ for some integers c_i not all of which are zero. This implies that $\sum_{i \notin I} c_i e_i^*$ belongs to the annihilator $M^\perp$ of M in $(\mathbb{Z}^n)^*$. If we identify $(\mathbb{Z}^n)^*$ with $\mathbb{Z}^n$, then $M^\perp$ is isomorphic to the submodule spanned by the rows $\bar{A}_i$ of the matrix A. Thus we can write

$$\sum_{i \notin I} c_i e_i = b_1 \bar{A}_1 + \cdots + b_r \bar{A}_r = \mathbf{b} \cdot A$$

for some $\mathbf{b} = (b_1, \ldots, b_r) \in \mathbb{Z}^r$. This implies that $\mathbf{b} \cdot \mathbf{a}_j = 0$ for $j \in I$.

Let us consider the one-parameter subgroup $\lambda_0 \in \mathcal{X}(T)^*$ corresponding to the vector $\mathbf{b}$. It is defined by

$$\lambda_0(t) = (t^{b_1}, \ldots, t^{b_r}).$$

For any $t \in k^*$ and $z \in k^n$ we have

$$\lambda_0(t) \cdot z = (t^{\mathbf{b} \cdot \mathbf{a}_1} z_1, \ldots, t^{\mathbf{b} \cdot \mathbf{a}_n} z_n). \tag{12.7}$$

Take a point $p = (z_1, \ldots, z_n)$, where $z_j = 1$ if $j \in I$ and $= 0$ otherwise. Since $Z_I(p) \neq 0$, we see that $p \in (\mathbb{A}^n)^{\mathrm{ss}}(L_{\mathbf{a}})$. On the other hand, $\mu(\lambda_0, p) = 0$ and hence p is not stable.

Conversely, assume that there exists a semi-stable but not stable point. Arguing as above, we find a one-parameter subgroup λ_0 such that $\lambda_0 \cdot \mathbf{a}_j = 0$ for all $j \in I$ where $\sigma_I \in \Sigma$. Then $(c_1, \ldots, c_n) = \lambda_0 \cdot A$ has not all coordinates c_j equal to zero for $j \notin I$ and $c_j = 0$ for all $j \in I$. This gives $\sum_{j \notin I} c_j \bar{e}_j^* = 0$, hence σ_I is not simplicial. $\square$

Since every line bundle on an affine variety is ample, we obtain that the toric varieties $X_\Sigma = (\mathbb{A}^n)^{\mathrm{ss}}(L_{\mathbf{a}})/\!\!/T$ are always quasi-projective. Let us find out when they are projective.

Definition. A fan Σ in a linear space V is called *complete* if

$$V = \bigcup_{\sigma \in \Sigma} \sigma.$$

For the proof of the following basic result we refer to [32].

Lemma 12.5. *A fan Σ is complete if and only if the toric variety X_Σ is complete.*

Theorem 12.2. *Assume that $L_{\mathbf{a}}$ is not the trivial linearized bundle (i.e., $\mathbf{a} \neq 0$) and $(\mathbb{A}^n)^{\mathrm{ss}}(L_{\mathbf{a}}) \neq \emptyset$. The toric variety $(\mathbb{A}^n)^{\mathrm{ss}}(L_{\mathbf{a}})/\!\!/T$ is projective if and only if 0 is not contained in the convex hull of the character vectors $\mathbf{a}_j, j = 1, \ldots, n$.*

Proof. It follows from the construction of $(\mathbb{A}^n)^{\mathrm{ss}}(L_{\mathbf{a}})/\!\!/T$ that it is equal to the projective spectrum Projm$(k[S])$, where S is the monoid of solutions of the system (12.1). We have $k[S]_0 = k[M \cap \mathbb{Z}^n_{\geq 0}]$ and the inclusion $k[S]_0 \subset k[S]$ defines a surjective map Projm$(k[S]) \to$ Spm$(k[S]_0)$. It is easy to see that Projm$(k[S])$ is projective if and only if this map is constant, i.e., $k[S]_0 = k$. The latter is equivalent to $M \cap \mathbb{Q}^n_{\geq 0} = \{0\}$, i.e., the only nonnegative rational combination of the columns of A which is equal to 0 must be the zero combination. If this is not true, then $0 = m_1\mathbf{a}_1 + \cdots + m_n\mathbf{a}_n$ for some nonnegative integers m_i, and dividing both sides by $\sum_i m_i$ we see that 0 is in the convex hull $C =$ c.h.$(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ of the vectors $\mathbf{a}_j$. Conversely assume that $0 \in C$. Without loss of generality we can assume that $\mathbf{a}_1, \ldots, \mathbf{a}_n$ span $\mathbb{R}^n$. We can subdivide C into simplices to assume that 0 belongs to the convex hull of r vectors $\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_r}$ such that n among them are linearly independent. Then the space of solutions of the system of linear equations $\sum_{j=1}^r \lambda_j \mathbf{a}_{i_j} = 0$ is one-dimensional and is generated by a vector $v \in \mathbb{Z}^n$. Since $0 \in C$, we can assume that v has nonnegative coordinates, and hence $k[S]_0 \neq k$. This proves the assertion. □

Assume $(\mathbb{A}^n)^{\mathrm{ss}}(L_{\mathbf{a}})/\!\!/T$ is projective. Since 0 is not in the convex hull of the character vectors $\mathbf{a}_i$, there exists a linear function $f : \mathbb{R}^r \to \mathbb{R}$ such that $f(\mathbf{a}_i) > 0, i = 1, \ldots, n$. This is a well-known assertion from the theory of convex sets (called the Theorem on a Supporting Hyperplane). Obviously we can choose f to be rational, i.e., defined by $f(x_1, \ldots, x_n) = b_1x_1 + \cdots + b_nx_n$ for some $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{Q}^n$. Assume that $k[S] \neq k$, i.e., there exists a solution of $A \cdot \mathbf{m} = d\mathbf{a}$ for some $d > 0$. Then $q = \mathbf{a} \cdot \mathbf{b} > 0$. Let

$$q_i = \mathbf{b} \cdot \mathbf{a}_i, \quad i = 1, \ldots, n.$$

We can choose $\mathbf{b}$ such that $(q_1, \ldots, q_n, q) \in \mathbb{Z}^{n+1}_+$. For any $\mathbf{m} \in S_d$ we have

$$m_1\mathbf{a}_1 + \cdots + m_n\mathbf{a}_n = d\mathbf{a}. \tag{12.8}$$

Taking the dot-product of both sides with $\mathbf{b}$, we obtain

$$m_1q_1 + \cdots + m_nq_n = dq. \tag{12.9}$$

Consider the action of T on the weighted projective space $\mathbb{P} = \mathbb{P}(1, q_1, \ldots, q_n)$ given by the formula

$$(t_1, \ldots, t_r) \cdot (x_0, x_1, \ldots, x_n) = (t^{-\mathbf{a}} x_0, t^{(q+1)\mathbf{a}_1 - q_1\mathbf{a}} x_1, \ldots, t^{(q+1)\mathbf{a}_n - q_n\mathbf{a}} x_n). \tag{12.10}$$

The restriction of this action to the open subset $D(X_0) \cong \mathbb{A}^n$ of $\mathbb{P}(1, q_1, \ldots, q_n)$ coincides with the action

$$(t_1, \ldots, t_n) \cdot (x_1, \ldots, x_n) = (t^{(q+1)\mathbf{a}_1} x_1, \ldots, t^{(q+1)\mathbf{a}_n} x_n).$$

This action contains in its kernel the finite subgroup H of T equal to the group of points $(t_1, \ldots, t_r)$ such that $t_i^{q+1} = 1, i = 1, \ldots, r$. The induced action of the torus $T' = T/H$ is isomorphic to our old action. Clearly each $F \in k[X_0, \ldots, X_n]_l^T$ is a linear combination of monomials $X_0^{m_0} \cdots X_n^{m_n}$ such that

$$\begin{aligned}
&m_0 + m_1 q_1 + \cdots + m_n q_n = l, \\
&m_0(-\mathbf{a}) + m_1((q+1)\mathbf{a}_1 - q_1\mathbf{a}) + \cdots + m_n((q+1)\mathbf{a}_n - q_n\mathbf{a}) \\
&= (q+1)\sum_{i=1}^{n} m_i \mathbf{a}_i - l\mathbf{a} = (d(q+1) - l)\mathbf{a} = 0.
\end{aligned}$$

Comparing this with equations (12.8) and (12.9) we find an isomorphism of vector spaces

$$k[S_d] \to H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d(q+1)))^T, \quad Z_1^{m_1} \cdots Z_n^{m_n} \mapsto X_0^d X_1^{m_1} \cdots X_n^{m_n},$$

and also an isomorphism of graded algebras

$$\bigoplus_{d=0}^{\infty} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d(q+1)))^T \cong k[S].$$

Thus we obtain

$$\mathbb{P}(1, q_1, \ldots, q_n)^{\mathrm{ss}}(\mathcal{O}_{\mathbb{P}}(q+1))/\!\!/T \cong (\mathbb{A}^n)^{\mathrm{ss}}(L_{\mathbf{a}})/\!\!/T. \tag{12.11}$$

Obviously $(\mathbb{A}^n)^{\mathrm{ss}}(L_{\mathbf{a}}) = \mathbb{P}(1, q_1, \ldots, q_n)^{\mathrm{ss}}(\mathcal{O}_{\mathbb{P}}(q+1))$ since each point in the weighted projective space $\mathbb{P}(1, q_1, \ldots, q_n)$ lying on the hyperplane $X_0 = 0$ is unstable (because each $F \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d(q+1)))^T$ with $d > 0$ is divisible by T_0). To summarize we obtain

Proposition 12.2. *Let C be the convex hull of the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$. Assume that $0 \notin C$. Then $(\mathbb{A}^n)^{\mathrm{ss}}(L_{\mathbf{a}})/\!\!/T$ is projective and*

$$(\mathbb{A}^n)^{\mathrm{ss}}(L_{\mathbf{a}}) = \mathbb{P}(1, q_1, \ldots, q_n)^{\mathrm{ss}}(\mathcal{O}_{\mathbb{P}}(q+1)),$$

where $q = \mathbf{b}\cdot\mathbf{a} > 0, q_i = \mathbf{b}\cdot\mathbf{a}_i > 0$ for some $\mathbf{b} \in \mathbb{Z}^r$ and T acts on $\mathbb{P}(1, q_1, \ldots, q_n)$ by the formula (12.10)*.*

Applying the numerical criterion of stability we can find the set of unstable points in $\mathbb{P}(1, q_1, \ldots, q_n)$. It follows from Chapter 9 (up to some modifications using a weighted projective linearization, i.e. a G-equivariant embedding of a variety into a weighted projective space) that a point $x = (x_0, \ldots, x_n)$ is unstable if and only if the set $I = \{i_1, \ldots, i_k\}$ such that $x_i \neq 0, i \in I$, satisfies the property that 0 does not belong to the convex hull of the vectors $-\mathbf{a}, (q+1)\mathbf{a}_1 - q_1\mathbf{a}, \ldots, (q+1)\mathbf{a}_n - q_n\mathbf{a}$.

12.3 Examples

Let us give some examples.

Example 12.1. Let $\mathbb{G}_m$ act on $\mathbb{A}^{n+1}$ by the formula

$$t \cdot (z_0, \ldots, z_n) = (tz_0, \ldots, tz_n),$$

We have

$$\begin{aligned} A &= \begin{pmatrix} 1 & \ldots & \ldots & 1 \end{pmatrix}, \\ M &= \Big\{(m_0, \ldots, m_n) \in \mathbb{Z}^{n+1} : \sum_{i=0}^{n} m_i = 0\Big\}. \end{aligned}$$

It is easy to see that vectors $v_i = e_i - e_{i+1}, i = 1, \ldots, n$, form a basis of M. If we choose the dual basis $(v_1^*, \ldots, v_n^*)$ of $N = M^*$, the vectors $\bar{e}_i^*$ are equal to

$$\bar{e}_1^* = v_1^*, \bar{e}_2^* = -v_1^* + v_2^*, \ldots, \bar{e}_n^* = -v_{n-1}^* + v_n^*, \bar{e}_{n+1}^* = -v_n^*.$$

We can take for a new basis of M^* the vectors $\bar{e}_i^*, i = 2, \ldots, n+1$. Then

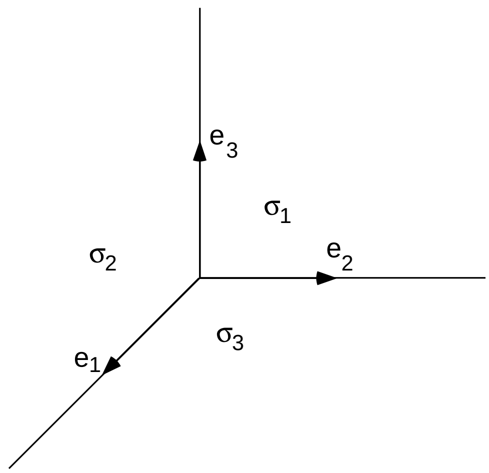
$$\bar{e}_1^* = -(\bar{e}_2^* + \cdots + \bar{e}_{n+1}^*).$$

Let us linearize the action by taking the line bundle L_a, where $a = 1$. Then we have an isomorphism of graded rings

$$\bigoplus_{d\geq 0} \Gamma(\mathbb{A}^{n+1}, L_1^{\otimes d})^{\mathbb{G}_m} = k[Z_0, \ldots, Z_n].$$

Obviously the minimal generators of the ideal $k[S]_{>0}$ are the unknowns Z_i. Thus the cones of our fan Σ are

$$\sigma_j = \text{span}\{\bar{e}_1^*, \ldots, \bar{e}_{j-1}^*, \bar{e}_{j+1}^*, \ldots, \bar{e}_{n+1}^*\}, \quad j = 1, \ldots, n+1.$$



This is the fan defining the projective space $\mathbb{P}^n$ (see [32]). Let us see the corresponding gluing. We can take for a basis of M the dual basis of $(\bar{e}_2^*, \ldots, \bar{e}_{n+1}^*)$ which is the set of vectors

$$e_2 - e_1, \ldots, e_{n+1} - e_1.$$

We easily find

$$k[\check{\sigma}_1 \cap M] = k[\tfrac{Z_1}{Z_0}, \ldots, \tfrac{Z_n}{Z_0}],\ \ldots,\ k[\check{\sigma}_{n+1} \cap M] = k[\tfrac{Z_0}{Z_n}, \ldots, \tfrac{Z_{n-1}}{Z_n}].$$

These are the coordinate rings of the standard open subsets of $\mathbb{P}^n$.

Example 12.2. Consider the action of $\mathbb{G}_m$ on $\mathbb{A}^4$ by the formula

$$t \cdot (z_1, z_2, z_3, z_4) = (tz_1, tz_2, t^{-1}z_3, t^{-1}z_4).$$

We have

$$A = \begin{pmatrix}1 & 1 & -1 & -1\end{pmatrix},$$

$$M = \{(m_1, m_2, m_3, m_4) \in \mathbb{Z}^4 : m_1 + m_2 - m_3 - m_4 = 0\}.$$

Let us choose the following basis of M:

$$v_1 = -e_1 + e_2, v_2 = e_1 + e_3, v_3 = e_1 + e_4.$$

We can express the vectors $\bar{e}_i^*$ in terms of the dual basis $(v_1^*, \ldots, v_3^*)$ of $N = M^*$ as follows:

$$\bar{e}_1^* = -v_1^* + v_2^* + v_3^*,\ \bar{e}_2^* = v_1^*,\ \bar{e}_3^* = v_2^*,\ \bar{e}_4^* = v_3^*.$$

Choose $L = L_1$ and consider the monoid S of nonnegative solutions of the equation

$$m_1 + m_2 - m_3 - m_4 - d = 0,\ m_i \geq 0, d > 0.$$

For any $(\mathbf{m}, d) \in S$ we have $d \leq m_1 + m_2$. If $d \leq m_1$ or $d \leq m_2$ we can subtract $d(1,0,0,0,1)$ or $d(0,1,0,0,1)$ from $(\mathbf{m}, d)$ to obtain a vector from S_0. If $d \geq m_1$ we have $d - m_1 \leq m_2$, and we do the same by subtracting $(d-m_1)(0,1,0,0,1) + m_1(1,0,0,0,1)$. This shows that $k[S]$ is generated over $k[S]_0$ by Z_1 and Z_2. This means that the unknowns Z_1, Z_2 are the minimal generators of the ideal $k[S]_{>0}$. Thus the fan Σ consists of two cones

$$\sigma_1 = \text{span}\{\bar{e}_2^*, \bar{e}_3^*, \bar{e}_4^*\},\ \sigma_2 = \text{span}\{\bar{e}_1^*, \bar{e}_3^*, \bar{e}_4^*\}.$$

The dual cones are

$$\check{\sigma}_1 = \text{span}\{-e_1 + e_2, e_1 + e_3, e_1 + e_4\},\ \check{\sigma}_2 = \text{span}\{-e_2 + e_1, e_2 + e_3, e_2 + e_4\}.$$

The quotient X_Σ is obtained by gluing together two nonsingular algebraic varieties with the coordinate algebras

$$\begin{aligned} k[\check{\sigma}_1 \cap M] &\cong k[Z_1Z_3, Z_1Z_4]\Big[\frac{Z_2}{Z_1}\Big], \\ k[\check{\sigma}_2 \cap M] &\cong k[Z_2Z_3, Z_2Z_4]\Big[\frac{Z_1}{Z_2}\Big]. \end{aligned}$$

Similarly if we take $L = L_{-1}$ we get that the fan Σ consists of two cones

$$\sigma_1 = \text{span}\{\bar{e}_1^*, \bar{e}_2^*, \bar{e}_4^*\},\ \sigma_2 = \text{span}\{\bar{e}_1^*, \bar{e}_2^*, \bar{e}_3^*\}.$$

The quotient X_Σ is obtained by gluing together two nonsingular algebraic varieties with the coordinate algebras

$$k[\check{\sigma}_1 \cap M] \cong k[Z_1Z_3, Z_2Z_3]\Big[\frac{Z_4}{Z_3}\Big],\ k[\check{\sigma}_2 \cap M] \cong k[Z_1Z_4, Z_2Z_4]\Big[\frac{Z_3}{Z_4}\Big].$$

If we now change the linearization by taking $L = L_0$ we get $L = L_0^{\otimes d} = L_0$ for all $d \geq 0$, hence $k[S]_{>0}$ is generated by 1. Then we have only one cone spanned by the four vectors $\bar{e}_i^*$. The toric quotient is isomorphic to the affine variety with the coordinate algebra

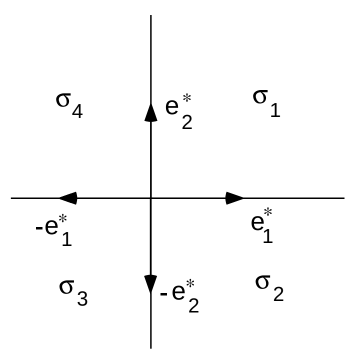
$$k[\check{\sigma} \cap M] \cong k[Z_1Z_3, Z_1Z_4, Z_2Z_3, Z_2Z_4] \cong k[T_1, T_2, T_3, T_4]/(T_1T_4 - T_2T_3).$$

One should compare this with our previous computation of this quotient in Example 8.6 from Chapter 8. We see here a general phenomenon: two toric varieties X_Σ and X'_Σ whose fans have the same set of one-dimensional edges of their cones (called the 1-*skeleton* of a fan) differ by a special birational modification. We refer the interested reader to [92] for more details.

Example 12.3. Let Σ consist of the following four cones in $\mathbb{R}^2$:

$$\begin{aligned} \sigma_1 &= \text{span}\{e_1, e_2\}, \quad \sigma_2 = \text{span}\{e_1, -e_2\}, \\ \sigma_3 &= \text{span}\{-e_1, -e_2\}, \quad \sigma_4 = \text{span}\{-e_1, e_2\}. \end{aligned}$$

This is shown in the following figure.



We have

$$U = \mathbb{A}^4 \setminus \{Z_3Z_4 = Z_1Z_2 = Z_2Z_3 = Z_1Z_4 = 0\},$$
$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad \mathbf{a} = (1,1).$$

hence the action is given by

$$(t_1,t_2)\cdot(z_1,z_2,z_3,z_4) = (t_1z_1, t_2z_2, t_1z_3, t_2z_4).$$

The variety X_Σ is obtained by gluing four affine planes with coordinate rings

$$k[Z_1,Z_2], k[Z_1,Z_2^{-1}], k[Z_1^{-1},Z_2^{-1}], k[Z_1^{-1},Z_2].$$

It is easy to see that X_Σ is isomorphic to the product $\mathbb{P}^1\times\mathbb{P}^1$. This is also seen by observing that

$$U/T = (\mathbb{A}^2\setminus\{Z_1=Z_3=0\})/\mathbb{G}_m \times (\mathbb{A}^2\setminus\{Z_2=Z_4=0\})/\mathbb{G}_m = \mathbb{P}^1\times\mathbb{P}^1.$$

Example 12.4. Recall that the coordinate ring of the Grassmannian $\mathrm{Gr}_{n,m-1}$ is isomorphic to $\mathrm{Pol}(\mathrm{Mat}_{n+1,m})^{\mathrm{SL}_{n+1}}$. It is generated by the bracket functions $p_I, I\subset\{1,\dots,m\}$. The torus of diagonal matrices $T\cong(\mathbb{G}_m)^m$ in GL_m acts naturally on $k[\mathrm{Mat}_{n+1,m}]$ by multiplying a matrix on the right by a diagonal matrix. It is easy to see that each function p_I spans an eigensubspace corresponding to the character $t\mapsto t^{\mathbf{e}_I}$, where $\mathbf{e}_I=\sum_{j\in I}e_j$. Consider the cone $\widetilde{\mathrm{Gr}}_{n,m-1}$ over $\mathrm{Gr}_{n,m-1}$ as a closed subvariety of $X=\mathbb{A}^{\binom{m}{n+1}}$. Then the torus T acts on X by multiplying each coordinate function p_I by $\mathbf{t}^{\mathbf{e}_I}$. Thus the action is given by the matrix A with columns equal to $\mathbf{e}_I$. Let the linearized line bundle be $L_{\mathbf{a}}$, where $\mathbf{a}=(1,\dots,1)$. It is easy to see that

$$\Gamma(X, L_{\mathbf{a}}^{\otimes d})^T \cong k[S_d],$$

where S_d is the set of vectors $e_{I_1}+\cdots+e_{I_w}$ where each $j\in\{1,\dots,m\}$ appears exactly d times in the sets $I_1,\dots,I_w$. In other words, S_d is in a bijective correspondence with the set of tableaux of degree d and size $(n+1)\times w$, where $(n+1)w=md$. Let $\overline{L}_{\mathbf{a}}$ be the restriction of $L_{\mathbf{a}}$ to $\widetilde{\mathrm{Gr}}_{n,m-1}$. Then

$$\Gamma(\widetilde{\mathrm{Gr}}_{n,m-1}, \overline{L}_{\mathbf{a}}^{\otimes d})^G \cong \mathrm{Pol}(\mathrm{Mat}_{n+1,m})^{\mathrm{SL}_{n+1}}_{d,\dots,d} \cong (\mathrm{Pol}_d(k^{n+1})^{\otimes m})^{\mathrm{SL}_{n+1}}.$$

This shows that

$$\widetilde{\mathrm{Gr}}_{n,m-1}/\!\!/T \cong P_n^m = ((\mathbb{P}^n)^m)^{\mathrm{ss}}/\!\!/\mathrm{SL}_{n+1}.$$

Also, we see that there is a natural closed embedding

$$P_n^m \hookrightarrow \left(\mathbb{A}^{\binom{m}{n+1}}\right)^{\mathrm{ss}}(L_{\mathbf{a}})/\!\!/T.$$

The latter quotient is a toric variety X_Σ of dimension $\binom{m}{n+1} - m$, where Σ depends only on (n, m). Let us denote it by $\Sigma(n, m)$. For example, take $n = 1, m = 4$. We have

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

It is easy to see that the monoid S_n of nonnegative integer solutions of the equation $A \cdot \mathbf{m} = n(1, 1, 1, 1)$ consists of vectors $(m_1, m_2, m_3, m_3, m_2, m_1)$ with $m_1, m_2, m_3 \geq 0, m_1 + m_2 + m_3 = n$. Thus $k[S]_n \cong k[X_1, X_2, X_3]_n$ and $k[S] \cong k[X_1, X_2, X_3]$. Thus

$$X_{\Sigma(1,4)} \cong \mathbb{P}^2.$$

The embedding $P_1^4 \to \mathbb{P}^2$ is of course the Veronese embedding.

One can go in the opposite direction by identifying any toric variety X_Σ with a categorical quotient of some open subset of an affine space. We state without proof the following result of D. Cox ([16]).

Theorem 12.3. *Let X_Σ be a toric variety determined by a $\mathbb{Z}^l$-fan Σ. To each one-dimensional edge of the 1-skeleton of Σ assign a variable Z_i and consider the polynomial algebra $k[Z_1, \ldots, Z_n]$ generated by these variables. For each cone $\sigma \in \Sigma$ let $Z_{I(\sigma)} \in k[Z_1, \ldots, Z_n]$, where $I(\sigma) \subset \{1, \ldots, n\}$ is the complementary set to the 1-skeleton of σ. Let $U = \mathbb{A}^n \setminus V(\{Z_{I(\sigma)}\}_{\sigma \in \Sigma})$. Let $\bar{e}_i^*$ be the primitive vectors of the lattice $\mathbb{Z}^l$ which span one-dimensional edges of the cones from Σ. Let B be the matrix whose columns are the vectors $\overline{e_i}^*, i = 1, \ldots, n$, and let A be an $(r \times n)$ matrix whose rows form a basis of the module $\mathrm{Null}(B) \cap \mathbb{Z}^n$. Assume that the vectors $\overline{e_i}^*$ span $\mathbb{Z}^l$. Then*

(i)

$$X_\Sigma \cong U/\!\!/T,$$

with the action of $T = (\mathbb{G}_m)^r$ given by the formula

$$t \cdot (z_1, \ldots, z_n) = (t^{\mathbf{a}_1} z_1, \ldots, t^{\mathbf{a}_n} z_n),$$

where $\mathbf{a}_j$ are the columns of A,

(ii) X_Σ is simplicial if and only if $U/\!\!/T = U/T$.

Remark 12.1. Note that applying this construction to the toric varieties X_Σ obtained as the quotients $(\mathbb{A}^n)^{\mathrm{ss}}(L_{\mathbf{a}})/\!\!/T$ we obtain $U = (\mathbb{A}^n)^{\mathrm{ss}}(L_{\mathbf{a}})$ and the action is isomorphic to the one we started with. However, in general, $U \neq (\mathbb{A}^n)^{\mathrm{ss}}(L_{\mathbf{a}})$ for any $\mathbf{a} \in \mathbb{Z}^r$. One reason for this is that our quotients are always quasi-projective and there are examples of nonquasi-projective toric varieties. Another reason is simpler. The fans we are getting from our quotient constructions are "full" in the following sense. One cannot extend them to larger fans with the same 1-skeleton.

The torus T which acts on U has a very nice interpretation. Its character group $\mathcal{X}(T)$ is naturally isomorphic to the group $\mathrm{Cl}(X_\Sigma)$ of classes of Weil divisors on X_Σ.

Also, if the vectors $\bar{e}_i$ do not span $\mathbb{Z}^l$, the assertion is true if we replace T by a diagonalizable algebraic group, an extension of T with the help of a finite abelian group.

Bibliographical notes

The theory of toric varieties is a subject of many books and articles. We refer to [32] and [84] for the bibliography. The fact that any toric variety can be obtained as a categorical quotient of an open subset of affine space was first observed by M. Audin ([3]) and D. Cox ([16]). The relationship between solutions of systems of linear integral equations, Gröbner bases and toric varieties is a subject of the book [113]. The systematic study of quotients of toric varieties by a torus can be found in [57]. We refer to [51] and [10] for the theory of variation of a torus quotient with respect to the linearization.

Exercises

12.1 Consider the action $t \cdot (z_1, z_2, z_3) = (tz_1, t^{-1}z_2, tz_3)$ and take $L = L_1$. Show that the quotient X_Σ is isomorphic to the blow-up of $\mathbb{A}^2$ at the origin. Draw the corresponding fan.

12.2 Let $T = (\mathbb{G}_m)^4$ act on $\mathbb{A}^6$ by the formula

$$t \cdot z = (t_1t_3^{-1}t_4z_1, t_2t_3t_4^{-1}z_2, t_4z_3, t_3z_4, t_2z_5, t_1z_6).$$

Take $L = L_{\mathbf{a}}$, where $\mathbf{a} = (1, 1, 1, 1, 1, 1)$. Show that the quotient is isomorphic to the blow-up of the projective plane at three points. Draw the picture of the fan.

12.3 Take a fan Σ in $\mathbb{R}^3$ formed by three one-dimensional cones spanned by the unit vectors e_1, e_2, e_3. Using Cox's Theorem represent the toric variety X_Σ as a geometric quotient.

12.4 A toric variety X_Σ is nonsingular if and only if each $\sigma \in \Sigma$ is spanned by a part of a basis of the lattice N. Show that $U/T = X_\Sigma$ is nonsingular if and only if the stabilizer of each point of U is equal to the same subgroup of T.

12.5 Describe the fan $\Sigma(1,5)$ and the corresponding toric variety $X_{\Sigma(1,5)}$.

12.6 Show that the moduli space of six lines in $\mathbb{P}^3$ is isomorphic to a double cover of the toric variety $X_{\Sigma(1,6)}$.

12.7 Consider the isomorphism $\mathrm{Gr}_{n,m-1} \cong \mathrm{Gr}_{m-n-2,m-1}$ defined by assigning to a linear subspace L of a linear space V its annihilator $L^\perp$ in the dual space V^*. Show that this isomorphism commutes with the action of the torus $\mathbb{G}_m^m$, and induces an isomorphism of the quotients $P_n^m \cong P_{m-n-1}^m$. Show that this isomorphism coincides with the association isomorphism defined in Chapter 11.

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